Complex analysis 1

Exercises 2

David Johansson

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Exercise 1

Compute $(1+i)^6$.

Solution:

First write $1+i$ in the form re^{iw}, so that it's easy to compute exponents. The modulus is $r = |1+i| = \sqrt{2}$ and the argument is $w = \mathrm{Arg}(1+i) = \frac{\pi}{4}$. So $1+i = \sqrt{2}e^{i\frac{\pi}{4}}$ and $(1+i)^6 = \sqrt{2}^6 e^{i\frac{6\pi}{4}} = 8e^{i\frac{3\pi}{2}} = -8i$ ■

Exercise 2

Compute all square roots of $-1+\sqrt{3}i$ and all cube roots of -8. Which of these roots are principal roots?

Hint: one of the angles in the right triangle with sides 1, $\sqrt{3}$ and 2 is $\pi/3$.

Solution:

Again, write the numbers in the form re^{iw}. The modulus of $-1+\sqrt{3}i$ is 2 and the argument is $\frac{2\pi}{3}$. So $-1+\sqrt{3}i = 2e^{i\frac{2\pi}{3}}$. The square root are given by $\sqrt{2}e^{i(\frac{\pi}{3}+\pi k)}$ for $k \in \{0,1\}$, the principal root corresponding to $k=0$. Explicitly, the two roots are

$$k = 0 \to \sqrt{2}e^{i\frac{\pi}{3}} = \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{\sqrt{2}}i$$
$$k = 1 \to \sqrt{2}e^{i\frac{4\pi}{3}} = \sqrt{2}e^{-i\frac{2\pi}{3}} = -\frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{\sqrt{2}}i.$$

Now we do the same for $-8 = 8e^{i(\pi+2\pi k)}$. So the cube roots are given by

$2e^{i(\frac{\pi}{3}+\frac{2\pi k}{3})}$ for $k \in \{0,1,2\}$, the principal root corresponding to $k=0$. We get

$$k = 0 \to 2e^{i\frac{\pi}{3}} = 1 + \sqrt{3}i$$
$$k = 1 \to 2e^{i\pi} = -2$$
$$k = 2 \to 2e^{-i\frac{\pi}{3}} = 1 - \sqrt{3}i.$$

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EXERCISE 3

Show the formulas for triple sines and cosines:

$$\begin{cases} \sin(3\alpha) = -\sin^3(\alpha) + 3\cos^2(\alpha)\sin(\alpha), & \alpha \in \mathbb{R} \\ \cos(3\alpha) = \cos^3(\alpha) - 3\cos(\alpha)\sin^2(\alpha), & \alpha \in \mathbb{R}. \end{cases}$$

Hint: Use Euler's formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

SOLUTION:

Despite the hint to use Euler's formula I'll just use the formulas for a sum of angles(Lemma 1.3.18) to prove this. My quick attempt at using Euler's formula led to a somewhat different looking triple angle formula.

For cosine we have

$$\begin{aligned} \cos(2\alpha) &= \cos(\alpha + 2\alpha) \\ &= \cos(\alpha)\cos(2\alpha) - \sin(\alpha)\sin(2\alpha) \\ &= \cos(\alpha)(\cos^2(\alpha) - \sin^2(\alpha)) - \sin(\alpha)(2\sin(\alpha)\cos(\alpha)) \\ &= \cos^3(\alpha) - 3\cos(\alpha)\sin^{\alpha} \end{aligned}$$

and for sine we have

$$\begin{aligned} \sin(3\alpha) &= \sin(\alpha + 2\alpha) \\ &= \sin(\alpha)\cos(2\alpha) + \sin(2\alpha)\cos(\alpha) \\ &= \sin(\alpha)(\cos^2(\alpha) - \sin^2(\alpha)) + (2\sin(\alpha)\cos(\alpha))\cos(\alpha) \\ &= 3\cos^2(\alpha)\sin(\alpha) - \sin^3(\alpha). \end{aligned}$$

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Exercise 4

Show that for any $z \in \mathbb{C}$ one has the formulas

$$\overline{e^z} = e^{\bar{z}}$$
$$\cos^2(z) + \sin^2(z) = 1.$$

Solution:

