Complex analysis 1

Lecture notes, Spring 2024

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"The natural development of this work soon led the geometers in their studies to embrace imaginary as well as real values of the variable... It came to appear that, between two truths of the real domain, the easiest and shortest path quite often passes through the complex domain." (Paul Painlevé 1900)

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Preface

This course gives an introduction to complex numbers and functions of a complex variable. Complex numbers arose in the 16th century as a way of finding "imaginary" solutions to equations. In the 19th century complex analysis became an important part of mathematics. Nowadays complex numbers and functions are regarded as very "real", and they appear naturally in many parts of mathematics, physics and engineering.

Topics include basic properties of complex numbers, analytic functions, complex derivatives, and complex integrals. We will also discuss (local) Cauchy's theorem and Cauchy integral formula, the maximum modulus principle, and the fundamental theorem of algebra. In period 4, it is possible to continue with the course *Complex analysis 2*.

The main reference for the course are these lecture notes, which borrow heavily (and directly) from earlier lecture notes by Tero Kilpeläinen [**Ki15**] and Tuomas Orponen [**Or23**]. Further explanations and illustrations will be given during the lectures, and therefore it is recommended to take notes also then.

The following textbooks may be useful additional reading:

- Bruce P. Palka: An introduction to complex function theory, Springer, 1990 (the course follows parts I.1.1-V.4.3)
- Elias M. Stein, Rami Shakarchi: Complex analysis, Princeton University Press, 2003 (less rigorous and proceeds quickly, but explains many things beautifully)
- Eberhard Freitag, Rolf Busam: Complex analysis, 2nd edition, Universitext, Springer, 2009 (a systematic treatment)

CHAPTER 1

Complex numbers and the complex plane

1.1. Motivation

We begin by motivating the notion of complex numbers and discuss briefly several applications in mathematics, physics and engineering. In this section we only give formal (=not rigorous) arguments. Precise definitions will be given in the later sections.

Example 1.1.1 (Quadratic equation). Consider the quadratic equation

$$ax^2 + bx + c = 0.$$

The coefficients a, b, c are real numbers, and one would like to find a real number x solving this equation. In high school we have learned the following *quadratic formula* for a solution:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{1.1.1}$$

The formula involves the *discriminant* $\Delta = b^2 - 4ac$. The square root $\sqrt{b^2 - 4ac}$ makes sense when $b^2 - 4ac \geq 0$. If $b^2 - 4ac > 0$ we know that there are two distinct solutions, and if $b^2 - 4ac = 0$ we know that there is only one solution.

We now ask: what happens when $b^2 - 4ac < 0$? A very simple equation where this happens is

$$x^2 + 1 = 0.$$

No *real* number x can solve this equation, since $x^2 + 1 \geq 1$ for any $x \in \mathbb{R}$. Undeterred by this fact, we consider the possibility that some *more general number* x could solve this equation. If some x is a solution, then $x^2 = -1$. Formally we could write

$$x = \sqrt{-1}.$$

Another solution could be $x = -\sqrt{-1}$. Thus, formally, we could consider an *imaginary unit*

$$i = \sqrt{-1}.$$

The imaginary unit is a "generalized number" that satisfies $i^2 = -1$. Assuming the existence of such a number, one can formally define a *complex* *number* z to be an expression of the form

$$z = a + bi$$

where $a, b \in \mathbb{R}$. One would then like to calculate with such numbers. (The name *complex number* was introduced by Gauss in 1831.)

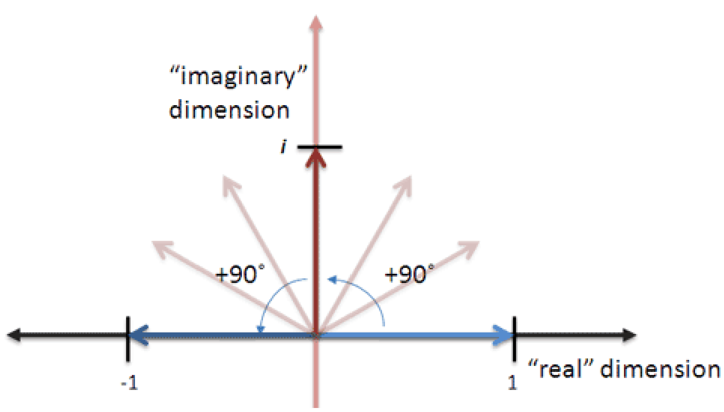
A natural way to add two such numbers $z = a+bi$ and $w = c+di$ where $a, b, c, d \in \mathbb{R}$ would be

$$z + w = (a + c) + (b + d)i. \tag{1.1.2}$$

In principle there are many choices for multiplying z and w. For instance one could try a product $z \times w = (ac)+(bd)i$. However, this product will not always have nice properties. It will turn out that there is a unique product that satisfies commutativity ($z \cdot w = w \cdot z$) and distributivity ($z \cdot (w + r) = z \cdot w + z \cdot r$). Formally

$$\begin{aligned} z \cdot w = (a + bi) \cdot (c + di) &= a \cdot (c + di) + bi \cdot (c + di) \\ &= ac + adi + bci + bdi^2 \\ &= (ac - bd) + (ad + bc)i. \end{aligned} \tag{1.1.3}$$

The formulas (1.1.2) and (1.1.3) are related to an *algebraic approach* to complex numbers. There is a perhaps more intuitive *geometric approach*, where a complex number $a+bi$ is identified with the vector (a, b) in $\mathbb{R}^2$. The vectors $(a, 0)$ are on the x-axis (real axis), whereas the vectors $(0, b)$ are on the y-axis which in this setting is called the *imaginary axis*.



Complex numbers were first brushed upon by the Italian mathematician Girolamo Cardano around 1545 in his treatise of solving cubic and quartic equations. Cardano writes (translation from Latin): "Dismissing mental tortures, and multiplying $5 + \sqrt{-15}$ by $5 - \sqrt{-15}$, we obtain $25 - (-15)$. Therefore the product is 40. .... and thus far does arithmetical subtlety go, of which this, the extreme, is, as I have said, so subtle that it is useless."

Example 1.1.2 (Higher order equations). The second order equation $ax^2+bx+c=0$ always has two complex solutions (which coincide if $b^2-4ac=0$), given by the quadratic formula (1.1.1). If $b^2-4ac<0$, we need to interpret the square root as an "imaginary number" as

$$\sqrt{b^2-4ac}=\sqrt{(4ac-b^2)\cdot(-1)}=\sqrt{4ac-b^2}\sqrt{-1}=i\sqrt{4ac-b^2}.$$

Next let $p,q\in\mathbb{R}$ and consider the cubic equation

$$t^3+pt+q=0.$$

Cardano published in 1545 the solution formula

$$t=\left(-\frac{q}{2}+\sqrt{\Delta}\right)^{1/3}+\left(-\frac{q}{2}-\sqrt{\Delta}\right)^{1/3}. \tag{1.1.4}$$

where $\Delta=\frac{p^3}{27}+\frac{q^2}{4}$ is the *discriminant*. If $\Delta<0$, then the equation has three real solutions, and all three solutions can be obtained from (1.1.4) by interpreting the roots as complex numbers. (If $\Delta\geq 0$ there are three complex solutions, at least one of which is real, obtained by a variation of (1.1.4).)

It was proved by Niels Abel in 1824 that a general equation

$$x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n=0$$

of order $n\geq 5$ cannot be solved in terms of radicals (roots etc). However, the *fundamental theorem of algebra* states that such an equation always has n complex solutions counting multiplicity. We will prove this theorem in the end of the course by using complex analysis.

Complex numbers have many other striking applications in several fields of mathematics and science. We list a few here.

Example 1.1.3 (Evaluation of integrals). Definite integrals such as

$$\int_{-\infty}^{\infty}\frac{1}{1+x^n}\,dx \quad\text{or}\quad \int_{-\pi}^{\pi}\frac{1}{3+\cos^2 t}\,dt$$

can be explicitly calculated by complex analysis and the powerful *residue theorem*, as will be discussed in *Complex analysis 2*.

Example 1.1.4 (Number theory). The *Riemann zeta function* is defined by

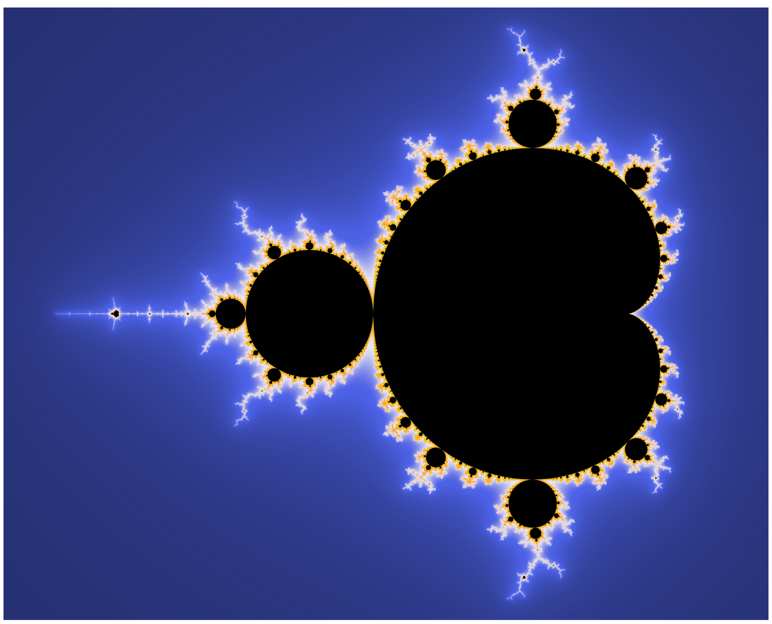
$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}.$$

The sum converges for $s>1$. It turns out that $\zeta(s)$ can be defined for complex numbers s and that this function has an intimate connection with *prime numbers*. For example, the *prime number theorem* which states that

$$\lim_{x\to\infty} \frac{\#\{\text{primes} \leq x\}}{\frac{\log x}{x}} = 1,$$

was first proved in 1896 by using complex analysis and the Riemann zeta function. The most famous unsolved problem in mathematics (and one of the $1,000,000 Millennium problems) is the *Riemann hypothesis*, which asks to prove that all nontrivial zeros of $\zeta(s)$ are of the form $s = \frac{1}{2} + it$ for $t \in \mathbb{R}$.

EXAMPLE 1.1.5 (Fractal geometry). The *Mandelbrot set* is the most famous example of a *fractal* (very irregular set). This set, like many other fractals, is generated via complex analysis, by iterating the function $f_c(z) = z^2 + c$ for different complex c.



EXAMPLE 1.1.6 (Physics). Many basic equations of physics, such as the *Schrödinger equation* in quantum mechanics,

$$i\partial_t \Psi + \Delta\Psi = 0,$$

or the *time-harmonic Maxwell equations* for electromagnetic waves,

$$\nabla \times E = -i\omega\mu H,$$
$$\nabla \times H = J + i\omega\varepsilon E,$$

explicitly involve complex numbers.

EXAMPLE 1.1.7 (Signal processing). A time-periodic audio signal $f(t)$ can be decomposed in its frequency components via *Fourier series*

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{ikt},$$

where the complex exponential e^{ix} is defined via the *Euler formula*

$$e^{ix} = \cos x + i \sin x.$$

We hope that these examples convince the reader that complex numbers are not "imaginary" or "useless", but rather a natural and powerful language for many applications in mathematics and science. We also hope that this course will show that complex analysis is a rich and beautiful subject in its own right. We should mention that complex analysis has been one of the most prominent fields in mathematics in Finland, including famous mathematicians such as Rolf Nevanlinna (creator of Nevanlinna theory, around 1925) and Lars Ahlfors (the only Finnish recipient of the Fields medal, in 1936).

1.2. Complex numbers: algebraic properties

Previously, we introduced complex numbers as objects $z = a + bi$ where $a, b \in \mathbb{R}$ and i is a special imaginary unit satisfying $i^2 = -1$. We make this definition rigorous by considering vectors $(a, b) \in \mathbb{R}^2$ and by taking the result of our formal computations, (1.1.2) and (1.1.3), as the *definitions* of sum and product.

1.2.1. Complex numbers, sums and products.

DEFINITION 1.2.1. A *complex number* is a vector $z = (a, b) \in \mathbb{R}^2$. The *sum* $z + w$ of two complex numbers $z = (a, b)$ and $w = (c, d)$ is defined to be

$$z + w = (a + c, b + d).$$

The *product* $zw = z \cdot w$ is defined as

$$zw = (ac - bd, ad + bc).$$

If $n \geq 1$ is an integer, the *nth power of* z is

$$z^n = \underbrace{z \cdot z \cdot \ldots \cdot z}_{n \text{ times.}}$$

The *set of complex numbers* is denoted by

$$\mathbb{C} = \{(a, b) \,:\, a, b \in \mathbb{R}\}.$$

EXAMPLE 1.2.2. We have

$$(1,3)+(-1,2)=(1+(-1),3+2)=(0,5)$$

and

$$(1,3)(-1,2)=(1\cdot(-1)-3\cdot 2, 1\cdot 2+3\cdot(-1))=(-7,-1).$$

REMARK 1.2.3. If $z=(a,b)$ and $w=(c,d)$ are complex numbers, then

$$z=w \quad\Longleftrightarrow\quad a=c \text{ and } b=d.$$

The next definition allows us to identify real numbers a with complex numbers $(a,0)$.

DEFINITION 1.2.4. If $a\in\mathbb{R}$, we identify a with the complex number $(a,0)$. In particular

$$\begin{aligned}0&=(0,0),\\ 1&=(1,0).\end{aligned}$$

If $z=(a,b)$ is a complex number, we write $-z=(-a,-b)$.

WARNING 1.2.5. Complex numbers, unlike real numbers, do not have a natural ordering. Thus whenever we write $a\le b$ etc, it is assumed that a and b are real numbers.

We now show that complex numbers satisfy many natural properties, just like the real numbers do. Those who have taken Algebra 1 may notice that the set $\mathbb{C}$ becomes a *commutative ring*.

THEOREM 1.2.6 (Ring properties). *Any $z,w,v\in\mathbb{C}$ satisfy*

$$\begin{array}{cl} z+w=w+z, \quad zw=wz & \textit{(commutativity)}\\ z+(w+v)=(z+w)+v, \quad z(wv)=(zw)v & \textit{(associativity)}\\ z(w+v)=zw+zv & \textit{(distributivity)}\\ z+(-z)=0 & \textit{(additive inverse)}\\ z+0=z, \quad z\cdot 1=z & \textit{(identity element).}\end{array}$$

PROOF. We only prove distributivity, and leave the other parts as exercises. Let $z=(a,b)$, $w=(c,d)$ and $v=(x,y)$. Then

$$\begin{aligned}z(w+v)&=(a,b)\cdot(c+x,d+y)\\ &=(a(c+x)-b(d+y),a(d+y)+b(c+x))\\ &=(ac-bd,ad+bd)+(ax-by,ay+bx)\\ &=(a,b)\cdot(c,d)+(a,b)\cdot(x,y)\\ &=zw+zv.\end{aligned}$$

□

1.2.2. Imaginary unit. Now, how does the imaginary unit i fit in the above scheme? Recall that i is supposed to be a complex number such that $i^2 = -1 = (-1, 0)$. It is easy to find such a number:

DEFINITION 1.2.7. The *imaginary unit* is the complex number

$$i = (0, 1).$$

The *real part* $\operatorname{Re}(z)$ and *imaginary part* $\operatorname{Im}(z)$ of $z = (a, b)$ are

$$\operatorname{Re}(z) = a,$$
$$\operatorname{Im}(z) = b.$$

We now make the connection between complex numbers $(a, b) \in \mathbb{R}^2$ and the numbers $a + bi$ as in Section 1.1.

THEOREM 1.2.8 (Basic facts).
(i) $i^2 = -1$.
(ii) *One has* $(a, b) = a + bi$ *and* $\operatorname{Re}(a + bi) = a$, $\operatorname{Im}(a + bi) = b$.
(iii) *Any complex number* z *can be represented uniquely as* $z = a + bi$ *for some* $a, b \in \mathbb{R}$.

PROOF. (i) One has

$$i^2 = (0, 1) \cdot (0, 1) = (0 \cdot 0 - 1 \cdot 1, 0 \cdot 1 + 1 \cdot 0) = (-1, 0) = -1.$$

(ii) Given any $(a, b) \in \mathbb{C}$, we have

$$(a, b) = (a, 0) + (0, b) = a + b \cdot (0, 1) = a + bi.$$

Consequently $\operatorname{Re}(a + bi) = a$ and $\operatorname{Im}(a + bi) = b$.

(iii) By (ii), any $z = (a, b) \in \mathbb{C}$ can be represented as $z = a + bi$. For uniqueness, if $z = a + bi = c + di$ for some $a, b, c, d \in \mathbb{R}$, then (ii) applied to (a, b) and (c, d) gives $(a, b) = (c, d)$, so $a = c$ and $b = d$. This shows that there is only one possible representation of $z \in \mathbb{C}$ as $z = a + bi$. The result follows. □

From now on, justified by Theorem 1.2.8, we will most often write complex numbers as $a + bi$ instead of (a, b), because this will be easier for computations. In particular, it is not necessary to remember the definition of product in Definition 1.2.1, but just the following rule:

> One computes with complex numbers $a + bi$ just like with real numbers, just collecting all terms with i and keeping in mind that $i^2 = -1$.

EXAMPLE 1.2.9. Powers of i can be computed as

$$\begin{aligned} i^2 &= -1, \\ i^3 &= i^2 \cdot i = (-1) \cdot i = -i, \\ i^4 &= i^3 \cdot i = (-i) \cdot i = 1, \\ i^5 &= i^4 \cdot i = (1) \cdot i = i, \\ &\vdots \end{aligned}$$

One also computes

$$(3+2i) + (4-3i) = (3+4) + (2-3)i = 7 - i$$

and

$$(3+2i)(4-3i) = 3 \cdot 4 - 3 \cdot 3i + 2i \cdot 4 - 2i \cdot 3i = 12 - 9i + 8i + 6 = 18 - i.$$

1.2.3. Division. Let us now consider how to *divide* by a complex number. If $z = a + bi$ is nonzero (meaning that $a \neq 0$ or $b \neq 0$), we formally compute

$$\frac{1}{z} = \frac{1}{a+bi} = \frac{a-bi}{(a+bi)(a-bi)} = \frac{a-bi}{a^2+b^2}. \tag{1.2.1}$$

We take this result as a definition (again, in practice it is enough to remember the previous computation and not the definition):

DEFINITION 1.2.10. If $z = a + bi \neq 0$, its inverse is

$$z^{-1} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i.$$

If $w \in \mathbb{C}$ we define $\frac{w}{z} = w \cdot z^{-1}$.

Combined with Theorem 1.2.6, the following result shows that $\mathbb{C}$ is a *field* in the language of Algebra 1:

THEOREM 1.2.11. *For any $z \in \mathbb{C}$ with $z \neq 0$, one has $z \cdot z^{-1} = 1$.*

PROOF. Exercise. $\square$

EXAMPLE 1.2.12. One has

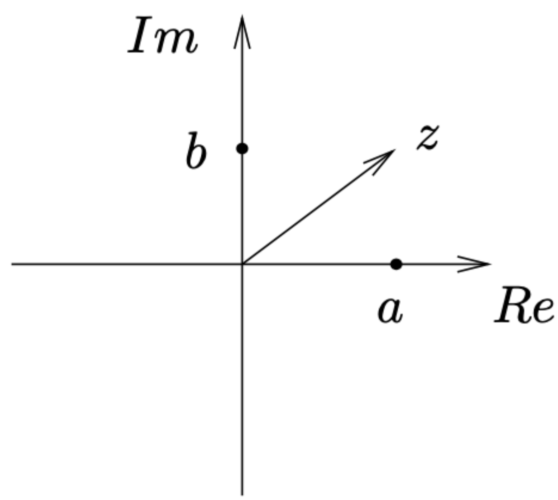
$$\frac{1}{3+4i} = \frac{3-4i}{(3+4i)(3-4i)} = \frac{3-4i}{3^2+4^2} = \frac{3}{25} - \frac{4}{25}i$$

and

$$\frac{2-5i}{3+4i} = (2-5i)\frac{1}{3+4i} = (2-5i)(\frac{3}{25} - \frac{4}{25}i) = -\frac{14}{25} - \frac{23}{25}i.$$

1.3. Complex numbers: geometric properties

Above we established the most basic algebraic properties of complex numbers. At this point it is useful to look at complex numbers from a geometric point of view. By definition the set $\mathbb{C}$ is just $\mathbb{R}^2$, and a complex number $z = a+bi$ with $a, b \in \mathbb{R}$ can be identified with the vector $(a, b) \in \mathbb{R}^2$. The set $\mathbb{C}$ is often called the *complex plane*, and we can draw complex numbers as vectors in the plane.



1.3.1. Modulus and complex conjugate. Next we introduce two useful quantities associated to every complex number $z \in \mathbb{C}$: its *modulus* and its *complex conjugate*. We begin with modulus. The reader has likely seen earlier the Euclidean norm $\|(a, b)\| = \sqrt{a^2 + b^2}$ of vectors in $\mathbb{R}^2$. For historical reasons, this quantity is often called "modulus" in complex analysis – but it is exactly the same thing.

Definition 1.3.1 (Modulus). Let $z = a + bi \in \mathbb{C}$. Then the *modulus* of z is the number

$$|z| := \|(a, b)\| = \sqrt{a^2 + b^2}.$$

Proposition 1.3.2. *If $z, w \in \mathbb{C}$, then $|z + w| \leq |z| + |w|$.*

Proof. This is just the triangle inequality for Euclidean norm $\|\cdot\|$! □

We then move on to the complex conjugate (see Figure 1):

Definition 1.3.3 (Complex conjugate). Let $z = a + bi \in \mathbb{C}$. Then the *complex conjugate of z* is the number $\bar{z} = a - bi \in \mathbb{C}$.

Example 1.3.4. If $z = 3 + 4i$, then $|z| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$ and $\bar{z} = 3 - 4i$.

There is a beautiful link between the modulus, conjugate, and complex inverse:

Proposition 1.3.5. *Let $z \in \mathbb{C}$. Then $z\bar{z} = |z|^2$. In particular $z^{-1} = \bar{z}/|z|^2$, if $z \neq 0$.*

Proof. Write $z = a + bi$, so $\bar{z} = a - bi$. Now,

$$z\bar{z} = (a+bi)(a-bi) = a^2 - abi + bai - i^2b^2 = a^2 + b^2 = |z|^2.$$

This is what we claimed. The formula $z^{-1} = \bar{z}/|z|^2$ follows by multiplying both sides of the equation by $z^{-1}/|z|^2$. (We have seen this formula for z^{-1} before in (1.2.1).) □

Complex conjugation and multiplication play nicely together:

Proposition 1.3.6. *Let $z, w \in \mathbb{C}$. Then $\overline{zw} = \bar{z}\bar{w}$.*

Proof. This is just a computation. Let $z = a + ib$ and $w = c + id$. Then,

$$\begin{aligned}
\overline{zw} &= \overline{(ac - bd) + i(ad + bc)} \\
&= (ac - bd) - i(ad + bc) \\
&= (ac - (-b)(-d)) + i(a(-d) + (-b)c) \\
&= (a - ib)(c - id) = \bar{z}\bar{w}.
\end{aligned}$$

This is what we claimed. □

For $x, y \in \mathbb{R}$, everyone knows that $|xy| = |x||y|$. It is rather striking that the same remains true for $z, w \in \mathbb{C}$:

Corollary 1.3.7. *Let $z, w \in \mathbb{C}$. Then $|zw| = |z||w|$.*

Proof. We use Propositions 1.3.5-1.3.6, and the commutativity of the complex product:

$$|zw|^2 = (zw)\overline{(zw)} = zw\bar{z}\bar{w} = (z\bar{z})(w\bar{w}) = |z|^2|w|^2.$$

The proof is completed by taking square roots. □

Here are a few further useful properties of the modulus and conjugate:

Proposition 1.3.8. *Let $z, w \in \mathbb{C}$. Then,*

$$\bar{\bar{z}} = z, \quad \overline{z+w} = \bar{z} + \bar{w}, \quad |z| = |\bar{z}|, \quad \textit{and}$$
$$\max\{|\mathrm{Re}(z)|, |\mathrm{Im}(z)|\} \leq |z|.$$

If $z \neq 0$, then $\overline{z^{-1}} = \bar{z}^{-1}$. Finally, the real and imaginary parts of z can be expressed in terms of the conjugate, as follows:

$$\mathrm{Re}(z) = \frac{z + \bar{z}}{2} \quad \textit{and} \quad \mathrm{Im}(z) = \frac{z - \bar{z}}{2i}. \tag{1.3.1}$$

PROOF. Exercise. □

EXAMPLE 1.3.9. We would like to describe geometrically the set

$$S = \{z \in \mathbb{C} : |z-1| = 2|z+1|\}.$$

Let $z = x + iy$. We have a chain of equivalences:

$$\begin{aligned}
z \in S \quad &\Longleftrightarrow \quad |z-1| = 2|z+1| \\
&\Longleftrightarrow \quad |z-1|^2 = 4|z+1|^2 \\
&\Longleftrightarrow \quad (z-1)\overline{(z-1)} = 4(z+1)\overline{(z+1)} \\
&\Longleftrightarrow \quad |z|^2 - (z+\bar{z}) + 1 = 4|z|^2 + 4(z+\bar{z}) + 4 \\
&\Longleftrightarrow \quad |z|^2 - 2\mathrm{Re}(z) + 1 = 4|z|^2 + 8\mathrm{Re}(z) + 4 \\
&\Longleftrightarrow \quad 3|z|^2 + 10\mathrm{Re}(z) + 3 = 0 \\
&\Longleftrightarrow \quad |z|^2 + 2\cdot\frac{5}{3}\cdot\mathrm{Re}(z) + 1 = 0 \\
&\Longleftrightarrow \quad x^2 + y^2 + 2\cdot\frac{5}{3}\cdot x + \left(\frac{5}{3}\right)^2 = \left(\frac{5}{3}\right)^2 - 1 = \frac{16}{9} \\
&\Longleftrightarrow \quad |(x,y) + (5/3,0)|^2 = (4/3)^2.
\end{aligned}$$

Thus S is a circle in $\mathbb{C} = \mathbb{R}^2$ with radius $4/3$ and centre $-5/3 = (-5/3, 0)$. We can also write $S = \{z \in \mathbb{C} : |z + 5/3|^2 = (4/3)^2$.

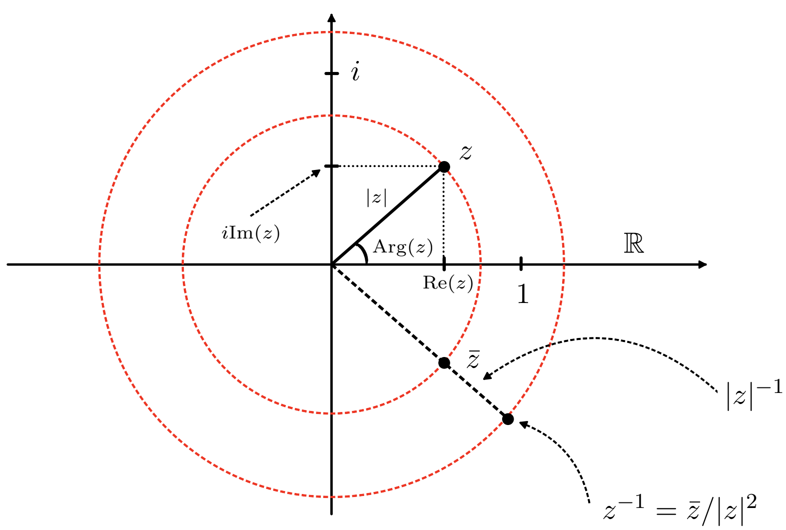


FIGURE 1. The points $z, \bar{z}$ and $z^{-1} = \bar{z}/|z|^2$.

The concepts introduced in the previous section have illustrative geometric interpretations, shown in Figure 1. Here are the key points to pay attention to in the figure:

- The modulus $|z|$ is just the length of the vector $z \in \mathbb{C}$.
- If $z = x + iy$, the complex conjugate $\bar{z} = x - iy$ is obtained by "mirroring" the point z over the real axis $\mathbb{R}$.
- The complex inverse $z^{-1} = \bar{z}/|z|^2$ is obtained by stretching the conjugate $\bar{z}$ by a factor of $|z|^{-2}$. In the figure $|z| < 1$, so the stretch factor is greater than one. Note that $|z^{-1}| = \frac{1}{|z|}$.

1.3.2. Argument. Figure 1 illustrates a new concept "Arg(z)". This is the angle (called *argument* in complex analysis), in radians, between the vector z and the positive x-axis. However, this angle is not uniquely defined, since changing the angle by an integer multiple of 2π gives the same vector z! Thus we need to take some care in the definition.

Given $z \in \mathbb{C} \setminus \{0\}$, we can interpret $\frac{z}{|z|}$ as a unit vector in $\mathbb{R}^2$. Now we recall the following fact from plane geometry.

LEMMA 1.3.10. *Any unit vector $v \in \mathbb{R}^2$ is of the form*

$$v = (\cos\theta, \sin\theta)$$

for some $\theta \in \mathbb{R}$. Moreover, one has $v = (\cos\alpha, \sin\alpha)$ if and only if $\alpha = \theta + 2\pi k$ for some $k \in \mathbb{Z}$.

DEFINITION 1.3.11 (Argument). If $z \neq 0$ is a complex number, then its (multi-valued) *argument* is the set

$$\arg(z) = \{\theta \in \mathbb{R} \, : \, \frac{z}{|z|} = (\cos\theta, \sin\theta)\}.$$

The *principal argument* $\mathrm{Arg}(z)$ is the unique number $\theta \in \arg(z)$ satisfying

$$\theta \in (-\pi, \pi], \qquad z \in \mathbb{C} \setminus \{0\}.$$

In other words, $\mathrm{Arg}(z)$ is the signed angle formed by the vector z and the positive real axis, with the convention that

$$\begin{cases} \mathrm{Arg}(z) \in [0, \pi] & \text{if } \mathrm{Im}(z) \geq 0, \\ \mathrm{Arg}(z) \in (-\pi, 0) & \text{if } \mathrm{Im}(z) < 0. \end{cases}$$

By Lemma 1.3.10, the set $\arg(z)$ can be expressed as

$$\arg(z) = \{\mathrm{Arg}(z) + 2\pi k \, : \, k \in \mathbb{Z}\}. \tag{1.3.2}$$

The principal argument Arg is just one way of selecting a unique representative from the many possible angles representing z.

EXAMPLE 1.3.12. $\mathrm{Arg}(i) = \pi/2$, $\mathrm{Arg}(-1) = \pi$, $\mathrm{Arg}(-i) = -\pi/2$.

REMARK 1.3.13. The function Arg is **not** continuous on the set $\mathbb{C} \setminus \{0\}$. It has a "jump" over the negative real axis $(-\infty, 0)$, where the value of $\mathrm{Arg}(z)$ changes abruptly from π to $-\pi$ (when arriving to the axis from above).

REMARK 1.3.14. By definition, one has

$$\frac{z}{|z|} = \cos\left(\operatorname{Arg}(z)\right) + i \sin\left(\operatorname{Arg}(z)\right). \tag{1.3.3}$$

Therefore one can (for example) write explicitly

$$\operatorname{Arg}(z) = \arccos\left(\tfrac{\operatorname{Re}(z)}{|z|}\right), \qquad z \in \mathbb{C} \setminus \{0\}. \tag{1.3.4}$$

This implies that Arg **is** a continuous function in $\mathbb{C} \setminus (-\infty, 0]$. However, when writing $\operatorname{Arg}(z)$ in this way, one has to be careful in choosing correctly the domain and range of arccos. We leave the verification of these facts to the reader.

REMARK 1.3.15. The function $\operatorname{Arg}(z)$ is often called the *principal branch* of the multi-valued (i.e. set-valued) argument $\arg(z)$. Let us explain this terminology a little bit.

If we have a set-valued function $F(z)$, e.g. $F(z) = \arg(z)$, in some open set $X \subset \mathbb{C}$, then a *branch* of the set-valued function $F(z)$ is a continuous function $f : X \to \mathbb{C}$ such that $f(z) \in F(z)$ for all $z \in X$. This means that a *branch* is just a way of selecting a unique representative from the set $F(z)$ in a continuous way.

For example, Arg is a branch of arg in $\mathbb{C}\setminus(-\infty, 0]$, since Arg is continuous in this set and $\operatorname{Arg}(z) \in \arg(z)$ for any $z \in \mathbb{C} \setminus (-\infty, 0]$. Branches are in general not unique. For instance, the function $\operatorname{Arg}_1 : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{R}$ such that $\operatorname{Arg}_1(z)$ is the unique point in $\arg(z) \cap (\pi, 3\pi]$ would be another branch of arg in $\mathbb{C} \setminus (-\infty, 0]$.

Often in complex analysis textbooks one talks about the principal branch of the argument, the principal branch of the nth root function, and the principal branch of the complex logarithm. This just means that we single out a certain unique representative from many possible solutions. Since the nth root and logarithm are defined in terms of the argument, this boils down to using the principal argument Arg that we have defined. This will be enough for us for the moment (we will return to this in Section 3.1.3).

1.3.3. The geometry of complex products. Let us next investigate how the complex product interacts with the argument. We first consider complex numbers in terms of polar coordinates.

TERMINOLOGY 1.3.16 (Polar coordinates). Let $z \in \mathbb{C} \setminus \{0\}$ and $\theta \in \arg(z)$. Then, according to Definition 1.3.11, we have

$$z = |z|(\cos\theta + i \sin\theta). \tag{1.3.5}$$

The formula (1.3.5) is called a *polar coordinate representation of z*.

Note that z has many polar coordinate representations, for each choice of $\theta \in \arg(z)$.

PROPOSITION 1.3.17 (Uniqueness of polar representations). *Let $r, \rho \geq 0$ and $\theta, \alpha \in \mathbb{R}$. One has*

$$r(\cos\theta + i\sin\theta) = \rho(\cos\alpha + i\sin\alpha)$$

if and only if

$$\begin{aligned} \rho &= r, \\ \alpha &= \theta + 2\pi k \end{aligned}$$

for some $k \in \mathbb{Z}$.

PROOF. If $r(\cos\theta + i\sin\theta) = \rho(\cos\alpha + i\sin\alpha)$, then in terms of vectors

$$r(\cos\theta, \sin\theta) = \rho(\cos\alpha, \sin\alpha).$$

Taking the Euclidean norm yields $r = \rho$, and then using Lemma 1.3.10 gives $\alpha = \theta + 2\pi k$ for some $k \in \mathbb{Z}$. Conversely, if $r = \rho$ and $\alpha = \theta + 2\pi k$, then $r(\cos\theta + i\sin\theta) = \rho(\cos\alpha + i\sin\alpha)$ since cos and sin are 2π-periodic. $\square$

Next we recall:

LEMMA 1.3.18 (Sum formulas for cos and sin). *Let $\alpha, \beta \in \mathbb{R}$. Then,*

$$\begin{aligned} \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta, \\ \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta. \end{aligned}$$

PROPOSITION 1.3.19 (Product in polar coordinates). *Consider two complex numbers*

$$\begin{aligned} z_1 =& |z_1|(\cos\theta_1 + i\sin\theta_1), \\ z_2 =& |z_2|(\cos\theta_2 + i\sin\theta_2). \end{aligned}$$

Then

$$z_1 z_2 = |z_1|\,|z_2|(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)).$$

PROOF. By Lemma 1.3.18 we have

$$\begin{aligned} &(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &\quad = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2) \\ &\quad = \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2). \end{aligned}$$

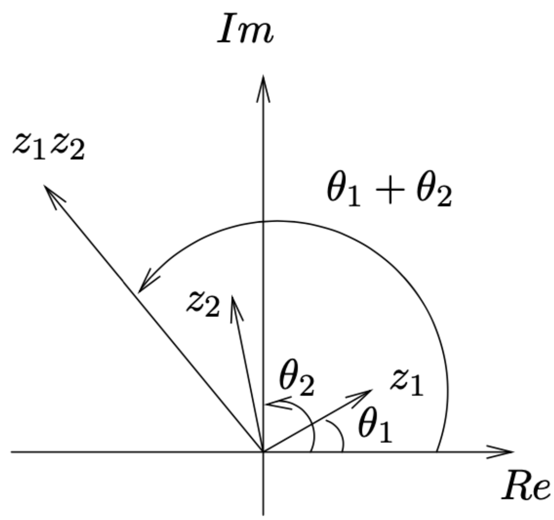
Then also

$$\begin{aligned} z_1 z_2 &= |z_1|\,|z_2|(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= |z_1|\,|z_2|(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)). \end{aligned}$$

$\square$

Proposition 1.3.19 allows us to describe the multiplication of two complex numbers in geometric terms. Here is an informal but easy to remember description:

> To multiply two complex numbers, multiply the moduli and add the angles.



REMARK 1.3.20. The only catch in this informal description is that the angle is only defined up to multiples of 2π. Thus even if $\theta_1, \theta_2 \in (-\pi, \pi]$ it may happen that $\theta_1 + \theta_2$ is outside of $(-\pi, \pi]$. In this case, to find the value of $\operatorname{Arg}(z_1 z_2)$, we have to add or subtract 2π to $\theta_1 + \theta_2$ (the choice is unique!) to bring it back inside $(-\pi, \pi]$.

EXAMPLE 1.3.21. Let's see how this works in a simple case like $z = -i = w$. Of course then we can directly calculate that $zw = (-i)(-i) = -1$, but let's deduce the same result with the geometric method of the previous remark.

First, obviously $|z| = 1 = |w|$, so also $|zw| = 1$. So, zw must be a point on the unit circle S^1. Second, $\operatorname{Arg}(-i) = -\pi/2$, so $\operatorname{Arg}(z) + \operatorname{Arg}(w) = -\pi/2 - \pi/2 = -\pi$. The result is no longer in $(-\pi, \pi]$, and we need to add 2π to bring it back inside $(-\pi, \pi]$. Consequently,

$$\operatorname{Arg}(zw) = -\pi + 2\pi = \pi.$$

Thus, zw is the point on S^1 whose angle with the positive $\mathbb{R}$-axis is π. This point is -1.

Example 1.3.22. The previous was perhaps a little uninteresting, since $(-i)(-i)$ is anyway easy to calculate directly. To increase the challenge, consider computing $(1-i)^8$. What a nightmare would it be to calculate this, based on the definition! Luckily there is a much quicker way. Note that $|1-i| = \sqrt{2}$, so $|(1-i)^8| = 2^4 = 16$. On the other hand, clearly $\operatorname{Arg}(1-i) = -\pi/4$, so $\arg((1-i)^8)$ contains the point $8 \cdot \operatorname{Arg}(1-i) = -2\pi$. It follows that

$$(1-i)^8 = |(1-i)|^8(\cos(-2\pi) + i\sin(-2\pi)) = 16.$$

1.4. Complex roots

1.4.1. De Moivre's formula. We have already seen in the proof of Proposition 1.3.19 that

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2). \tag{1.4.1}$$

In particular, the special case $\theta_1 = \theta_2 = \theta$ reads

$$(\cos\theta + i\sin\theta)^2 = \cos(2\theta) + i\sin(2\theta), \qquad \theta \in \mathbb{R}.$$

Using formula (1.4.1), we can continue this inductively:

$$\begin{aligned}(\cos\theta + i\sin\theta)^3 &= \big(\cos(2\theta) + i\sin(2\theta)\big)(\cos\theta + i\sin\theta)\\ &= \cos(3\theta) + i\sin(3\theta)\\ &\dots\\ (\cos\theta + i\sin\theta)^{n+1} &= \big(\cos(n\theta) + i\sin(n\theta)\big)(\cos\theta + i\sin\theta)\\ &= \cos\big((n+1)\theta\big) + i\sin\big((n+1)\theta\big)\end{aligned}$$

This argument proves the cases $n \in \mathbb{N}$ of the following important theorem:

Theorem 1.4.1 (De Moivre's formula). *Let $n \in \mathbb{Z}$ and $\theta \in \mathbb{R}$. Then,*

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

Proof. We already proved the cases $n \geq 1$. For $n = 0$, the formula is also true under the convention $z^0 = 1$:

$$(\cos\theta + i\sin\theta)^0 = 1 = \cos(0\cdot\theta) + i\sin(0\cdot\theta).$$

Regarding $n < 0$, let's first consider the case $n = -1$. Recall that $z^{-1} = \bar{z}/|z|^2$ for $z \in \mathbb{C} \setminus \{0\}$. For $z = \cos\theta + i\sin\theta$ we have $|z|^2 = (\cos\theta)^2 + (\sin\theta)^2 = 1$. Therefore,

$$(\cos\theta + i\sin\theta)^{-1} = \overline{\cos\theta + i\sin\theta} = \cos\theta - i\sin\theta = \cos(-\theta) + i\sin(-\theta),$$

using that cos is even and sin is odd. This proves the case $n = -1$. For general $n \in \mathbb{Z}$ with $n < 0$, we finally have

$$\begin{aligned}(\cos\theta + i\sin\theta)^n &= \left((\cos\theta + i\sin\theta)^{-1}\right)^{-n} \\ &= (\cos(-\theta) + i\sin(-\theta))^{-n} \\ &= \cos(n\theta) + i\sin(n\theta),\end{aligned}$$

applying the first part of the proof to $-n \in \mathbb{N}$. $\square$

REMARK 1.4.2. Later, when we define the exponential function e^z, we will establish Euler's formula

$$e^{ix} = \cos x + i\sin x, \qquad x \in \mathbb{R}.$$

For now, one could use the notation e^{ix} as a convenient *shorthand* for $\cos x + i\sin x$. Using this shorthand, De Moivre's formula takes the form $(e^{i\theta})^n = e^{in\theta}$ which is easier to remember.

1.4.2. Complex roots. We have already seen that one can multiply and divide complex numbers (as long as we do not divide by zero) and take powers z^n. Now we will show that one can also take nth roots of complex numbers.

DEFINITION 1.4.3 (nth root). Let $z \in \mathbb{C}$ and let $n \geq 2$ be an integer. A number $w \in \mathbb{C}$ is said to be an *nth root* of z if $w^n = z$.

Let us first consider taking the nth roots of 0. If $w^n = 0$, then taking the modulus gives $|w|^n = 0$, so $|w| = 0$ and $w = 0$. Thus the only nth root of 0 is 0. However, it is a fact of life that nth roots of $z \neq 0$ are not unique, in the same way that the equation $x^2 = a$ for $a > 0$ has two distinct solutions $\sqrt{a}$ and $-\sqrt{a}$.

THEOREM 1.4.4 (Complex roots). *Let $z \in \mathbb{C}$, $z \neq 0$, and let $n \geq 1$ be an integer. Then the equation*

$$w^n = z$$

has precisely n distinct solutions $w \in \mathbb{C}$. The solutions are given by

$$w = w_k = \sqrt[n]{|z|}\left(\cos\left(\frac{\mathrm{Arg}(z) + 2\pi k}{n}\right) + i\sin\left(\frac{\mathrm{Arg}(z) + 2\pi k}{n}\right)\right) \tag{1.4.2}$$

where $k = 0, 1, \ldots, n-1$.

PROOF. We start by writing z in polar coordinates as in (1.3.3),

$$z = |z|(\cos(\mathrm{Arg}(z)) + i\sin(\mathrm{Arg}(z))). \tag{1.4.3}$$

We look for a solution w of $w^n = z$ also in polar coordinate form, i.e.

$$w = r(\cos\theta + i\sin\theta).$$

By De Moivre's formula, we have

$$w^n = r^n(\cos\theta + i\sin\theta)^n = r^n(\cos(n\theta) + i\sin(n\theta)). \tag{1.4.4}$$

Comparing (1.4.3) and (1.4.4), we have $w^n = z$ if and only if

$$r^n(\cos(n\theta) + i\sin(n\theta)) = |z|(\cos(\mathrm{Arg}(z)) + i\sin(\mathrm{Arg}(z))).$$

By uniqueness of polar coordinate representations (Proposition 1.3.17), r and θ must satisfy

$$\begin{aligned} r^n &= |z|, \\ n\theta &= \mathrm{Arg}(z) + 2\pi k \end{aligned}$$

for some $k \in \mathbb{Z}$. The only positive number satisfying $r^n = |z|$ is

$$r = \sqrt[n]{z}.$$

It follows that all the solutions of $w^n = z$ are of the form (1.4.2) for some $k \in \mathbb{Z}$.

It might seem that there are infinitely many such numbers w, one for each $k \in \mathbb{Z}$. However, since cos and sin are 2π-periodic, we see that changing k to $k + mn$ for some $m \in \mathbb{Z}$ gives the same complex number w. Hence it is enough to look at the cases $k = 0, 1, \ldots, n-1$. It is an easy exercise to check that the numbers w_k for $k = 0, 1, \ldots, n-1$ are all distinct, using the fact that $\mathrm{Arg}(z) \in (-\pi, \pi]$. □

Remark 1.4.5. Using e^{ix} as a shorthand for $\cos x + i\sin x$ as in Remark 1.4.2, it is easy to remember how to solve $w^n = z$. Writing $z = |z|e^{i\mathrm{Arg}(z)}$, one can formally take the nth root in the equation $w^n = z = |z|e^{i\mathrm{Arg}(z)}$ to obtain one solution

$$w_0 = \sqrt[n]{|z|}e^{i\frac{\mathrm{Arg}(z)}{n}}.$$

Theorem 1.4.4 shows that all solutions are given by

$$w_k = \sqrt[n]{|z|}e^{i(\frac{\mathrm{Arg}(z)}{n} + \frac{2\pi k}{n})}, \qquad k = 0, 1, \ldots, n-1.$$

Next, how do the solutions of $w^n = z$ look like in the complex plane? Clearly they all lie on the circle of radius $r = \sqrt[n]{|z|}$ centred at the origin. Moreover, they form an evenly spaced set of cardinality "n" on that circle.

Example 1.4.6. Let's illustrate this with a picture in a simple case like $z = -1$, so

$$|z| = 1 \quad \text{and} \quad \mathrm{Arg}(z) = \pi.$$

Now, with $n = 4$ (for example), the solutions to the equation $w^4 = -1$ are given by

$$w_k = \cos(\pi/4 + 2\pi k/4) + i\sin(\pi/4 + 2\pi k/4), \qquad k \in \{0, \ldots, 3\}.$$

These four points are drawn in Figure 2. Notice that none of them lies on the real axis: of course there is no $r \in \mathbb{R}$ satisfying $r^4 = -1$! Since $\cos(\pi/4) = 1/\sqrt{2}$ etc, the solutions also have the explicit formulas

$$\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}, \quad -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}, \quad -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}, \quad \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}.$$

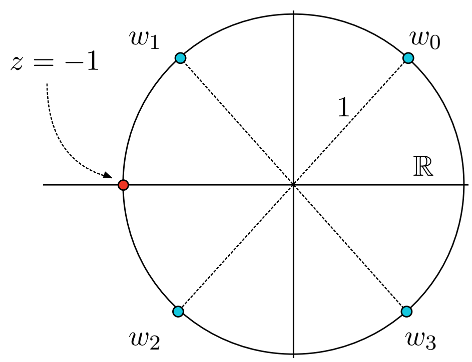


FIGURE 2. All solutions to the equation $w^4 = -1$.

Recall that in Definition 1.3.11 we chose a unique representative, the principal argument $\mathrm{Arg}(z)$, of the multi-valued argument $\arg(z)$. Similarly, we single out a unique choice of nth root by choosing $k = 0$ in Theorem 1.4.4:

DEFINITION 1.4.7 (Principal n^{th} root). Let $n \geq 2$. We define the $\mathbb{C}$-valued function $\sqrt[n]{\cdot} \colon \mathbb{C} \to \mathbb{C}$ by setting

$$\sqrt[n]{z} := \begin{cases} 0, & z = 0, \\ \sqrt[n]{|z|}(\cos(\mathrm{Arg}(z)/n) + i\sin(\mathrm{Arg}(z)/n)), & z \neq 0. \end{cases}$$

The function $\sqrt[n]{\cdot}$ is called the *principal n^{th} root.* For $n = 2$, we abbreviate $\sqrt[2]{z} =: \sqrt{z}$.

EXAMPLE 1.4.8. Following Example 1.4.6, the principal 4th root of -1 is

$$\sqrt[4]{-1} = \cos(\pi/4) + i\sin(\pi/4) = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}.$$

REMARK 1.4.9. If $r \in \mathbb{R}$ with $r \geq 0$, then $\mathrm{Arg}(r) = 0$. Therefore the principal n^{th} root $\sqrt[n]{r}$ agrees with the familiar definition of $\sqrt[n]{r}$ from the real line.

WARNING 1.4.10. You may be familiar with the rule $\sqrt{rs} = \sqrt{r}\sqrt{s}$ valid for $r, s \in [0, \infty)$. The principal square root also satisfies this for $r, s \in [0, \infty)$

(since it agrees with the familiar square root on $[0,\infty)$), but it does **not** satisfy a similar equation for all complex numbers:

$$\sqrt{-1}\sqrt{-1} = e^{i\mathrm{Arg}(-1)/2}e^{i\mathrm{Arg}(-1)/2} = e^{i\pi/2}e^{i\pi/2} = e^{i\pi} = -1 \neq \sqrt{1}.$$

Remark 1.4.11. Recall that Arg (the principal argument) has a jump discontinuity along the negative real axis $(-\infty,0)$. Essentially for this reason, the principal n^{th} root $\sqrt[n]{\cdot}\colon \mathbb{C} \to \mathbb{C}$ is also discontinuous in the set $(-\infty,0)$, for every $n \geq 2$. The functions $\sqrt[n]{\cdot}$ are, however, continuous on the set $\mathbb{C} \setminus (-\infty,0)$.

1.5. The complex exponential

Up to now we have seen expressions like

$$z^n, \qquad \frac{1}{z}, \qquad \sqrt[n]{z}.$$

Next we wish to define expressions like

$$e^z, \qquad \sin z, \qquad \cos z, \qquad \log z, \qquad z^w$$

for (many) complex numbers z and w.

1.5.1. Complex exponential. We know the exponential function e^x for $x \in \mathbb{R}$, with Taylor series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots.$$

We formally replace x by iy with $y \in \mathbb{R}$. Using that $i^2 = -1$, so that $i^{2k} = (-1)^k$ for $k \geq 1$, and recalling the Taylor series of cos and sin, this would give

$$\begin{aligned} e^{iy} &= 1 + iy - \frac{y^2}{2!} - \frac{iy}{3!} + \frac{y^4}{4!} + \dots \\ &= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \dots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + \dots\right) \\ &= \cos y + i \sin y. \end{aligned}$$

This gives further evidence that Euler's formula $e^{iy} = \cos y + i\sin y$ is a reasonable definition of e^{iy} for $y \in \mathbb{R}$. If we expect that the rule $e^{x+iy} = e^x e^{iy}$ also works for complex numbers, we arrive at the following definition.

Definition 1.5.1 (Exponential). Let $z = x + iy \in \mathbb{C}$ where $x, y \in \mathbb{R}$. We define

$$e^z = e^x(\cos y + i \sin y).$$

Example 1.5.2. $e^{4+i\pi/2} = e^4(\cos(\pi/2) + i\sin(\pi/2)) = e^4 i$.

The following proposition collects basic properties of the complex exponential.

PROPOSITION 1.5.3 (Properties of e^z).
(a) *If $z = x \in \mathbb{R}$, then e^z coincides with the usual exponential e^x.*
(b) *For any $z, w \in \mathbb{C}$ one has*

$$\begin{aligned} e^{z+w} &= e^z e^w, \\ (e^z)^{-1} &= e^{-z}. \end{aligned}$$

(c) *If $z = x + iy$, then $|e^z| = e^x$. In particular,*

$$|e^{iy}| = 1, \qquad y \in \mathbb{R},$$

and $e^z \neq 0$ for any $z \in \mathbb{C}$.

PROOF. (a) If $z = x + iy$ with $y = 0$, then $e^z = e^x(\cos 0 + i \sin 0) = e^x$.
(b) Let $z = x + iy$ and $w = a + ib$. Then by (1.4.1)

$$\begin{aligned} e^{z+w} &= e^{x+a+i(y+b)} = e^{x+a}(\cos(y+b) + i\sin(y+b)) \\ &= e^x e^a(\cos y + i \sin y)(\cos b + i \sin b) = e^z e^w. \end{aligned}$$

Then $e^z e^{-z} = e^{z+(-z)} = e^0 = 1$, which gives $(e^z)^{-1} = e^{-z}$.
(c) We have $|e^z| = |e^x(\cos y + i \sin y)| = e^x\sqrt{\cos^2 y + \sin^2 y} = e^x$. This gives $|e^{iy}| = 1$ and $e^z \neq 0$ for any z. □

REMARK 1.5.4. Later we will show that e^z is an analytic function of z, and the following fact further justifies Definition 1.5.1: *the map $z \mapsto e^z$ is the unique analytic function on $\mathbb{C}$ which agrees with $x \mapsto e^x$ on $\mathbb{R}$*. In fact, a result in *Complex analysis 2* will show that if two analytic functions $\mathbb{C} \to \mathbb{C}$ agree on $\mathbb{R}$, then they agree everywhere.

Now that we have properly defined $e^{i\theta} = \cos\theta + i \sin\theta$, we can use the $e^{i\theta}$ notation and rewrite some facts that we have seen earlier:

- (Polar coordinates) Any $z \in \mathbb{C} \setminus \{0\}$ may be written as
$$z = |z|e^{i\theta}$$
for any $\theta \in \arg(z)$. In particular, we may write
$$z = |z|e^{i\operatorname{Arg}(z)}.$$
- (Products) If $z_1 = |z_1|e^{i\theta_1}$ and $z_2 = |z_2|e^{i\theta_2}$, then
$$z_1 z_2 = |z_1|\,|z_2|e^{i(\theta_1+\theta_2)}.$$

- (Roots) If $z \in \mathbb{C} \setminus \{0\}$ and $n \geq 2$, then $z = |z|e^{i\operatorname{Arg}(z)}$ has principal nth root
$$\sqrt[n]{z} = \sqrt[n]{|z|}e^{i\frac{\operatorname{Arg}(z)}{n}}.$$
All the nth roots of z are given by
$$\sqrt[n]{|z|}e^{i\frac{\operatorname{Arg}(z)}{n}}, \quad \sqrt[n]{|z|}e^{i(\frac{\operatorname{Arg}(z)}{n}+\frac{2\pi}{n})}, \quad \ldots \quad \sqrt[n]{|z|}e^{i(\frac{\operatorname{Arg}(z)}{n}+\frac{(n-1)2\pi}{n})}$$

REMARK 1.5.5. If you have trouble remembering the sum formulas for sin and cos, a good way to recall them is the relation $e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta}$:
$$\begin{aligned}
&\cos(\alpha+\beta) + i\sin(\alpha+\beta) = e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta} \\
&= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) \\
&= \cos\alpha\cos\beta - \sin\alpha\sin\beta + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta).
\end{aligned}$$

We will next investigate the "mapping properties" of $z \mapsto e^z$. Here are two illustrative properties (see Figure 3):

PROPOSITION 1.5.6. *Let $a, b \in \mathbb{R}$. The complex exponential maps the horizontal line $\{z : \operatorname{Im}(z) = a\}$ to the ray $\{re^{ia} : r > 0\}$ and the vertical line $\{z : \operatorname{Re}(z) = b\}$ to the circle with radius e^b.*

PROOF. The first statement is is clear from $e^{x+ia} = e^x e^{ia}$, and noting that e^x takes all values on $(0, \infty)$ when x ranges in $\mathbb{R}$.

The second statement follows from $|e^{b+iy}| \equiv e^b$, and noting that $e^{iy} = \cos y + i\sin y$ takes all values on the unit circle when y ranges in $\mathbb{R}$. $\square$

PROPOSITION 1.5.7. *The complex exponential is $(2\pi i)$-periodic:*
$$e^{z+2\pi i} = e^z, \qquad z \in \mathbb{C}.$$
Moreover,
$$e^z = e^w \quad \Longleftrightarrow \quad w = z + 2\pi ik \text{ for some } k \in \mathbb{Z}.$$

PROOF. The $(2\pi i)$-periodicity is simple:
$$e^{z+2\pi i} = e^z e^{2\pi i} = e^z,$$
since $e^{2\pi i} = \cos(2\pi) + i\sin(2\pi) = 1$. Assume then that $e^z = e^w$. Multiplying by e^{-z} and using Proposition 1.5.3 gives
$$e^{w-z} = 1.$$
If we write $w - z = x + iy$, we have $1 = |e^{x+iy}| = e^x$, and therefore $x = 0$. Consequently,
$$\cos y + i\sin y = e^{iy} = e^{w-z} = 1,$$
which forces $y \in 2\pi\mathbb{Z}$. Therefore $w - z = iy \in 2\pi i\mathbb{Z}$, as claimed. $\square$

By the $(2\pi i)$-periodicity just established, the complex exponential e^z attains all of its values in any *strip* of the form

$$S_h := \{z \in \mathbb{C} : \mathrm{Im}(z) \in (h - \pi, h + \pi]\}, \qquad h \in \mathbb{R}, \tag{1.5.1}$$

see Figure 3. In other words,

$$\{e^z : z \in \mathbb{C}\} = \{e^z : z \in S_h\} \text{ for all } h \in \mathbb{R}.$$

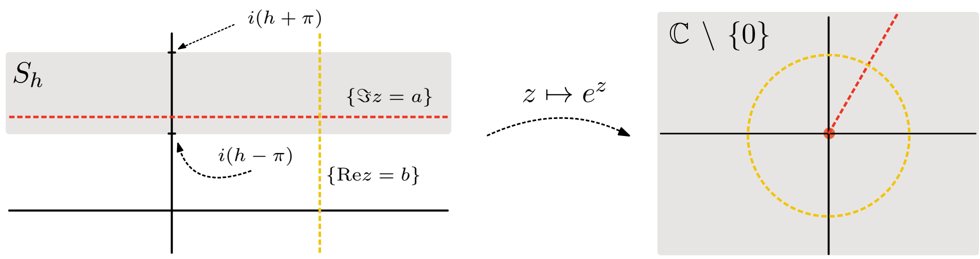


FIGURE 3. The mapping properties of $z \mapsto e^z$. The grey strip S_h maps to $\mathbb{C} \setminus \{0\}$. The red horizontal line $\{\mathrm{Im}(z) = a\}$ maps to a ray $\{re^{ia} : r > 0\}$ emanating from 0. The beige vertical line $\{\mathrm{Re}(z) = b\}$ maps to the circle with radius e^b.

PROPOSITION 1.5.8. *Let $h \in \mathbb{R}$. The complex exponential $z \mapsto e^z$ is a bijection $S_h \to \mathbb{C} \setminus \{0\}$.*

PROOF. Let us first prove the injectivity. Let $z_1, z_2 \in S_h$. If $e^{z_1} = e^{z_2}$, then $e^{z_1 - z_2} = e^{z_1}/e^{z_2} = 1$, which implies by Proposition 1.5.7 that $z_1 - z_2 \in 2\pi i\mathbb{Z}$. However, the strip S_h has been chosen to be exactly so "narrow" that this forces $z_1 = z_2$.

We then prove the surjectivity: $\{e^z : z \in S_h\} = \mathbb{C} \setminus \{0\}$. We already saw that $e^z \neq 0$ for all $z \in \mathbb{C}$. Now if $w \in \mathbb{C} \setminus \{0\}$, then w has a polar coordinate representation

$$w = |w|e^{i\theta}, \qquad \theta \in \arg(w).$$

In (1.3.2) we showed that $\arg(w) = \mathrm{Arg}(w) + 2\pi\mathbb{Z}$. In particular, there exists $\theta \in \arg(w)$ satisfying $h < \theta \leq h + 2\pi$. Now

$$z := \log|w| + i\theta \in S_h,$$

and $e^z = e^{\log|w|}e^{i\theta} = |w|e^{i\theta} = w$. (Here "log" refers to the logarithm in base e on the positive real line.) $\square$

1.5.2. Sine and cosine functions in $\mathbb{C}$. Back in (1.3.1) we recorded that

$$\mathrm{Re}(z) = \frac{z+\bar{z}}{2} \quad \text{and} \quad \mathrm{Im}(z) = \frac{z-\bar{z}}{2i}, \qquad z \in \mathbb{C}.$$

In particular,

$$\cos\theta = \mathrm{Re}(e^{i\theta}) = \frac{e^{i\theta}+e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \mathrm{Im}(e^{i\theta}) = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

for all $\theta \in \mathbb{R}$. Now that we have defined e^z for all $z \in \mathbb{C}$, the formulas above suggest a neat way to extend the sine and cosine functions to the whole complex plane:

Definition 1.5.9 (Sine, cosine, tangent). For $z \in \mathbb{C}$, we define

$$\cos z := \frac{e^{iz}+e^{-iz}}{2} \quad \text{and} \quad \sin z := \frac{e^{iz}-e^{-iz}}{2i}. \tag{1.5.2}$$

We also define the complex tangent function $\tan z := (\sin z)/(\cos z)$ whenever $\cos z \neq 0$.

Many of the familiar properties of the cosine and sine functions on the real line have counterparts in $\mathbb{C}$:

Proposition 1.5.10. *Let $z \in \mathbb{C}$. Then,*

$$(\cos z)^2 + (\sin z)^2 = 1, \quad \cos(2z) = (\cos z)^2 - (\sin z)^2, \quad \cos\left(\tfrac{\pi}{2} - z\right) = \sin z.$$

Also the following addition rules hold for all $z, w \in \mathbb{C}$:

$$\sin(z+w) = \sin z \cos w + \cos z \sin w,$$
$$\cos(z+w) = \cos z \cos w - \sin z \sin w.$$

Proof. Exercise. $\square$

But not everything is so familiar. On the real axis, the functions sin and cos are bounded by 1 in absolute value. This is completely different in $\mathbb{C}$:

Example 1.5.11. Both cos and sin are surjective $\mathbb{C} \to \mathbb{C}$ (but not injective). In particular, their moduli are unbounded. The first claim takes a little effort to prove, and we omit it here, but the second one can be observed easily by considering the values of cos or sin on the imaginary axis:

$$\lim_{t\to\infty} \cos(it) = \lim_{t\to\infty} \frac{e^{iit}+e^{-iit}}{2} = \lim_{t\to\infty} \frac{e^{-t}+e^{t}}{2} = \infty.$$

Similarly $\sin(it) \to -\infty$ as $t \to \infty$.

As a side remark, the map $t \mapsto \cos(it)$ is real-valued (as we just saw), and is known as the *hyperbolic cosine*. Similarly, $t \mapsto -i\sin(ix)$ is also real-valued, and is known as the *hyperbolic sine*.

EXAMPLE 1.5.12. Starting from the definitions, one can quite easily find that all the solutions to $\sin z = 0$ lie on the real line, and we all know that $\sin x = 0$ for $x \in \pi\mathbb{Z}$. Similarly, all the solutions of $\cos z = 0$ also lie on the real line, more precisely in the set $\pi\mathbb{Z} + \frac{\pi}{2}$.

1.6. The complex logarithm

For $x > 0$, the logarithm $\log x$ is defined as the number $t \in \mathbb{R}$ satisfying $e^t = x$. (In this course, log always means the natural logarithm in base e.) Similarly, we might like to define $\log z$ as a complex number w with $e^w = z$. However, like in the case of the argument, such a logarithm is not unique since $z \mapsto e^z$ is not injective on $\mathbb{C}$.

If $z \in \mathbb{C} \setminus \{0\}$, we use the polar coordinate representation

$$z = |z|e^{i\mathrm{Arg}(z)} = e^{\log|z|}e^{i\mathrm{Arg}(z)} = e^{\log|z|+i\mathrm{Arg}(z)}$$

where $\log|z|$ is the logarithm of the real number $|z| > 0$. Thus $w_0 = \log|z| + i\mathrm{Arg}(z) \in \mathbb{C}$ satisfies $e^{w_0} = z$. However, since e^w is $2\pi i$-periodic, the numbers $w_k = \log|z| + i(\mathrm{Arg}(z) + 2\pi k)$ for $k \in \mathbb{Z}$ also satisfy $e^{w_k} = z$ and there are no other such numbers (this follows from Proposition 1.5.7).

DEFINITION 1.6.1 (Logarithm). If $z \in \mathbb{C} \setminus \{0\}$, then its (multi-valued) logarithm is the set

$$\log z = \{w \in \mathbb{C} \,:\, e^w = z\}.$$

The *principal logarithm* of z is the number

$$\mathrm{Log}\, z = \log|z| + i\mathrm{Arg}(z).$$

Thus, like the principal argument Arg or the principal nth root $\sqrt[n]{\cdot}$, the principal logarithm $\mathrm{Log}\, z$ selects a unique representative from the many possible logarithms of z. Of course there are other possible choices (e.g. $\mathrm{Log}_1 z = \mathrm{Log}\, z + 2\pi i$). To be explicit, we note that one has

$$e^{\mathrm{Log}\, z} = z \quad \text{for } z \in \mathbb{C} \setminus \{0\}$$

and

$$\mathrm{Log}\, x = \log x \quad \text{for } x > 0.$$

The previous discussion proves that for $z \neq 0$,

$$\log z = \{\mathrm{Log}\, z + 2\pi i k \,:\, k \in \mathbb{Z}\}.$$

Note that the logarithm of 0 is not defined (just like in the real case), since there is no $w \in \mathbb{C}$ with $e^w = 0$.

The mapping properties of the principal logarithm are contained in Figure 4 and verified in the next proposition.

Proposition 1.6.2. *Let* $\operatorname{Log} z = \log|z| + i\operatorname{Arg}(z)$ *be the principal logarithm, defined in* $\mathbb{C} \setminus (-\infty, 0]$. *Then* Log *maps* $\mathbb{C} \setminus (-\infty, 0]$ *bijectively onto the open strip*

$$\mathcal{S} = \{z : \operatorname{Im}(z) \in (-\pi, \pi)\}.$$

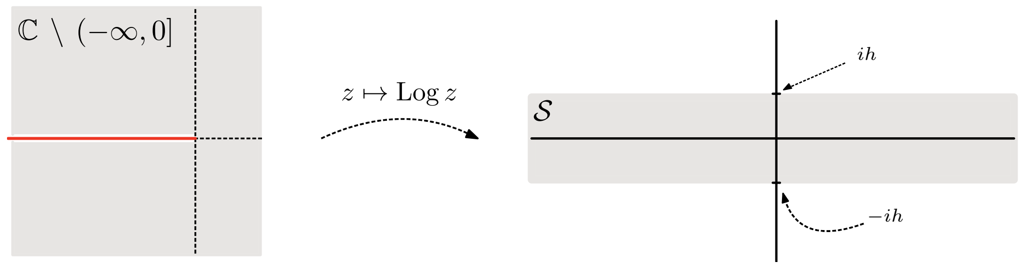


Figure 4. The principal logarithm Log maps $\mathbb{C} \setminus (-\infty, 0]$ bijectively onto the open strip $\mathcal{S}$.

Proof. The easiest way to see this is straight from the formula: when $z \in \mathbb{C} \setminus (-\infty, 0]$ varies, $\log|z|$ takes all possible values in $\mathbb{R}$, and $\operatorname{Arg}(z)$ takes all values in $(-\pi, \pi)$. Therefore the image of Log of the set $\mathbb{C} \setminus (-\infty, 0]$ is the product $\mathbb{R} \times (-\pi, \pi) = \mathcal{S}$. The fact that $\operatorname{Log}\colon \mathbb{C} \setminus (-\infty, 0] \to \mathcal{S}$ is bijective is also easy to see from the formula. $\square$

Remark 1.6.3. By Proposition 1.6.2, the map $\operatorname{Log} : \mathbb{C} \setminus (-\infty, 0] \to \mathcal{S}$ is the inverse function of $f : \mathcal{S} \to \mathbb{C} \setminus (-\infty, 0]$, $f(z) = e^z$.

Warning 1.6.4. If $s, t > 0$, the real logarithm satisfies $\log(st) = \log(s) + \log(t)$. We have seen earlier that sometimes $\operatorname{Arg}(zw) \neq \operatorname{Arg}(z) + \operatorname{Arg}(w)$ and $\sqrt{zw} \neq \sqrt{z}\sqrt{w}$, so it should not be a surprise that also sometimes $\operatorname{Log}(zw) \neq \operatorname{Log} z + \operatorname{Log} w$ (for the principal logarithms). We leave finding concrete examples of z, w as an exercise.

The complex logarithm for $z, w \in \mathbb{C} \setminus \{0\}$ does satisfy the following equality in the sense of sets (exercise):

$$\log(zw) = \log(z) + \log(w).$$

Here we write $A + B = \{a + b : a \in A, b \in B\}$. We warn the readers that in general one needs to be careful with such set equalities (for instance, one has $\mathbb{Z} - \mathbb{Z} = \mathbb{Z}$ which may seem counterintuitive).

1.6.1. Complex powers. Recall that if $t > 0$ and $a \in \mathbb{R}$, one defines

$$t^a := e^{a \log t}.$$

We can make a similar definition for complex numbers, but again we have to take care with the possibly multi-valued logarithm. We will use the principal logarithm Log.

DEFINITION 1.6.5 (Principal branch of complex powers). If $z \in \mathbb{C} \setminus \{0\}$ and $w \in \mathbb{C}$, we define

$$z^w := e^{w \operatorname{Log} z}.$$

EXAMPLE 1.6.6.

$$i^{2\pi} = e^{2\pi \operatorname{Log} i} = e^{2\pi \operatorname{Log}(e^{i\pi/2})} = e^{2\pi(i\pi/2)} = e^{i\pi^2} = \cos(\pi^2) + i\sin(\pi^2),$$
$$i^{2\pi i} = e^{2\pi i \operatorname{Log} i} = e^{2\pi i(i\pi/2)} = e^{-\pi^2}.$$

We can check that the above definition coincides with definitions that we already know (below z^w is as in Definition 1.6.5):

- If $n \geq 1$ is an integer and $z \neq 0$, then by Proposition 1.5.3

$$z^n = e^{n \operatorname{Log} z} = e^{\operatorname{Log} z} \cdot \ldots \cdot e^{\operatorname{Log} z} = \underbrace{z \cdot \ldots \cdot z}_{n \text{ times}}$$

- If $n \geq 2$ is an integer and $z \neq 0$, then by the definition of $\operatorname{Log} z$ and Proposition 1.5.3 we see that $z^{\frac{1}{n}}$ is the principal nth root:

$$z^{\frac{1}{n}} = e^{\frac{1}{n}\operatorname{Log} z} = e^{\frac{1}{n}(\log|z| + i\operatorname{Arg}(z))} = e^{\frac{1}{n}\log|z|} e^{\frac{1}{n} i \operatorname{Arg}(z)} = |z|^{\frac{1}{n}} e^{i\frac{\operatorname{Arg}(z)}{n}} = \sqrt[n]{z}$$

For any $z \neq 0$ and $w, v \in \mathbb{C}$, we also have the familiar rule

$$\begin{aligned} z^{w+v} &= e^{(w+v)\operatorname{Log} z} = e^{w \operatorname{Log} z + v \operatorname{Log} z} = e^{w \operatorname{Log} z} e^{v \operatorname{Log} z} \\ &= z^w z^v. \end{aligned}$$

However, in general one has

$$(zw)^v \neq z^v w^v, \qquad (z^w)^v \neq z^{wv}.$$

WARNING 1.6.7. In general, be extremely careful when computing with the maps $z \mapsto \operatorname{Log} z$ or $z \mapsto \sqrt[n]{z}$ or $z \mapsto z^w$ when $z \in \mathbb{C}$ or $w \in \mathbb{C}$, or both. Some rules from the real line remain valid, others do not. In contrast, the map $z \mapsto e^z$ has many of the good properties of $x \mapsto e^x$. This is, heuristically, because the definition of e^z does **not** involve making choices of the multi-valued argument.

Summary of Chapter 1. Here is a list of some key topics.

- The definition of the complex product zw and inverse z^{-1}.
- Modulus, complex conjugate, argument, and how these concepts interact with the complex product and complex inverse.
- De Moivre's formula.
- The representation of $z \in \mathbb{C} \setminus \{0\}$ in polar coordinates.
- Solving the equation $z^n = w$, and the definition of $\sqrt[n]{w}$.
- The complex exponential and its basic properties.
- The complex (principal) logarithm.

CHAPTER 2

Topology of the complex plane

The main topic of this course will be the notion of an analytic function $f : U \to \mathbb{C}$, where U is an open subset of $\mathbb{C}$. In order to study such functions we need to talk about open sets, continuity and differentiability. We will collect some required facts in this chapter.

Many concepts will likely be familiar to the reader from previous courses such as *Vector analysis 1*. In particular, all the metric and topological concepts in $\mathbb{C}$ (open and closed sets, distance between points, limits, continuity) are **exactly** the same as in $\mathbb{R}^2$. So, if you are confident with metric and topological concepts in $\mathbb{R}^2$, you can just skim through this chapter.

2.1. Open and closed sets

Definition 2.1.1 (Distance). Let $z, w \in \mathbb{C}$. The *distance* between z and w is $|z - w|$, namely the modulus of $z - w$. In coordinates,

$$|(a + ib) - (x + iy)| = \sqrt{(x - a)^2 + (y - b)^2}.$$

This is the same number as the Euclidean distance between the vectors $(a, b), (x, y) \in \mathbb{R}^2$.

Remark 2.1.2. The following inequalities are useful: if $z, w \in \mathbb{C}$, then

$$||z| - |w|| \le |z + w| \le |z| + |w|.$$

The second inequality is the *triangle inequality* $|z + w| \le |z| + |w|$ from Proposition 1.3.2. The first one is the *reverse triangle inequality* proved as follows:

$$|z| = |z - w + w| \le |z - w| + |w| \quad \Longrightarrow |z| - |w| \le |z - w|,$$

and similarly

$$|w| = |w - z + z| \le |z - w| + |z| \quad \Longrightarrow \quad |w| - |z| \le |z - w|,$$

so $||z| - |w|| \le |z - w|$.

Definition 2.1.3 (Open and closed discs, circles). Let $z \in \mathbb{C}$ and $r > 0$. We write

$$D(z, r) := \{w \in \mathbb{C} : |z - w| < r\}.$$

This is the *open disc with centre z and radius r*. We also write $\bar{D}(z,r) := \{w \in \mathbb{C} : |z-w| \le r\}$ for the *closed disc with centre z and radius r*. Finally, we write

$$S(z,r) := \{w \in \mathbb{C} : |z-w| = r\}$$

for the *circle with centre z and radius r*. One has $S(z,r) = \bar{D}(z,r) \setminus D(z,r)$.

DEFINITION 2.1.4 (Interior, exterior, boundary and closure). Let $A \subset \mathbb{C}$. We say that $z \in \mathbb{C}$ is

- an *interior point* of A if there is $r > 0$ such that $D(z,r) \subset A$;
- an *exterior point* of A if z is an interior point of $\mathbb{C} \setminus A$;
- a *boundary point* of A if it is not an interior or exterior point of A.

The sets of interior, exterior and boundary points are denoted by $\mathrm{int}(A)$, $\mathrm{ext}(A)$, and ∂A (the *interior*, *exterior*, and *boundary* of A), respectively. The *closure* of A is

$$\overline{A} = A \cup \partial A.$$

REMARK 2.1.5. It follows from the definition that $z \in \partial A$ if and only if for any $r > 0$ one has

$$A \cap D(z,r) \neq \emptyset \quad \text{and} \quad A \cap (\mathbb{C} \setminus D(z,r)) \neq \emptyset.$$

For any $A \subset \mathbb{C}$ the sets $\mathrm{int}(A)$, ∂A and $\mathrm{ext}(A)$ are disjoint. One always has

$$\mathbb{C} = \mathrm{int}(A) \cup \partial A \cup \mathrm{ext}(A).$$

DEFINITION 2.1.6 (Open and closed sets). A set $U \subset \mathbb{C}$ is *open* if any $z \in U$ is an interior point of U, i.e. $U = \mathrm{int}(U)$. A set $F \subset \mathbb{C}$ is *closed* if $\mathbb{C} \setminus F$ is open.

EXAMPLE 2.1.7. The open disc $D(z,r)$ is open, and the closed disc $\bar{D}(z,r)$ is closed. To see that $D(z,r)$ is open, you need to show that if $w \in D(z,r)$, then there exists a radius $s > 0$ such that $D(w,s) \subset D(z,r)$. Why is this true? To show that $\bar{D}(z,r)$ is closed, you need to show that $\mathbb{C} \setminus \bar{D}(z,r)$ is open. Why is this true? One has (exercise)

$$\begin{aligned} \mathrm{int}(D(z,r)) &= D(z,r), \\ \partial D(z,r) &= S(z,r), \\ \overline{D(z,r)} &= \bar{D}(z,r). \end{aligned}$$

EXAMPLE 2.1.8. For any $A \subset \mathbb{C}$ the sets $\mathrm{int}(A)$ and $\mathrm{ext}(A)$ are open, and the sets ∂A and $\overline{A}$ are closed (exercise).

EXAMPLE 2.1.9. The sets $\emptyset$ and $\mathbb{C}$ are both open and closed. These are the only subsets of $\mathbb{C}$ which are both open and closed. (This is related to the notion of *connectedness* discussed later.)

WARNING 2.1.10. Many sets are neither open nor closed. For example, $D(0,1) \cup \{2\}$ (the union of an open disc and a singleton) is neither open nor closed.

We next consider the unions and intersections of open and closed sets.

PROPOSITION 2.1.11. *Let $U_j \subset \mathbb{C}$, $j \in \mathcal{J}$, be an **arbitrary** collection of open sets. The index set $\mathcal{J}$ need not be finite, or even countable! Then the union $\bigcup_{j\in\mathcal{J}} U_j$ is also open.*

*Let $V_1, \ldots, V_n \subset \mathbb{C}$ be a **finite** family of open sets. Then $V_1 \cap \ldots \cap V_n$ is open.*

PROOF. Let us first prove the openness of the union $U = \bigcup_{j\in\mathcal{J}} U_j$. Fix $z \in U$. Then in particular $z \in U_j$ for some $j \in \mathcal{J}$. