Complex analysis 1

Exercises 4

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Exercise 1

For each of the following functions determine the largest open set in $\mathbb{C}$ where the function is analytic, and use the differentiation rules in the lecture notes to calculate its complex derivative:

(a) $f(z) = z^4 - 2iz^3 + 1$

(b) $g(z) = (z - i)/(z + i)$

(c) $h(z) = z^4(1 - z)^6$

(d) $k(z) = \sqrt{2z}$

Solution:

From the definition of the complex derivative we see that the functions $z \mapsto z$ and $z \mapsto 1$ are differentiable everywhere in $\mathbb{C}$, with derivatives 1 and 0 respectively, and are therefore analytic everywhere. By using proposition 3.1.7 it then follows that complex polynomials in the variable z are analytic everywhere and that the differentiation formulas for real polynomials also hold for complex polynomials. So $f(z), h(z)$ are analytic everywhere and

$$f'(z) = 4z^3 - 6iz^2$$

and

$$h'(z) = 4z^3(1 - z)^6 - 6z^4(1 - z)^5 = 2z^3(1 - z)^5(2 - 5z).$$

Since $z \mapsto z - 1$ and $z \mapsto z + i$ are analytic everywhere and $z + i \neq 0$ in $\mathbb{C} \setminus \{-i\}$, $g(z)$ is analytic in $\mathbb{C} \setminus \{-i\}$ with derivative $g'(z) = \frac{2i}{(z+1)^2}$.

Lastly, if $p(z) = 2z$ then $k(z) = \sqrt{\cdot} \circ p(z) = \sqrt{2z}$ where $\sqrt{\cdot}$ is the principal square root. The principal square root is analytic in $\mathbb{C} \setminus (-\infty, 0]$ and $p(z)$ is analytic everywhere so the composition $k(z)$ is analytic in $\mathbb{C} \setminus (-\infty, 0]$ by proposition 3.1.12 and the derivative is $k'(z) = 1/\sqrt{2z}$. Also, $\mathbb{C} \setminus (-\infty, 0]$ is the largest open set in which $k(z)$ is analytic. Let $q(z) = z/2$ then q is analytic everywhere and then $k \circ q(z) = \sqrt{z}$ is analytic where k is analytic. So since $\sqrt{z}$ is only analytic in $\mathbb{C} \setminus (-\infty, 0]$, $k(z)$ cannot be analytic in a larger domain than this. ■

Exercise 2

Let $f : U \to \mathbb{C}$, $g : U \to \mathbb{C}$ and $h : U \to \mathbb{C}$ be analytic in U with $h(z) \neq 0$ for all $z \in U$. Prove that fg and f/h are analytic in U and

$$\begin{aligned} (fg)'(z) &= f'(z)g(z) + f(z)g'(z), \\ (f/h)'(z) &= \frac{f'(z)h(z) - f(z)h'(z)}{h(z)^2}. \end{aligned}$$

Solution:

In order to avoid using the $\varepsilon - \delta$ definition of a limit, I'll use proposition 3.1.6, namely that for each function $p \in \{f, g, h\}$ there exists a corresponding function ε_p such that $p(w) = p(z) + p'(z)(w-z) + \varepsilon_p(w)(w-z)$ and $\varepsilon_p(w) \to 0$ as $w \to z$. Then

$$\begin{aligned} f(w)g(w) =& f(z)g(z) + f'(z)g(z)(w-z) + g'(z)f(z)(w-z) \\ &+ \varepsilon_f(w)[g(z) + g'(z)(w-z)](w-z) \\ &+ \varepsilon_g(w)[f(z) + f'(z)(w-z)](w-z) \\ &+ [\varepsilon_f(w)\varepsilon_g(w) + f'(z)g'(z)](w-z)^2 \end{aligned}$$