If $z = x + iy$ with $x, y \in \mathbb{R}$ then we have, using the definitino $e^z = e^x e^{iy}$, that

$$\overline{e^z} = \overline{e^x e^{iy}} = e^x e^{-iy} = e^{\bar{z}}.$$

Next, we have by definition that $\cos(z) = \frac{e^{iz}+e^{-iz}}{2}$ and $\sin(z) = \frac{e^{iz}-e^{-iz}}{2i}$, so

$$\cos^2(z) = \frac{1}{4}(e^{i2z} + 2 + e^{-i2z})$$
$$\sin^2(z) = -\frac{1}{4}(e^{i2z} - 2 + e^{-i2z}).$$

Adding them up we get

$$\cos^2(z) + \sin^2(z) = \frac{1}{4}[(e^{i2z} + 2 + e^{-i2z}) - (e^{i2z} - 2 + e^{-i2z})] = 1.$$

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Exercise 5

Prove, by completing the square, that the solutions $z \in \mathbb{C}$ of a quadratic equation $az^2 + bz + c = 0$ where $a, b, c \in \mathbb{C}$ with $a \neq 0$ are given by the standard quadratic formula

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Here $\sqrt{w}$ denotes the principal square root of $w \in \mathbb{C}$, as usual.

Solution:

By adding and subtracting $\frac{b^2}{4a}$ and factorizing we have

$$az^2 + bz + c = a[(z + \frac{b}{2a})^2 - \frac{b^2 - 4ac}{4a^2}].$$

Since $a \neq 0$, $az^2 + bz + c = 0$ is therefore equivalent to

$$(z + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2}. \tag{1}$$

In polar form, $\frac{b-4ac}{4a^2} = re^{iw}$, with $r, w \in \mathbb{R}$. So the principal root is

$$\sqrt{\frac{b^2 - 4ac}{4a^2}} = \sqrt{r}e^{i\frac{w}{2}}$$

and the other root is

$$\sqrt{r}e^{i\frac{w+2\pi}{2}} = \sqrt{r}e^{i\frac{w}{2}}e^{i\pi} = -\sqrt{r}e^{i\frac{w}{2}} = -\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

So taking the square roots in (1) we find

$$z + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}. \tag{2}$$

Next we want to conclude that

$$z + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

but doing this require a little more work, since $\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$ is not true in general when $x, y \in \mathbb{C}$. Let's write in polar form:

$$\begin{aligned} \frac{b^2 - 4ac}{4a^2} &= re^{iw}, \\ b^2 - 4ac &= pe^{iu}, \\ 4a^2 &= qe^{iv}, \end{aligned}$$

where $-\pi < u, v, w \leq \pi$ and $p, q, r \in \mathbb{R}$. Now $re^{iw} = pq^{-1}e^{i(u-v)}$. The principal square root $\sqrt{r}e^{i\frac{w}{2}}$ is equal to the principal square root of $pq^{-1}e^{i(u-v)}$. It's tempting to say that the principal square root of $pq^{-1}e^{i(u-v)}$ is $pq^{-1}e^{i\frac{u-v}{2}}$, but this is only true if $-\pi < u - v \leq \pi$ which need not hold. What we know for certain is that $-2\pi < u - v < 2\pi$. If

$-2\pi < u - v < -\pi$ then $\text{Arg}(pq^{-1}e^{i(u-v)}) = u - v + 2\pi$, in which case

$$\begin{aligned}
\sqrt{\frac{b^2 - 4ac}{4a^2}} &= \sqrt{pq^{-1}e^{i(u-v+2\pi)}} = \frac{\sqrt{p}}{\sqrt{q}}e^{i(\frac{u-v}{2}+\pi)} \\
&= \frac{\sqrt{p}}{\sqrt{q}}e^{i(\frac{u-v}{2}}e^{i\pi)} = -\frac{\sqrt{p}}{\sqrt{q}}e^{i\frac{u-v}{2}} \\
&= -\frac{\sqrt{p}e^{i\frac{w}{2}}}{\sqrt{q}e^{i\frac{v}{2}}} = -\frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}} \\
&= -\frac{\sqrt{b^2 - 4ac}}{2a}.
\end{aligned}$$

