Complex analysis 1

Exercises 5, 16.2.2024

(suomeksi toisella sivulla)

1. (Complex l'Hôpital's rule) Let $U \subset \mathbb{C}$ be open, let $f, g\colon U \to \mathbb{C}$ be complex differentiable at z with $g'(z) \neq 0$, and let $f(z) = 0 = g(z)$. Show that

$$\lim_{\substack{w \to z \\ w \in \mathbb{C} \setminus \{z\}}} \frac{f(w)}{g(w)} = \frac{f'(z)}{g'(z)}.$$

(*Hint:* Proposition 3.1.6.)

2. Let $U \subset \mathbb{C}$ be open and connected and $f : U \to \mathbb{C}$ analytic. Assume that one of the following two functions is constant on U:

$$u = \mathrm{Re}(f), \qquad \text{or } v = \mathrm{Im}(f).$$

Show that f is constant in U. (*Hint:* Cauchy-Riemann equations.)

3. Let $U \subset \mathbb{C}$ be open, and let $f = u + iv\colon U \to \mathbb{C}$ be analytic. Assume additionally that u, v are twice continuously differentiable.[1] Let Δ be the *Laplace operator* $\Delta := \partial_{xx} + \partial_{yy}$. Show that u and v are *harmonic*, i.e.

$$(\Delta u)(z) = 0 = (\Delta v)(z), \qquad z \in U. \tag{1}$$

Does the converse hold (i.e. if $f = u + iv$ and (1) holds, is f analytic)?

4. Show that the complex path integral is (complex) linear: if $\gamma\colon [a, b] \to \mathbb{C}$ is a piecewise C^1-path, $f, g\colon \gamma^* \to \mathbb{C}$ are continuous, and $\alpha, \beta \in \mathbb{C}$, then

$$\int_\gamma (\alpha f + \beta g)(z)\, dz = \alpha \int_\gamma f(z)\, dz + \beta \int_\gamma g(z)\, dz.$$

5. Calculate the following complex path integrals:

$$\int_{[-i,1+2i]} \mathrm{Im}(z)\, dz \quad \text{and} \quad \int_{\partial D(z_0,r)} \bar{z}\, dz,$$

where $\partial D(z_0, r)$ refers to the path $\gamma(t) = z_0 + re^{it}$, $t \in [0, 2\pi]$.

6. We know that $f\colon \mathbb{C} \to \mathbb{C}$ is analytic, $f(0) = i$, and $u = \mathrm{Re}(f)$ is given by

$$u(x, y) = 2x^3y - 2xy^3 + x^2 - y^2.$$

Find $v = \mathrm{Im}(f)$. (*Hint:* Cauchy-Riemann equations.)

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[1] We will see later that this follows automatically from the analyticity of U.

Kompleksianalyysi 1

Demo 5, 16.2.2024

(English on the other page)

1. (Kompleksinen l'Hôpitalin sääntö) Olkoon $U \subset \mathbb{C}$ avoin ja $f, g : U \to \mathbb{C}$ kompleksidifferentioituvia pisteessä z. Olkoon $g'(z) \neq 0$ and $f(z) = 0 = g(z)$. Osoita, että
$$\lim_{\substack{w \to z \\ w \in \mathbb{C} \setminus \{z\}}} \frac{f(w)}{g(w)} = \frac{f'(z)}{g'(z)}.$$
(*Vihje:* Propositio 3.1.6.)

2. Olkoon $U \subset \mathbb{C}$ avoin ja yhtenäinen ja $f : U \to \mathbb{C}$ analyyttinen. Oletetaan, että toinen seuraavista funktioista on vakio joukossa U:
$$u = \mathrm{Re}(f), \qquad \text{tai } v = \mathrm{Im}(f).$$
Osoita, että f on vakio joukossa U. (*Vihje:* Cauchy-Riemannin yhtälöt.)

3. Olkoon $U \subset \mathbb{C}$ avoin ja $f = u + iv \colon U \to \mathbb{C}$ analyyttinen. Olkoot lisäksi u, v kahdesti jatkuvasti derivoituvia. Olkoon Δ *Laplace-operaattori* $\Delta := \partial_{xx} + \partial_{yy}$. Osoita, että u ja v ovat *harmonisia*, ts.
$$(\Delta u)(z) = 0 = (\Delta v)(z), \qquad z \in U. \tag{2}$$
Päteekö käänteinen väite (ts. jos $f = u + iv$ ja (1) pätee, onko f analyyttinen)?

4. Jos $\gamma\colon [a, b] \to \mathbb{C}$ on paloittain C^1-polku, $f, g\colon \gamma^* \to \mathbb{C}$ ovat jatkuvia ja $\alpha, \beta \in \mathbb{C}$, osoita että
$$\int_\gamma (\alpha f + \beta g)(z)\, dz = \alpha \int_\gamma f(z)\, dz + \beta \int_\gamma g(z)\, dz.$$

5. Laske seuraavat kompleksiset polkuintegraalit:
$$\int_{[-i, 1+2i]} \mathrm{Im}(z)\, dz \quad \text{ja} \quad \int_{\partial D(z_0, r)} \bar{z}\, dz,$$
missä $\partial D(z_0, r)$ on polku $\gamma(t) = z_0 + re^{it}$, $t \in [0, 2\pi]$.

6. Olkoon $f\colon \mathbb{C} \to \mathbb{C}$ analyyttinen, $f(0) = i$, ja $u = \mathrm{Re}(f)$ on muotoa
$$u(x, y) = 2x^3y - 2xy^3 + x^2 - y^2.$$
Määritä $v = \mathrm{Im}(f)$. (*Vihje:* Cauchy-Riemannin yhtälöt.)