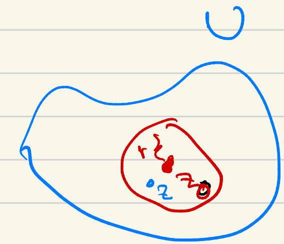
Review:



__Thm__ (Cauchy integral formula) If $f: U \to \mathbb{C}$ analytic, $\overline{D}(z_0, r) \subset U$, then

$$f(z) = \frac{1}{2\pi i} \int_{\partial D(z_0, r)} \frac{f(\xi)}{\xi - z} \, d\xi, \quad z \in D(z_0, r).$$

Cauchy int. formula

$\Rightarrow$ analytic functions are infinitely differentiable

$$\Rightarrow f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial D(z_0, r)} \frac{f(\xi)}{(\xi - z)^{n+1}} \, d\xi$$

Cauchy's estimates

$$\Rightarrow |f^{(n)}(z_0)| \leq \frac{n!}{2\pi} \int_{\partial D(z_0, r)} \frac{|f(\xi)|}{|\xi - z|^{n+1}} |d\xi| \leq \frac{n! \left(\sup_{\xi \in \partial D(z_0, r)} |f(\xi)| \right)}{r^n}$$

$\Rightarrow$ any bounded $\overbrace{\text{analytic } f: \mathbb{C} \to \mathbb{C}}^{\text{entire function}}$ is constant.

Liouville's theorem

(proof: for $z_0 \in \mathbb{C}$, $|f'(z_0)| \le \frac{\sup_{\partial D(z_0,r)} |f(\zeta)|}{r} \le \frac{M}{r}$, let $r \to \infty$

$\Rightarrow f'(z_0) = 0 \ \forall z_0 \in \mathbb{C} \Rightarrow f$ constant)

Recall that we started this course by looking at

$$ax^2 + bx + c = 0$$

This has no real solutions if $b^2 - 4ac < 0$, but it always has two complex solutions.

<u>Thm</u> (Fundamental thm of algebra)

If $p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ where $a_n, \dots, a_0 \in \mathbb{C}$ $(n \ge 1)$ and $a_n \ne 0$, then $\exists\ z_1 \in \mathbb{C}$ with $p(z_1) = 0$.

Moreover, there are precisely n numbers $z_1, \dots, z_n \in \mathbb{C}$ with $p(z_j) = 0$ (some of the z_j can be the same).

One can factorize $p(z) = a_n (z - z_1)(z - z_2) \cdots (z - z_n)$.

__Pf__ First we show: $\exists\, z_1 \in \mathbb{C}$ with $p(z_1) = 0$.

We argue by contradiction and suppose that $p(z) \neq 0 \ \forall z \in \mathbb{C}$.

Then $f(z) := \frac{1}{p(z)}$ is an analytic function $f : \mathbb{C} \to \mathbb{C}$.

We claim that f is bounded. To see this, note that

$$|p(z)| = |a_n z^n + a_{n-1} z^{n-1} + \dots + a_0| \geq |a_n z^n| - |a_{n-1} z^{n-1} + \dots + a_0|.$$

If $|z| \geq R$ for some large $R \geq 1$,

$$\left(|z| \geq R \Rightarrow \frac{1}{|z|} \leq \frac{1}{R} \Rightarrow -\frac{1}{|z|} \geq -\frac{1}{R} \right)$$

$$\begin{aligned} |p(z)| &\geq |a_n||z|^n - \left(|a_{n-1}||z|^{n-1} + \dots + |a_0| \right) \\ &= |z|^n \left(|a_n| - \frac{|a_{n-1}|}{|z|} - \dots - \frac{|a_0|}{|z|^n} \right) \\ &\geq |z|^n \left(|a_n| - \frac{C}{R} \right) \geq \frac{|a_n|}{2} |z|^n \geq 1 \end{aligned}$$

when R is large enough.

Then $$|p(z)| \geq \begin{cases} 1, & |z| \geq R \\ c, & |z| \leq R \end{cases}$$ $\leftarrow$ $|p(z)|$, $z \in \bar{D}(0, R)$ is a cont. function on a compact set

for some positive c.

Thus $|p(z)| \geq c_0 > 0 \quad \forall z \in \mathbb{C}$, so

$$|f(z)| = \frac{1}{|p(z)|} \leq \frac{1}{c_0}, \quad z \in \mathbb{C}.$$

So $f: \mathbb{C} \to \mathbb{C}$ is a bounded analytic function.

Liouville's thm $\Rightarrow$ $f = \text{const}$ and $p(z) = \text{const}$.

