# Complex analysis 1

## Exercises 3

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**Exercise 1**

Express in the form $x + iy$:

a) $\mathrm{Log}(-e^2)$

b) $\mathrm{Log}(-1 - i\sqrt{3})$

c) $i^{\mathrm{Log}(i)}$

d) $(\sqrt{3} + i)^{6-i}$

**Solution:**

**Part a):**

The principal argument is $\mathrm{Arg}(-e^2) = \pi$, so we get

$$\begin{aligned}\mathrm{Log}(-e^2) &= \log(|-e^2|) + i\,\mathrm{Arg}(-e^2)\\ &= 2 + i\pi\end{aligned}$$

**Part b):** The principal argument is $\mathrm{Arg}(1 - i\sqrt{3}) = -\frac{\pi}{3}$ and the modulus $|1 - i\sqrt{3}| = 2$, so we get

$$\mathrm{Log}(1 - i\sqrt{3}) = \log(2) - i\frac{\pi}{3}$$

**Part c):** First we have $\mathrm{Log}(i) = i\frac{\pi}{2}$. Now we have from the definition of complex power functions that

$$i^{\mathrm{Log}(i)} = e^{\mathrm{Log}(i)^2} = e^{i^2\frac{\pi^2}{4}} = e^{-\frac{\pi^2}{4}}$$

**Part d):** Using the definition of complex powers,

$$\begin{aligned}(\sqrt{3}+i)^{6-i} &= e^{(6-i)\,\mathrm{Log}(\sqrt{3}+i)} \\ &= e^{(6-i)(\log(2)+i\frac{\pi}{2})} \\ &= e^{\log(2^6)}e^{-i\log(2)}e^{i\pi}e^{\frac{\pi}{6}} \\ &= -64e^{\frac{\pi}{6}}e^{-i\log(2)}.\end{aligned}$$

This isn't on the form $x + iy$ but I'll consider it close enough since it's in polar form and most of the expression has been simplified. ■

EXERCISE 2

Find complex numbers $z, w$ such that $\mathrm{Arg}(zw) \neq \mathrm{Arg}(z) + \mathrm{Arg}(w)$ and $\mathrm{Log}(zw) \neq \mathrm{Log}(z) + \mathrm{Log}(w)$.

SOLUTION:

For $z = -1, w = i$ we have $\mathrm{Arg}(zw) = \mathrm{Arg}(-i) = -\frac{\pi}{2}$ which is not equal to $\mathrm{Arg}(z) + \mathrm{Arg}(w) = \pi + \frac{\pi}{2} = \frac{3\pi}{2}$. For the logarithms we have

$$\mathrm{Log}(z) + \mathrm{Log}(w) = \mathrm{Log}(-1) + \mathrm{Log}(i) = i\pi + i\frac{\pi}{2} = i\frac{3\pi}{2}$$

but this is not equal to

$$\mathrm{Log}(-i) = -i\frac{\pi}{2}.$$

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EXERCISE 3

Verify that $\mathrm{Log}(1 - z^2) = \mathrm{Log}(1 - z) + \mathrm{Log}(1 + z)$ when $|z| < 1$. What can be said about $\mathrm{Log}[(1 - z)/1 + z]$? (Hint: it may help to draw a picture)

SOLUTION:

From $|\mathrm{Re}(z)| \leq |z|$ and $|z| < 1$ we get $-1 < \mathrm{Re}(z) < 1$ and from this it follows that

$$\begin{aligned}\operatorname{Re}(1-z) &= 1-\operatorname{Re}(z) > 1-1 = 0,\\ \operatorname{Re}(1+z) &= 1+\operatorname{Re}(z) > 1-1 = 0,\\ \operatorname{Re}(1-z^2) &= 1-\operatorname{Re}(z)^2+\operatorname{Im}(z)^2 \geq 1-\operatorname{Re}(z)^2 > 1-1 = 0.\end{aligned}$$

From this we conclude that $\operatorname{Arg}(1-z), \operatorname{Arg}(1+z) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ so that $\operatorname{Arg}(1-z)+\operatorname{Arg}(1+z) \in (-\pi, \pi)$. Then

$$\operatorname{Arg}(1-z)+\operatorname{Arg}(1+z) = \operatorname{Arg}(1-z^2)$$

from which we conclude that

$$\begin{aligned}\operatorname{Log}(1-z^2) &= \log(|(1-z)(1+z)|) + i\operatorname{Arg}(1-z^2)\\ &= \log(|1-z|)+\log(|1+z|)+i(\operatorname{Arg}(1-z)+\operatorname{Arg}(1+z))\\ &= \operatorname{Log}(1-z)+\operatorname{Log}(1+z).\end{aligned}$$

Next we have

$$\operatorname{Re}(\frac{1}{z+z}) = \operatorname{Re}(\frac{1+\bar{z}}{|1+z|^2}) = \frac{1}{|1+z|^2}\operatorname{Re}(1+\bar{z}) > 0,$$

so we again get $\operatorname{Arg}(\frac{1}{1+z}) \in (-\frac{\pi}{2}.\frac{\pi}{2})$ and conclude that

$$\operatorname{Arg}(\frac{1-z}{1+z}) = \operatorname{Arg}(1-z)+\operatorname{Arg}(\frac{1}{1+z}).$$