Using this in the definition of the derivative we get

$$\begin{aligned} (fg)'(z) &= \lim_{w \to z} \frac{f(w)g(w) - f(z)g(z)}{w-z} \\ &= \lim_{w \to z} \frac{f(z)g(z) + f'(z)g(z)(w-z) + g'(z)f(z)(w-z) - f(z)g(z)}{w-z} \\ &\quad + \lim_{w \to z} \frac{\varepsilon_f(w)[g(z) + g'(z)(w-z)](w-z)}{w-z} \\ &\quad + \lim_{w \to z} \frac{\varepsilon_g(w)[f(z) + f'(z)(w-z)](w-z)}{w-z} \\ &\quad + \lim_{w \to z} \frac{[\varepsilon_f(w)\varepsilon_g(w) + f'(z)g'(z)](w-z)^2}{w-z} \end{aligned}$$

$$
\begin{aligned}
&= f'(z)g(z) + g'(z)f(z) \\
&\quad + \lim_{w\to z} \varepsilon_f(w)[g(z) + g'(z)(w-z)] \\
&\quad + \lim_{w\to z} \varepsilon_g(w)[f(z) + f'(z)(w-z)] \\
&\quad + \lim_{w\to z} [\varepsilon_f(w)\varepsilon_g(w) + f'(z)g'(z)](w-z) \\
&= f'(z)g(z) + g'(z)f(z).
\end{aligned}
$$

Since this works for any z, the product fg is analytic.

Let's now obtain the derivative of f/h. First, we have

$$
\begin{aligned}
\left(\frac{1}{h}\right)'(z) &= \lim_{w\to z} \frac{\frac{1}{h(w)} - \frac{1}{h(z)}}{w-z} \\
&= \lim_{w\to z} \frac{h(z) - h(w)}{h(z)h(w)(w-z)} \\
&= \lim_{w\to z} \frac{h(z) - h(z) - h'(z)(w-z) - \varepsilon_h(w)(w-z)}{h(z)[h(z) + h'(z)(w-z) + \varepsilon_h(w)(w-z)](w-z)} \\
&= \lim_{w\to z} \frac{-h'(z)}{h(z)^2 + h(z)[h'(z)(w-z) + \varepsilon_h(w)(w-z)]} \\
&+ \lim_{w\to z} \frac{-\varepsilon_h(w)}{h(z)^2 + h(z)[h'(z)(w-z) + \varepsilon_h(w)(w-z)]} \\
&= \frac{-h'(z)}{h(z)^2}.
\end{aligned}
$$

Then we have, using the product rule, that

$$
\left(\frac{f}{h}\right)'(z) = f'(z)\frac{1}{h(z)} + f(z)\frac{-h'(z)}{h(z)^2} = \frac{f'(z)h(z) - h'(z)f(z)}{h(z)^2}.
$$

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Exercise 3

Prove that $f : \mathbb{C} \to \mathbb{C}$, $f(z) = \bar{z}$ is not complex differentiable at any point of $\mathbb{C}$.

Solution:

We prove that the limit

$$
\lim_{w\to z} \frac{f(w) - f(z)}{w-z} = \lim_{w\to z} \frac{\bar{w} - \bar{z}}{w-z}
$$

doesn't exist. To do this we construct two sequences that result in different limits. The sequences $w_n = z + \frac{1}{n}$ and $u_n = z + i\frac{1}{n}$ have limit z. Then

$$\frac{\bar{w}_n - \bar{z}}{w_n - z} = \frac{\frac{1}{n}}{\frac{1}{n}} = 1$$

but

$$\frac{\bar{u}_n - \bar{z}}{u_n - z} = \frac{-i\frac{1}{n}}{i\frac{1}{n}} = -1$$

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EXERCISE 4

Let $f : U \to \mathbb{C}$ be analytic and define $g(z) = \overline{f(\bar{z})}$. Show that g is analytic in $U^* = \bar{z} : z \in U\}$ and $g'(z) = \overline{f'(\bar{z})}$.

