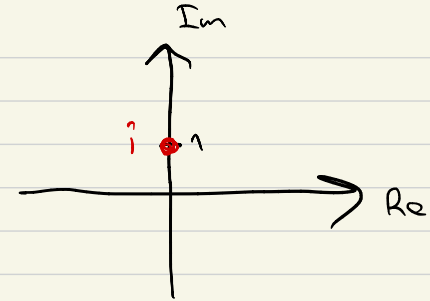
Review of course material:



Complex number $(a,b) \in \mathbb{R}^2$

$(a,b)+(c,d) = (a+c,\ b+d)$

$(a,b)(c,d) = (ac-bd,\ ad+bc)$

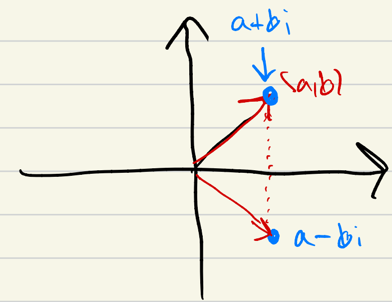
Imaginary unit $i = (0,1)$

Any complex number uniquely written as $a+bi$ where $a,b \in \mathbb{R}$

$\boxed{\text{Computing with complex numbers: group terms with } i\text{, remember } i^2=-1}$

$$(a+bi)(c+di) = ac + adi + bci + bd\underbrace{(i^2)}_{=-1} = ac-bd + i(ad+bc)$$

Division by $z \neq 0$: $\frac{1}{z} = \frac{1}{a+bi} = \frac{a-bi}{(a+bi)(a-bi)} = \frac{a-bi}{a^2+b^2}$



Geometric approach: $z = a+bi \longleftrightarrow (a,b)$

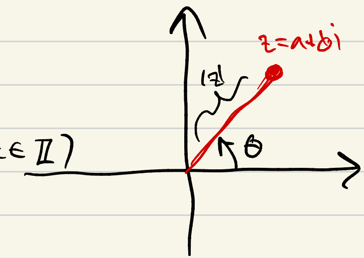
Modulus $|z| = \sqrt{a^2+b^2}$ $(= \|(a,b)\|)$

Conjugate $\bar{z} = a-bi$

$|z|^2 = z\bar{z}$, $|zw| = |z||w|$, $\overline{zw} = \bar{z}\,\bar{w}$, ...

Any $z = a+bi$, $z \neq 0$, can be written in polar coordinates as

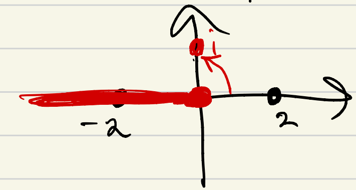
$$z = |z| e^{i\theta}, \qquad e^{i\theta} = \cos\theta + i \sin\theta, \quad \theta \in \arg(z)$$



Angle θ not uniquely defined (defined up to $2\pi k$, $k \in \mathbb{Z}$)

<u>Principal argument</u> $\operatorname{Arg}(z)$ $(z \neq 0)$ is the unique angle in $(-\pi, \pi]$

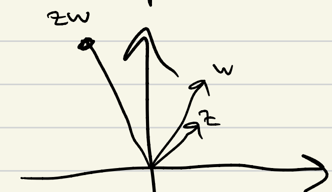
Ex. $\operatorname{Arg}(i) = \frac{\pi}{2}$, $\operatorname{Arg}(2) = 0$, $\operatorname{Arg}(-2) = \pi$



$(\arg(z) = \{\operatorname{Arg}(z) + 2\pi k : k \in \mathbb{Z}\})$

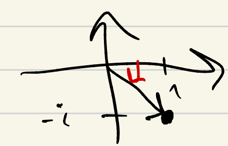
If $z = |z| e^{i\alpha}$, $w = |w| e^{i\beta}$, then

$$zw = |z||w| e^{i\alpha} e^{i\beta} = |z||w| e^{i(\alpha+\beta)}$$



<u>Ex.</u> $(1-i)^8 = (\sqrt{2}\, e^{-i\frac{\pi}{4}})^8 = (\sqrt{2})^8 e^{-i 8 \cdot \frac{\pi}{4}} = 16 e^{-i2\pi} = 16$

<u>Def.</u> $w \in \mathbb{C}$ is an <u>nth root</u> of $z \in \mathbb{C}$ if $z^n = w$.

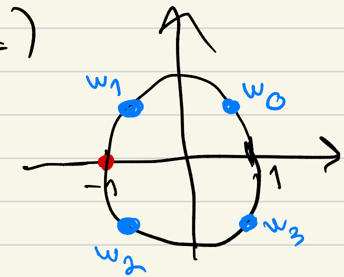


<u>Thm</u> Any $z \in \mathbb{C}$, $z \neq 0$, has n distinct nth roots $w_0, \dots, w_{n-1} \in \mathbb{C}$.

Pf $\Rightarrow$ $z = |z| e^{i\operatorname{Arg}(z)}$

$w_0 = |z|^{\frac{1}{n}} e^{i\frac{\operatorname{Arg}(z)}{n}}$ $\quad \left(w_0^n = \left(|z|^{\frac{1}{n}}\right)^n \left(e^{i\frac{\operatorname{Arg}(z)}{n}}\right)^n = |z| e^{i\operatorname{Arg}(z)} = z \right)$

$$\boxed{w_k = |z|^{\frac{1}{n}} e^{i\frac{\operatorname{Arg}(z) + 2\pi k}{n}}} \quad 0 \le k \le n-1 \quad (w_k^n = z)$$



Ex. All 4th roots of -1 are: since $-1 = 1 \cdot e^{i\pi}$,

$w_0 = 1^{\frac{1}{4}} e^{i\frac{\pi}{4}} = e^{i\frac{\pi}{4}}$, $\quad w_1 = e^{i\left(\frac{\pi+2\pi}{4}\right)} = e^{i\frac{3\pi}{4}}$

$w_2 = e^{i\left(\frac{\pi+4\pi}{4}\right)} = e^{i\frac{5\pi}{4}}$, $\quad w_3 = e^{i\frac{7\pi}{4}}$

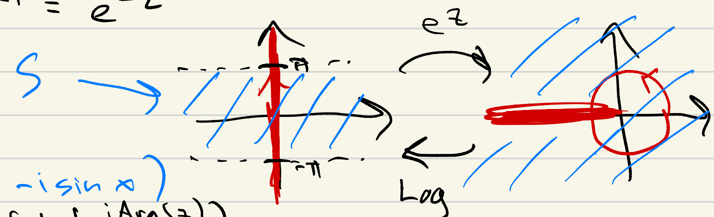
Principal nth root $\quad \sqrt[n]{z} = w_0 = |z|^{\frac{1}{n}} e^{i\frac{\operatorname{Arg}(z)}{n}}$ if $z \neq 0$, $\sqrt[n]{0} = 0$

Exponential: if $z = x+iy$, define $e^z = e^{x+iy} = e^x(\cos y + i \sin y)$

$e^{z+w} = e^z e^w$, $\quad (e^z)^{-1} = e^{-z}$

Periodic: $e^{z+2\pi i} = e^z$

$\exp(S) = \mathbb{C} \setminus (-\infty, 0]$



(Euler: $e^{ix} = \cos x + i \sin x$, $\quad e^{-ix} = \cos x - i \sin x$)

Log : $\mathbb{C} \setminus \{0\} \to \mathbb{C}$, $\quad \operatorname{Log}(z) = \operatorname{Log}(|z| e^{i\operatorname{Arg}(z)})$

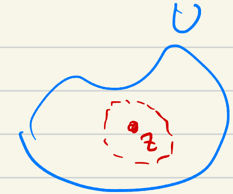
$$\boxed{\operatorname{Log}(z) := \log|z| + i \operatorname{Arg}(z)}$$

$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$

$\cos z = \frac{e^{iz} + e^{-iz}}{2}$

$(\log(z) = \{w \in \mathbb{C} : e^w = z\} = \{\operatorname{Log}(z) + 2\pi i k : k \in \mathbb{Z}\})$

2. Topology of $\mathbb{C}$

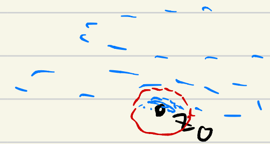


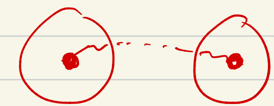
Open set, closed set, bounded set, compact set, connected set

closed and bounded
$\parallel$
compact set

connected set
$\parallel$
any two points can be be joined by path in that set

$z_n \to z_0$ if $\forall \varepsilon\ \exists n_0 :\ z_n \in D(z_0, \varepsilon)$ for $n \geq n_0$