The same conclusion holds in the case $\pi < u - v < 2\pi$, but the argument is $\text{Arg}(u - v) = u - v - 2\pi$. In the case $-\pi < u - v < \pi$,

$$\begin{aligned}
\sqrt{\frac{b^2 - 4ac}{4a^2}} &= \sqrt{pq^{-1}e^{i(u-v)}} = \frac{\sqrt{p}}{\sqrt{q}}e^{i(\frac{u-v}{2})} \\
&= \frac{\sqrt{p}e^{i\frac{w}{2}}}{\sqrt{q}e^{i\frac{v}{2}}} = \frac{\sqrt{b^2 - 4ac}}{2a}.
\end{aligned}$$

The conclusion is that the principal square root $\sqrt{\frac{b^2-4ac}{4a^2}}$ is either $\frac{\sqrt{b^2-4ac}}{2a}$ or $-\frac{\sqrt{b^2-4ac}}{2a}$. In the former case the other square root is $-\frac{\sqrt{b^2-4ac}}{2a}$ and in the latter case the other square root is $\frac{\sqrt{b^2-4ac}}{2a}$. So the two square roots of $\frac{b^2-4ac}{4a^2}$ are $\pm\frac{\sqrt{b^2-4ac}}{2a}$. So it indeed follows from (2) that

$$z + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

and in turn

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

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Exercise 6

For any $z_1, z_2, w_1, w_2 \in \mathbb{C}$ prove Lagrange's identity

$$|z_1w_1 + z_2w_2|^2 = (|z_1|^2 + |z_2|^2)(|w_1|^2 + |w_2|^2) - |z_1\bar{w}_2 - z_2\bar{w}_1|^2.$$

Use this to prove the Cauchy-Schwarz inequality

$$|z_1w_1 + z_2w_2| \leq (|z_1|^2 + |z_2|^2)^{1/2}(|w_1|^2 + |w_2|^2)^{1/2}.$$

Solution:

Let's start with Lagrange's identity. Here we just add and subtract both $|z_2|^2|w_1|^2$ and $|z_1|^2|w_2|^2$ and factorize. This leads to

$$\begin{aligned}|z_1w_1 + z_2w_2|^2 =&|z_1|^2|w_1|^2 + z_1\bar{z}_2w_1\bar{w}_2 + \bar{z}_1z_2\bar{w}_1w_2 + |z_2|^2|w_2|^2\\ =&(|z_1|^2 + |z_2|^2)|w_1|^2 - |z_2|^2|w_1|^2\\ &+ (|z_1|^2 + |z_2|^2)|w_2|^2 - |z_1|^2|w_2|^2\\ &+ z_1\bar{z}_2w_1\bar{w}_2 + \bar{z}_1z_2\bar{w}_1w_2\\ =&(|z_1|^2 + |z_2|^2)(|w_1|^2 + |w_2|^2)\\ &- |z_2|^2|w_1|^2 - |z_1|^2|w_2|^2 + z_1\bar{z}_2w_1\bar{w}_2 + \bar{z}_1z_2\bar{w}_1w_2\\ =&(|z_1|^2 + |z_2|^2)(|w_1|^2 + |w_2|^2)\\ &- (|z_2|^2|w_1|^2 - z_1\bar{z}_2w_1\bar{w}_2 - \bar{z}_1z_2\bar{w}_1w_2 + |z_1|^2|w_2|^2)\\ =&(|z_1|^2 + |z_2|^2)(|w_1|^2 + |w_2|^2) - |z_2\bar{w}_1 - z_1\bar{w}_2|^2.\end{aligned}$$

We get the Cauchy-Schwarz inequality as follows,

$$\begin{aligned}|z_1w_1 + z_2w_2| &= [|z_1w_1 + z_2w_2|^2]^{1/2}\\ &= [(|z_1|^2 + |z_2|^2)(|w_1|^2 + |w_2|^2) - |z_2\bar{w}_1 - z_1\bar{w}_2|^2]^{1/2}\\ &\leq [(|z_1|^2 + |z_2|^2)(|w_1|^2 + |w_2|^2)]^{1/2}\\ &= (|z_1|^2 + |z_2|^2)^{1/2}(|w_1|^2 + |w_2|^2)^{1/2}.\end{aligned}$$

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