Combining this with $\operatorname{Arg}(\frac{1}{w}) = \operatorname{Arg}(\frac{\bar{w}}{|w|^2}) = \operatorname{Arg}\bar{w} = -\operatorname{Arg}(w)$ we get

$$\operatorname{Arg}(\frac{1-z}{1+z}) = \operatorname{Arg}(1-z)-\operatorname{Arg}(1+z)$$

and from this we finally conclude that

$$\operatorname{Log}(\frac{1-z}{1+z}) = \operatorname{Log}(1-z)-\operatorname{Log}(1+z)$$

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**Exercise 4**

Which of the following sets are open, closed, org neither open nor closed?

a) $A = \{z \in \mathbb{C} : 1 < |z| < 2\}$

b) $B = \{z \in \mathbb{C} : -\pi < \operatorname{Im}(z) \leq \pi\}$

c) $C = \{z \in \mathbb{C} : |\mathrm{Re}(z)| + |\mathrm{Re}(z)| \leq 1\}$

d) $D = \{z \in \mathbb{C} : \mathrm{Re}(z) \text{ and } \mathrm{Im}(z) \text{ are rational}\}$

SOLUTION:

**Part a)**

The set $A$ is open since for any $z_0 \in A$, the open ball $B(z_0, r/2)$ centered at $z_0$ is a neighborhood of $z_0$ and is a subset of $A$ if we pick $r = \mathrm{dist}(z_0, \{|z| = 1\} \cup \{|z| = 2\})$.

**Part b)**

The set $B$ is not open since the sequence $z_n = i(-\pi + \frac{1}{n})$ belongs to $B$ but the limit $-i\pi$ is not in $B$. It is also not open since for any radius $r > 0$ the open ball $B(i\pi, r)$ around $i\pi$ is not a subset of $B$.

**Part c)**

The set $C$ is closed. Let $z_n \to z$ be an arbitrary convergent sequence with $z_n \in C$. Then for any $\varepsilon > 0$

$$\begin{aligned} |\mathrm{Re}(z)| + |\mathrm{Im}(z)| &\leq |\mathrm{Re}(z_n)| + |\mathrm{Im}(z_n)| + |z - z_n| \\ &\leq 1 + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get $|\mathrm{Re}(z)| + |\mathrm{Im}(z)| \leq 1$ and $C$ is closed.

**Part d)**

Note that $0 \in D$ and for any $n \in \mathbb{N}$ the ball $B(0, \frac{1}{n})$ centered at 0 contains $\frac{\pi}{4n}$ but $D \cap B(0, \frac{1}{n})$ doesn't. Hence 0 is not an interior point of $D$ and therefore $D$ is not open. In the same way we see that $D^c$ is not open, by noting that $\pi \in D^c$ but $B(\pi, \frac{1}{n})$ contains $\pi + \frac{\pi}{4n}$ while $D \cap B(\pi, \frac{1}{n})$ doesn't. So $\pi$ is not an interior point of $D^c$ and hence $D$ is not closed. ∎

EXERCISE 5

a) Show that $z \mapsto \bar{z}$ and $z \mapsto |z|$ are continuous in $\mathbb{C}$.

b) Show that $z \mapsto \frac{1}{z}$ is continuous in $\mathbb{C} \setminus \{0\}$.

c) Show that the principal square root $\sqrt{\cdot} : \mathbb{C} \to \mathbb{C}$ is continuous in $\mathbb{C} \setminus (-\infty, 0]$, but not continuous in $\mathbb{C} \setminus \{0\}$. How about continuity at 0? Finally, is $\sqrt{\cdot} : (-\infty, 0) \to \mathbb{C}$ continuous?

SOLUTION:

**Part a)**

Continuity of $z \mapsto \bar{z}$ follows from

$$|\bar{z} - \bar{w}| = |z - w|.$$

Next, it follows from the triangle inequality that

$$||z| - |w|| \leq |z - w|$$

so $z \mapsto |z|$ is continuous.

**Part b)**

To show that $z \mapsto \frac{1}{z}$ is continuous, let $z_n \in \mathbb{C} \setminus \{0\}$ be an arbitrary convergent sequence with limit $z \neq 0$. First I want to conclude that $\frac{1}{|z_n|}$ is bounded. Suppose to the contrary that it isn't, so that for every $M \in \mathbb{N}$ there exists $n_M \in \mathbb{N}$ such that $\frac{1}{|z_{n_M}|} > M$. But then $|z_{n_M}| < \frac{1}{M}$. Since $M$ is arbitrary, this show that $z_{n_M} \to 0$, which is a contradiction. So $\frac{1}{|z_n|}$ is a bounded sequence. Let's denote an upper bound by $K$. Then

$$|\frac{1}{z} - \frac{1}{z_n}| = |\frac{z_n - z}{z_n z}| = \frac{1}{|z|}\frac{1}{|z_n|}|z - z_n| \leq \frac{K}{|z|}|z - z_n| \to 0$$

Since the sequence $z_n$ is arbitrary, this shows continuity of $z \mapsto \frac{1}{z}$.