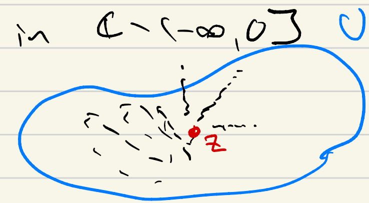




$f : A \to \mathbb{C}$ cont. at z_0 if ($A \subset \mathbb{C}$)
$\forall \varepsilon\ \exists \delta :\ |f(z) - f(z_0)| < \varepsilon$ whenever $|z - z_0| < \delta,\ z \in A$

f cont. at $z_0 \iff \mathrm{Re}(f)$ and $\mathrm{Im}(f)$ cont. at $z_0 \iff f(z_n) \to f(z_0)$ when $z_n \to z_0$

$z^n,\ \bar{z},\ e^z$ cont. in $\mathbb{C}$, but $\mathrm{Arg}(z),\ \sqrt[n]{z},\ \mathrm{Log}\, z$ cont. in $\mathbb{C} \setminus (-\infty, 0]$



3. Analytic functions

$f : U \to \mathbb{C}$ is complex. diff. at $z \in U$ if

$\lim_{w \to z} \dfrac{f(w) - f(z)}{w - z}$ exists $\forall$ seq. $w_n \to z$, $w_n \in U \setminus \{z\}$, and is independent of seq.

Then $\boxed{f'(z) = \lim_{w \to z} \dfrac{f(w) - f(z)}{w - z}}$.

$f : U \to \mathbb{C}$ is analytic if it is complex diff. $\forall z \in U$.

f complex diff. at z

$\Longleftrightarrow \quad f(w) = f(z) + f'(z)(w-z) + \varepsilon(w)(w-z), \qquad \varepsilon(w) \to 0$ as $w \to z$

Rules: $\quad (fg)'(z) = f'(z)g(z) + f(z)g'(z)$

$$\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}$$

$$(a_n z^n + \dots + a_1 z + a_0)' = n a_n z^{n-1} + (n-1)a_{n-1} z^{n-2} + \dots + a_1$$

$$(e^z)' = e^z, \qquad z \in \mathbb{C}$$

$$(\operatorname{Log} z)' = \frac{1}{z}, \qquad z \in \mathbb{C} \setminus (-\infty, 0]$$

Chain rule: $\quad (f \circ g)'(z) = f'(g(z))\, g'(z)$

Inverse: $\quad$ if $f(g(v)) = v \ \forall v \in V$, and $f'(g(w)) \neq 0$, then

$$g'(w) = \frac{1}{f'(g(w))}$$

u, v real valued

$f = u + iv$ is complex diff. at z $\Longleftrightarrow$ $\begin{cases} u_x(z) = v_y(z) \\ u_y(z) = -v_x(z) \end{cases}$ and $f'(z) = u_x(z) + i v_x(z)$
$= v_y(z) - i u_y(z)$

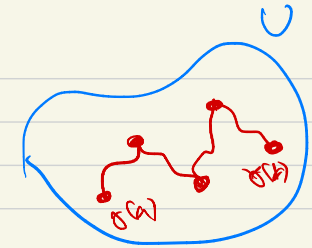
connected

Thm If $f: U \to \mathbb{C}$ is analytic and $\operatorname{Re}(f) = \text{const}$, then $f = \text{const}$.

Pf $\quad f = u + iv, \quad u \equiv \text{const} \Rightarrow u_x \equiv 0,\ u_y \equiv 0, \overset{(C\text{-}R)}{\ } v_x \equiv 0,\ v_y \equiv 0 \Rightarrow f = u + iv$ const. $\square$

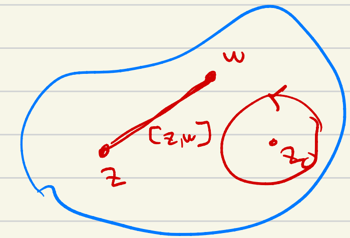
4. Complex integration

A path is a (cont.) map $\gamma : [a,b] \to \mathbb{C}$
(piecewise C^1) (piecewise C^1)



Path integral

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

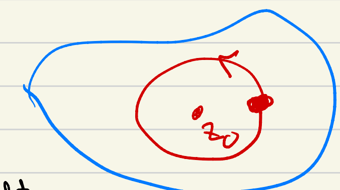


Segment $[z,w](t) = z + t(w-z)$

Circle path $\gamma_n(t) = z_0 + r e^{nit}$, $n \in \mathbb{Z}$

$\overleftarrow{\gamma}$ reverse, $\gamma * \eta$ composition

$$\int_{\partial D(z_0,r)} \frac{1}{z}\,dz = 2\pi i$$



Arc length integral $\int_\gamma f\,|dz| := \int_a^b f(\gamma(t))\,|\gamma'(t)|\,dt$