SOLUTION:

Note that if we denote $u = \bar{w}, v = \bar{z}$ then

$$\begin{aligned}
\left|\frac{f(w) - f(z)}{w - z} - f'(z)\right| &= \left|\frac{f(\bar{\bar{w}}) - f(\bar{\bar{z}})}{w - z} - f'(\bar{\bar{z}})\right| \\
&= \left|\overline{\frac{f(\bar{\bar{w}}) - f(\bar{\bar{z}})}{w - z} - f'(\bar{\bar{z}})}\right| \\
&= \left|\frac{\overline{f(\bar{\bar{w}})} - \overline{f(\bar{\bar{z}})}}{\bar{w} - \bar{z}} - \overline{f'(\bar{\bar{z}})}\right| \\
&= \left|\frac{\overline{f(\bar{u})} - \overline{f(\bar{v})}}{u - v} - \overline{f'(\bar{v})}\right| \\
&= \left|\frac{g(u) - g(v)}{u - v} - g'(v)\right|.
\end{aligned}$$

So if $\delta > 0$ is such that $|w - z| < \delta$ implies

$$\left|\frac{f(w) - f(z)}{w - z} - f'(z)\right| < \varepsilon$$

then $|u - v| = |w - z| < \delta$ implies

$$\left|\frac{g(u) - g(v)}{u - v} - g'(v)\right| = \left|\frac{f(w) - f(z)}{w - z} - f'(z)\right| < \varepsilon.$$

So g is indeed differentiable. ■

Exercise 5

Let $S = \{z \in \mathbb{C} : |\mathrm{Re}| < \pi/2\}$ and $f : S \to \mathbb{C} \setminus [1, \infty)$, $f(z) = \sin(z)$. Assuming that f is known to be bijective with continuous inverse, define

$$\mathrm{Arcsin}(w) \coloneqq f^{-1}(w), \quad w \in \mathbb{C} \setminus [1, \infty).$$

Show that Arcsin is analytic in $\mathbb{C} \setminus [1, \infty)$.

Solution:

Differentiability of $\mathrm{Arcsin}(w)$ follows from proposition 3.1.13, provided $f(z) = \sin(z)$ is differentiable at $z = \mathrm{Arcsin}(w)$ with nonzero derivative,

$$f'(z) = f'(\mathrm{Arcsin}(w)) \neq 0.$$

But $f'(z) = \cos(z)$ and for $z = x + iy \in S$,

$$\mathrm{Re}(f'(z)) = \mathrm{Re}(\cos(z)) = \frac{1}{2}(e^{-y} + e^{y})\cos(x) \neq 0.$$

So $f'(z) \neq 0$ for all $z \in S$. So $\mathrm{Arcsin}(w)$ is analytic and

$$\mathrm{Arcsin}'(w) = \frac{1}{\cos(\mathrm{Arcsin}(w))}.$$

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Exercise 6

Assume the formula

$$\mathrm{Arcsin}(w) = -i \,\mathrm{Log}\left[\sqrt{1 - w^2} + iw\right], \quad w \in \mathbb{C} \setminus [1, \infty).$$

Prove that

$$\mathrm{Arcsin}'(w) = \frac{1}{\sqrt{1 - w^2}}.$$

Solution:

From the previous exercise we know that

$$\mathrm{Arcsin}'(w) = \frac{1}{\cos(\mathrm{Arcsin}(w))}$$

Using the provided formula for $\mathrm{Arcsin}(w)$ and the definition of $\cos(z)$, we have

$$
\begin{aligned}
\cos(\operatorname{Arcsin}(w)) &= \cos(-i\operatorname{Log}\left[\sqrt{1-w^2}+iw\right]) \\
&= \frac{1}{2}\left(e^{i[-i\operatorname{Log}(\sqrt{1-w^2}+iw)]}+e^{-i[-i\operatorname{Log}(\sqrt{1-w^2}+iw)]}\right) \\
&= \frac{1}{2}\left(e^{\operatorname{Log}(\sqrt{1-w^2}+iw)}+e^{-\operatorname{Log}(\sqrt{1-w^2}+iw)}\right) \\
&= \frac{1}{2}\left(\sqrt{1-w^2}+iw+\frac{1}{\sqrt{1-w^2}+iw}\right) \\
&= \frac{1}{2}\left(\sqrt{1-w^2}+iw+\frac{\sqrt{1-w^2}-iw}{(\sqrt{1-w^2}-iw)(\sqrt{1-w^2}+iw)}\right) \\
&= \frac{1}{2}\left(\sqrt{1-w^2}+iw+\sqrt{1-w^2}-iw\right) \\
&= \sqrt{1-w^2}.
\end{aligned}
$$

We indeed get

$$\operatorname{Arcsin}'(w) = \frac{1}{\sqrt{1-w^2}}.$$

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