This is a contradiction (since $n \geq 1$ and $a_n \neq 0$). $\lightning$

To prove the second part, we have z_1 with $p(z_1) = 0$. Write

remainder $r_0 \in \mathbb{C}$

$$p(z) = (z - z_1)(a_n z^{n-1} + b_{n-2} z^{n-2} + \dots + b_1 z + b_0) + r_0$$

(compare: $23 = 7 \cdot 3 + 2$)

remainder

But since $p(z_1) = 0$, $r_0 = 0$

$$\Rightarrow p(z) = (z - z_1)\, p_2(z) \Rightarrow \dots \Rightarrow p(z) = a_n (z - z_1) \cdots (z - z_n).$$

poly order $n-1$

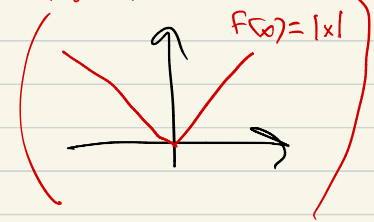
$\square$

Thm (Morera's theorem) Let $f: U \to \mathbb{C}$ cont. satisfy

$$\int_{\partial\Delta} f(z)\,dz = 0$$

for any triangle $\Delta \subset U$. Then f is analytic.

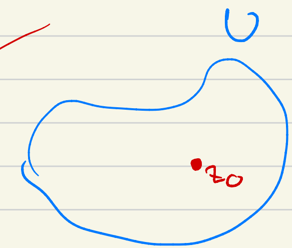
Not true on real line: $f(x) = |x|$



Pf Cor. 5.1.6 $\Rightarrow$ f has a primitive F in U.

Then $F'(z) = f(z)$ and F is analytic.

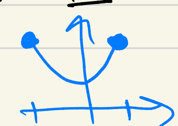
Thm 5.3.1 $\Rightarrow$ $F' = f$ is also analytic. $\square$

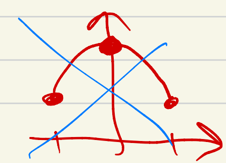


Thm (Analytic continuation to a point)

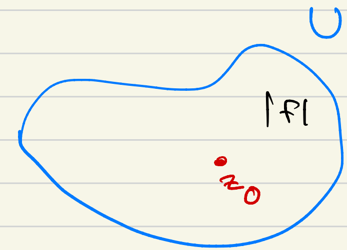
If $f: U \to \mathbb{C}$ cont. in U and analytic in $U \setminus \{z_0\}$ for some $z_0 \in U$. Then f is analytic in U.

Pf Cauchy's theorem for triangles $\Rightarrow \int_{\partial\Delta} f(z)\,dz = 0$ for any triangle $\Delta \subset U \overset{\text{Morera}}{\Longrightarrow} f$ analytic in U. $\square$





Thm (Maximum modulus principle 1) Let $f : U \to \mathbb{C}$ analytic where U open and connected. If $|f|$ reaches its maximum at some $z_0 \in U$, then f is constant.



Thm (Mean value principle) If $f : U \to \mathbb{C}$ analytic and $\bar{D}(z, r) \subset U$, then

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{it})\, dt.$$

Pf Cauchy's integral formula: if $\gamma(t) = z + re^{it}$, then $\gamma'(t) = ire^{it}$

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z}\, d\xi = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(\gamma(t))}{\gamma(t) - z} \gamma'(t)\, dt$$

$$= \frac{1}{2\pi \cancel{i}} \int_0^{2\pi} \frac{f(z + re^{it})}{\cancel{re^{it}}} \cancel{i}\cancel{re^{it}}\, dt. \qquad \square$$

Pf of max. modulus principle Assume $M = |f(z_0)| = \max_{z \in U} |f(z)|$, $z_0 \in U$.

We show that $V := \{ z \in U : |f(z)| = M \}$ is all of U.

Then $|f|$ is constant $\overset{\text{Cor. 33.2}}{\Longrightarrow}$ f is constant.

We first show that V is open. Let $z \in V$ and let $\overline{D}(z, r_0) \subset U$. We claim that actually $D(z, r_0) \subset V$.

Fix $r \in (0, r_0)$, use mean value principle:

$$M = |f(z)| = \left| \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{it})\, dt \right| \leq \frac{1}{2\pi} \int_0^{2\pi} \overbrace{|f(z+re^{it})|}^{\leq M}\, dt = M$$

Thus $|f(z+re^{it})| = M \quad \forall\, t \in [0, 2\pi]$ so $\partial D(z, r) \subset V$.

This holds $\forall\, r \in (0, r_0)$, so $D(z, r_0) \subset V$.

Next we prove $V = U$. Connectedness argument:

let $z \in U$ arbitrary, we claim $|f(z)| = M$.

Connectedness $\Rightarrow \exists$ path γ joining z_0 and z.

Show that $|f(\gamma(t))| = M \quad \forall\, t \in [0, 1]$. Use that $|f(\gamma(0))| = |f(z_0)| = M$, and openness of V to show the claim. $\square$

Thm (Maximum modulus principle 2) Let $U \subset \mathbb{C}$ open, connected and bounded. If $f : \overline{U} \to \mathbb{C}$ is cont. and if f is analytic in U, then

$$\max_{z \in \overline{U}} |f(z)| = \max_{z \in \partial U} |f(z)|.$$