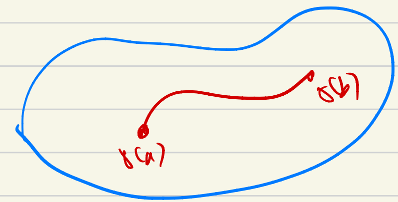
$$\boxed{\left|\int_\gamma f(z)\,dz\right| \le \int_\gamma |f(z)|\,|dz|} \le \left(\sup_{z\in\gamma^*} |f(z)|\right) \underbrace{\mathrm{length}(\gamma)}_{=\int_\gamma |dz|}$$

Def. $F: U \to \mathbb{C}$ is a primitive of $f: U \to \mathbb{C}$ if F is analytic in U, and $F'(z) = f(z)$.

Thm If $f: U \to \mathbb{C}$ is cont. and F is a primitive of f in U, then

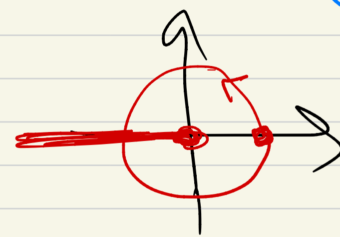
$$\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a))$$

$\forall$ piecewise C^1 path γ in U.



$$\int_{\partial D(0,1)} \frac{1}{z}\,dz$$

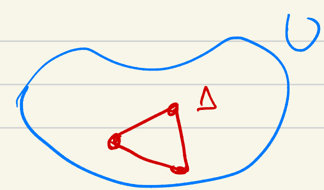
$$(\log z)' = \frac{1}{z}$$



5. Cauchy's thm and applications

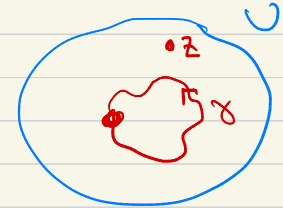
Cauchy for triangles: if $f: U \to \mathbb{C}$ is analytic in $U \setminus \{w_0\}$ and cont. in U,

$$\int_{\partial \Delta} f(z)\,dz = 0 \quad \forall \text{ triangle } \Delta \subset U.$$



Cauchy in convex sets: if $f : U \to \mathbb{C}$ is analytic, U convex, then

$$\int_\gamma f(z)\,dz = 0 \quad \forall \text{ piecewise } C^1 \text{ path } \gamma \text{ in } U.$$



Cauchy's integral formula: if $f : U \to \mathbb{C}$ is analytic, U convex, then

$$\boxed{f(z) \cdot n_\gamma(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z}\, d\xi} \quad \forall \gamma \text{ closed piecewise } C^1 \text{ path in } U.$$

winding number $\quad n_\gamma(z) = \frac{1}{2\pi i} \int_\gamma \frac{1}{\xi - z}\, d\xi, \quad n_\gamma(z) \in \mathbb{Z}$

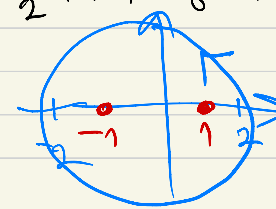
Compute integrals:

$$\int_{\partial D(0,2)} \frac{e^z}{z^2-1}\, dz = \int_{\partial D(0,2)} \frac{e^z}{(z-1)(z+1)}\, dz = \int_{\substack{\partial D(0,2) \\ f(z)=e^z}} \left(\frac{1}{2} \frac{e^z}{z-1} - \frac{1}{2} \frac{e^z}{z+1} \right) dz$$

$$= 2\pi i \left(\frac{1}{2} f(1) \cdot n_\gamma(1) - \frac{1}{2} f(-1) \cdot n_\gamma(-1) \right)$$

$$= 2\pi i \left(\frac{1}{2} e^1 - \frac{1}{2} e^{-1} \right)$$

$$= \pi i e - \pi i e^{-1}$$



Thus:

- analytic functions are infinitely diff.
- Cauchy formula for derivatives
- Cauchy estimates $|f^{(n)}(z_0)| \leq \frac{n! \, \|f\|_{L^\infty(D(z_0, r))}}{r^n}$
- Liouville: bounded analytic $f: \mathbb{C} \to \mathbb{C}$ is constant
- Fundamental thm of algebra
- Morera's theorem