1a) $(1+5i)+(-2+3i) = -1+8i$

b) $(2-i)(1+4i) = 2+8i-i-4i^2 = 6+7i$

c) $(3-4i)^2 = 9 - 24i + 16i^2 = -7-24i$

d) $(1+i)^{-1} = \frac{1}{1+i} = \frac{1-i}{(1+i)(1-i)} = \frac{1}{2} - \frac{1}{2}i$

2) Write $z = a+bi$. Then

$z + i = a + (b+1)i$,

$2(1-\bar{z}) = 2(1-a) + 2bi$.

These are equal iff the real and imaginary parts are equal:

$$\begin{cases} a = 2(1-a) \\ b+1 = 2b \end{cases} \Rightarrow a = \frac{2}{3}, \quad b = 1$$

So the solution is

$$z = \frac{2}{3} + i.$$

3) commutativity:

$$\begin{aligned} zw &= (z_r w_r - z_i w_i,\ z_r w_i + z_i w_r) \\ &= (w_r z_r - w_i z_i,\ w_i z_r + w_r z_i) \\ &= wz \end{aligned}$$

associativity:

$$\begin{aligned} z(wv) &= z(w_r v_r - w_i v_i,\ w_r v_i + w_i v_r) \\ &= \big(z_r(w_r v_r - w_i v_i) - z_i(w_r v_i + w_i v_r),\ z_r(w_r v_i + w_i v_r) + z_i(w_r v_r - w_i v_i)\big) \\ &= \big((z_r w_r - z_i w_i)v_r - (z_r w_i + z_i w_r)v_i,\ (z_r w_i + z_i w_r)v_r + (z_r w_r - z_i w_i)v_i\big) \\ &= (z_r w_r - z_i w_i,\ z_r w_i + z_i w_r)v \\ &= (zw)v \end{aligned}$$

multiplicative identity:

$$z \cdot 1 = (z_r \cdot 1 - z_i \cdot 0,\ z_r \cdot 0 + z_i \cdot 1) = (z_r, z_i) = z$$

4)

* $\max\{|Re(z)|, |Im(z)|\} = \left(\max\{|Re(z)|, |Im(z)|\}^2\right)^{1/2}$
$\leq \left(\max\{|Re(z)|, |Im(z)|\}^2 + \min\{|Re(z)|, |Im(z)|\}^2\right)^{1/2}$
$= \left(|Re(z)|^2 + |Im(z)|^2\right)^{1/2} = |z|$

* $\overline{w^{-1}} = \overline{\left(\frac{1}{w}\right)} = \overline{\left(\frac{\bar{w}}{w\bar{w}}\right)} = \frac{\bar{\bar{w}}}{|w|^2} = \frac{w}{|w|^2} = \frac{w}{w\bar{w}} = \frac{1}{\bar{w}} = \bar{w}^{-1}$

* $(zw)(z^{-1}w^{-1}) = (zw)(w^{-1}z^{-1}) = z(ww^{-1})z^{-1} = zz^{-1} = 1 \Rightarrow z^{-1}w^{-1} = (zw)^{-1}$

* $\frac{z+\bar{z}}{2} = \frac{z_r + iz_i + z_r - iz_i}{2} = \frac{2z_r}{2} = z_r = Re(z)$

* $\frac{z-\bar{z}}{2i} = \frac{z_r + iz_r - z_r + iz_i}{2i} = \frac{2iz_i}{2i} = z_i = Im(z)$

5) $w = w_r + iw_i \in \mathbb{C}$ is given. Let $z = a + ib$ for some $a, b \in \mathbb{R}$.
For $z^2 = w$ to hold we must have

$$* \quad \begin{cases} a^2 - b^2 = w_r \\ 2ab = w_i \end{cases}$$

If $w_i \neq 0$ then $a \neq 0$ and $b \neq 0$, otherwise the second equation is violated.
We get $b = \frac{w_i}{2a}$. Inserting this in the first equation gives

$$a^2 - \frac{w_i^2}{4a^2} = w_r$$

which is equivalent to

$$a^4 - w_r a^2 - \frac{w_i^2}{4} = 0.$$

Substituting $x = a^2$ gives

$$x^2 - w_r x - \frac{w_i^2}{4} = 0.$$

This has solutions

$$x = \frac{w_r}{2} \pm \sqrt{\frac{w_r^2}{4} + \frac{w_i^2}{4}} = \frac{w_r \pm |w|}{2}.$$

Since $\frac{w_r - |w|}{2} < 0$ (since $w_i \neq 0$) there doesn't exist $a \in \mathbb{R}$ satisfying

$$a^2 = \frac{w_r - |w|}{2}.$$

But $\frac{w_r + |w|}{2} > 0$, so find 2 solutions

$$a = \pm\sqrt{\frac{w_r + |w|}{2}}.$$

So when $w_i \neq 0$, $z \in \left\{ \sqrt{\frac{w_r + |w|}{2}} + \frac{w_i}{\sqrt{2(w_r + |w|)}} i, -\sqrt{\frac{w_r + |w|}{2}} - \frac{w_i}{\sqrt{2(w_r + |w|)}} i \right\}$ satisfies

$$z^2 = w$$

If instead $w_i = 0$ then the second equation in $*$ says that at least one of a, b must be 0 and from the first equation we find the other.
If $w_r \geq 0$ then $b = 0$ and we get from the first equation in $*$ that

$$a = \pm\sqrt{w_r}.$$

In this case $z \in \{\sqrt{w_r}, -\sqrt{w_r}\}$ satisfies

$$z^2 = w.$$

If $w_r < 0$ then $a = 0$ and

$$b = \pm\sqrt{-w_r}$$

and $z \in \{i\sqrt{-w_r}, -i\sqrt{-w_r}\}$ satisfies

$$z^2 = w.$$

6) Since we're dealing with positive numbers,

$$\left|\frac{z-a}{1-\bar{a}z}\right| \leq 1$$

is equivalent to

$$\left|\frac{z-a}{1-\bar{a}z}\right|^2 \leq 1$$

which is equivalent to

$$|z-a|^2 - |1-\bar{a}z|^2 \leq 0. \qquad **$$

But

$$|z-a|^2 = |z|^2 - z\bar{a} - \bar{z}a + |a|^2$$

and

$$|1-\bar{a}z|^2 = 1 - \bar{a}z - \bar{z}a + |a|^2|z|^2.$$

Using ** we have

$$\begin{aligned} 0 &\geq |z-a|^2 - |1-\bar{a}z|^2 \\ &= |z|^2 - z\bar{a} - \bar{z}a + |a|^2 - 1 + \bar{a}z + a\bar{z} - |a|^2|z|^2 \\ &= |z|^2 + |a|^2 - 1 - |a|^2|z|^2 \\ &= (1-|a|^2)(|z|^2-1). \end{aligned}$$

Since $1-|a|^2 > 0$, we get

$$0 \geq |z|^2 - 1,$$

which is equivalent to

$$|z| \leq 1.$$

Since all steps led to equivalent expressions, this proves that

$$\left|\frac{z-a}{1-\bar{a}z}\right| \leq 1 \iff |z| \leq 1.$$