**Part c)**

The principal square root is $\sqrt{z} = \sqrt{|z|}e^{i\frac{\operatorname{Arg}(z)}{2}}$ when $z \neq 0$ and $\sqrt{0} = 0$ and its continuity on $\mathbb{C} \setminus (-\infty, 0]$ follows by the fact that products and compositions preserve continuity. More specifically $z \mapsto |z|$, $\mathbb{R} \ni \theta \mapsto e^{i\theta}$ are continuous everywhere and $z \mapsto \operatorname{Arg}(z)$ is continuous on $\mathbb{C} \setminus (-\infty, 0]$. Then we see that $\sqrt{z}$ is continuous on $\mathbb{C} \setminus (-\infty, 0]$. To see that it is not continuous across the negative real axis, let $z_n = -1 - i\frac{1}{n}$. The sequence is convergent, $z_n \to -1$ and

$$\lim_{n\to\infty} \operatorname{Arg}(z_n) = -\pi$$

but

$$\operatorname{Arg}(\lim_{n\to\infty} z_n) = \operatorname{Arg}(-1) = \pi.$$

So

$$\lim_{n\to\infty} \sqrt{z_n} = \lim_{n\to\infty} \sqrt{|z_n|}e^{i\frac{\operatorname{Arg}(z_n)}{2}} = e^{-i\frac{\pi}{2}} = -i$$

but

$$\sqrt{\lim_{n\to\infty} z_n} = \sqrt{|\lim_{n\to\infty} z_n|}e^{i\frac{\operatorname{Arg}(\lim_{n\to\infty} z_n)}{2}} = e^{i\frac{\pi}{2}} = i$$

and it is therefore not continuous at $-1$. Replacing $-1$ with any real number $x < 0$ should prove that it's not continuous anywhere on $(-\infty, 0)$. However, it is continuous at 0, which follows by the continuity of $z \mapsto |z|$

and the continuity of the real square root $\sqrt{\cdot} : [0, \infty) \to [0, \infty)$ at 0. To see this, let $z_n \to 0$. Then $|z_n| \to 0$ and

$$|\sqrt{z_n}| = |\sqrt{|z_n|} e^{i\frac{\mathrm{Arg}(z_n)}{2}}| = \sqrt{|z_n|} \to 0.$$

Lastly when $\sqrt{\cdot} : (-\infty, 0) \to \mathbb{C}$ is restricted to the negative real axis,

$$\sqrt{z} = \sqrt{|z|} e^{-i\frac{\pi}{2}} = i\sqrt{|z|}$$

and this is continuous. ■

EXERCISE 6

A square root function $g : X \to \mathbb{C}$ is any function satisfying $g(w)^2 = w$ for all $w \in X \subseteq \mathbb{C}$. Show that there does *not* exist a continuous square root function $g : S^1 \to \mathbb{C}$, where $S^1$ is the unit circle.(Hint: one approach is to try to solve $g(w)^2 = w$ and to derive a contradiction)

SOLUTION:

We know that the principal square root $w \mapsto \sqrt{w}$ is not continuous on $S^1$ since it's discontinuous at $-1$ and the same for the other square root $w \mapsto -\sqrt{w}$. But for any fixed $w \in S^1$, the only two square roots are $\sqrt{w}$ and $-\sqrt{w}$. So any square root function $g(w)$ must assign one of $\sqrt{w}$ and $-\sqrt{w}$ to each $w$. So let $C_1$ be the set on which $g(w) = \sqrt{w}$ and $C_2$ be the set on which $g(w) = -\sqrt{w}$,

$$g(w) = \begin{cases} \sqrt{w}, & w \text{ in } C_1, \\ -\sqrt{w}, & w \text{ in } C_2, \end{cases}$$

and suppose that $g$ is continuous. Then $C_1, C_2$ are a partition of $S_1$, so $S^1 = C_1 \cup C_2$ and $C_1 \cap C_2 = \emptyset$. I claim that there always exist a point $z_0 \in S^1$ in one of the sets which is a limit point of the other. If this is not true then every point of both $C_1$ and $C_2$ are interior points so that both $C^1$ and $C^2$ are open. But this contradicts the assumption that $C_1, C_2$ are a partition of $S_1$ because if $C_1$ is open then $C_2 = S^1 \setminus C_1$ is closed and $C_2$ cannot be both open and closed since $S^1$ is connected. Suppose without loss of generality that $z_0 \in C_1$(the case $z_0 \in C_2$ is identical) and let $w_n \in C_2$ be a sequence converging to $z_0$. Using the continuity of $g$,

$$\sqrt{z_0} = g(z_0) = \lim_{n\to\infty} g(w_n) = \lim_{n\to\infty} -\sqrt{z_n} = -\sqrt{z_0}$$

which is a contradiction. ■