Complex analysis 1

Exercises 5

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February 16, 2024

Exercise 1

Let $U \subset \mathbb{C}$ be open, let $f, g : U \to \mathbb{C}$ be complex differentiable at z with $g'(z) \neq 0$, and let $f(z) = 0 = g(z)$. Show that

$$\lim_{\substack{w \to z \\ w \in \mathbb{C} \setminus \{z\}}} \frac{f(w)}{g(w)} = \frac{f'(z)}{g'(z)}.$$

Solution:

Since we will divide by $g(w)$ and $g(z) = 0$, I want to first make sure that $g(w) \neq 0$ for w close enough to z. This follows from proposition 3.1.6:

$$g(w) = g(z) + g'(z)(w - z) + \varepsilon_g(w)(w - z).$$

Since $g(z) = 0$ and $\varepsilon_g(w) \to 0$ as $w \to z$ then

$$\begin{aligned} |g'(z)(w - z)| &\leq |g'(z)(w - z) + \varepsilon_g(w)(w - z)| + |\varepsilon_g(w)(w - z)| \\ &= |g(w)| + |e_g(w)(w - z)| \end{aligned}$$

and from this we get

$$\begin{aligned} |g(w)| &\geq |g'(z)(w - z)| - |\varepsilon_g(w)(w - z)| \\ &= (|g'(z)| - |\varepsilon_g(w|)|w - z| \\ &> 0 \end{aligned}$$

for w close enough to z, since $\varepsilon_g(w) \to 0$ as $w \to z$. So for w close enough to z, $g(w) \neq 0$ and $f(w)/g(w)$ therefore makes sense.

Again from proposition 3.1.6 we get

$$f(w) = f(z) + f'(z)(w - z) + \varepsilon_f(w)(w - z).$$

Using this and $f(z) = 0 = g(z)$ we have

$$\begin{aligned} \lim_{w \to z} \frac{f(w)}{g(w)} &= \lim_{w \to z} \frac{f'(z)(w-z) + \varepsilon_f(w)(w-z)}{g'(z)(w-z) + \varepsilon_g(w)(w-z)} \\ &= \lim_{w \to z} \frac{f'(z) + \varepsilon_f(w)}{g'(z) + \varepsilon_g(w)} \\ &= \frac{f'(z)}{g'(z)}. \end{aligned}$$

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Exercise 2

Let $U \subset \mathbb{C}$ be open and connected and $f : U \to \mathbb{C}$ analytic. Assume that one of the following two functions is constant on U:

$$u = \mathrm{Re}(f), \quad v = \mathrm{Im}(f).$$

Show that f is constant in U.

Solution:

Since f is analytic, u, v satisfy the Cauchy-Riemann equations,

$$\begin{cases} \partial_x u = \partial_y v \\ \partial_y u = -\partial_x v. \end{cases}$$

and we see that if one of u, v have partial derivatives equal to 0 then the other has it too. So if either one of u, v is constant then the other must be as well. Now we get from corollary 3.2.7 that $f'(z) = 0$ for all $z \in U$ and then theorem 3.3.1 says that $f(z)$ is constant in U. ■

Exercise 3

Let $U \subset \mathbb{C}$ be open, and let $f = u + iv : U \to \mathbb{C}$. Assume additionally that u, v are twwice continuously differentiable. Let Δ be the Laplace operator. Show that u and v are harmonic:

$$(\Delta u)(z) = 0 = (\Delta v)(z), \quad \forall z \in U.$$

Does the converse hold?

SOLUTION:

Since u, v are twice continuously differentiable, $\partial_{xy} u = \partial_{yx} u$ and $\partial_{xy} v = \partial_{yx} v$. We again use the Cauchy-Riemann equations,

$$\begin{cases} \partial_x u = \partial_y v \\ \partial_y u = -\partial_x v. \end{cases}$$

Then we see that

$$\partial_{xx} u = \partial_{xy} v = \partial_{yx} v = -\partial_{yy} u$$

which is after a little rearrangement says that $\Delta u = 0$. Similarly, for v,

$$\partial_{xx} v = -\partial_{xy} u = -\partial_{yx} u = -\partial_{yy} v$$

and we get $\Delta v = 0$.

The converse is not true. Take for example $f(z) = \text{Re}(z)$. Then $\partial_x u = 1, \partial_y u = 0, \partial_x v = 0, \partial_y v = 0$ so the Cauchy-Riemann equations are not satisfied and f is therefore not analytic, but u is harmonic, $\Delta u = 0$. ■

EXERCISE 4

Show that the complex path integral is linear: if $\gamma : [a, b] \to \mathbb{C}$ is a piecewise C^1-path, $f, g : \gamma^* :\to \mathbb{C}$ are continuous, and $\alpha, \beta \in \mathbb{C}$, then

$$\int_\gamma (\alpha f + \beta g)(z)\, dz = \alpha \int_\gamma f(z)\, dx + \beta \int_\gamma g(z)\, dz.$$

SOLUTION:

I'll show the multiplication and addition properties separately, but by combining them we get the desired formula. We mostly reuse linearity from real analysis, but have to pay attention to the complex multiplication when factorizing. To make the notation shorter, let $r = \text{Re}(\alpha), s = \text{Re}(\alpha)$ and $u(t) = \text{Re}(f(\gamma(t))\gamma'(t)), v(t) = \text{Im}(f(\gamma(t))\gamma'(t))$ then r, s are real and u, v are real-valued and take a real parameter. Then

$$\begin{aligned} \int_\gamma \alpha f(z)\, dz &= \int_a^b \text{Re}[\alpha f(\gamma(t))\gamma'(t)]\, dt + i \int_a^b \text{Im}[\alpha f(\gamma(t))\gamma'(t)]\, dt \\ &= \int_a^b ru(t) - sv(t)\, dt + i \int_a^b rv(t) + su(t)\, dt \end{aligned}$$

$$
\begin{aligned}
&= r\int_a^b u(t)\,dt - s\int_a^b v(t)\,dt + ir\int_a^b v(t)\,dt + is\int_a^b u(t)\,dt \\
&= (r+si)\cdot\int_a^b u(t)+iv(t)\,dt = \alpha\int_a^b f(\gamma(t))\gamma'(t)\,dt \\
&= \alpha\int_\gamma f(z)\,dz.
\end{aligned}
$$

Next, we have the additive property,

$$
\begin{aligned}
\int_\gamma f(z)+g(z)\,dz &= \int_a^b \mathrm{Re}[f(\gamma(t))\gamma'(t)+g(\gamma(t))\gamma'(t)]\,dt \\
&+ i\int_a^b \mathrm{Im}[f(\gamma(t))\gamma'(t)+g(\gamma(t))\gamma'(t)]\,dt \\
&= \int_a^b \mathrm{Re}[f(\gamma(t))\gamma'(t)]\,dt + \int_a^b \mathrm{Re}[g(\gamma(t))\gamma'(t)]\,dt \\
&+ i\Big(\int_a^b \mathrm{Im}[f(\gamma(t))\gamma'(t)]\,dt + \int_a^b \mathrm{Re}[g(\gamma(t))\gamma'(t)]\,dt\Big) \\
&= \int_a^b \mathrm{Re}[f(\gamma(t))\gamma'(t)]\,dt + i\int_a^b \mathrm{Im}[f(\gamma(t))\gamma'(t)]\,dt \\
&+ \int_a^b \mathrm{Re}[g(\gamma(t))\gamma'(t)]\,dt + i\int_a^b \mathrm{Re}[g(\gamma(t))\gamma'(t)]\,dt \\
&= \int_\gamma f(z)\,dz + \int_\gamma g(z)\,dz.
\end{aligned}
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Exercise 5

Calculate the following complex path integrals:

$$
\int_{[-i,1+2i]} \mathrm{Im}(z)\,dz \quad \text{and} \quad \int_{\partial D(z_0,r)} \bar{z}\,dz
$$

where $\partial D(z_0,r)$ refers to the path $\gamma(t) = z_0 + re^{it}$, $t \in [0,2\pi]$.

Solution:

If $\gamma(t) = (1-t)(-i) + t(1+2i)$ for $t \in [0,1]$ then $\gamma'(t) = 1+3i$ and

$$
\int_\gamma \mathrm{Im}(z)\,dz = \int_0^1 \mathrm{Im}(\gamma(t))\gamma'(t)\,dt = \int_0^1 (3t-1)(1+3i)\,dt =
$$

$$= \frac{3}{2} - 1 + i(\frac{9}{2} - 3) = \frac{1}{2} + i\frac{3}{2}.$$

For the other integral, $\gamma(t) = z_0 + re^{it}$ for $t \in [0, 2\pi]$ and $\gamma'(t) = ire^{it}$,

$$\int_\gamma \bar{z}\, dz = \int_0^{2\pi} \overline{\gamma(t)}\gamma'(t)\, dt = \int_0^{2\pi} (z_0 + re^{-it})ire^{it}\, dt$$
$$= \int_0^{2\pi} z_0 ire^{it} + ir^2\, dt = \int_0^{2\pi} ir^2\, dt = i2\pi r^2.$$

Note that the integral of $z_0 ire^{it}$ is 0 since e^{it} is 2π-periodic. ■

EXERCISE 6

We know that $f : \mathbb{C} \to \mathbb{C}$ is analytic, $f(0) = i$ and $u = \mathrm{Re}(f)$ is given by

$$u(x, y) = 2x^3y - 2xy^3 + x^2 - y^2.$$

Find $v = \mathrm{Im}(f)$.

SOLUTION:

Since f is analytic, u, v satisfy the Cauchy-Riemann equations,

$$\begin{cases} 6x^2y - 2y^3 + 2x = \partial_x u = \partial_y v, \\ 2x^3 - 6xy^2 - 2y = \partial_y u = -\partial_x v. \end{cases}$$

Integrating the first equation in y, we get

$$v(x, y) = 3x^2y^2 - \frac{y^4}{2} + 2xy + C(x).$$

Inserting this expression into the second of the C-R equations gives

$$2x^3 - 6xy^2 - 2y = -(6xy^2 + 2y + C'(x))$$

which simplifies to

$$C'(x) = -2x^3$$

from which we get $C(x) = -\frac{x^4}{2} + D$ for some constant D. From $f(0) = i$ we conclude that $1 = v(0) = C(0) = D$. Finally, $v(x, y)$ is given by

$$v(x, y) = 3x^2y^2 - \frac{y^4 + x^4}{2} + 2xy + 1.$$

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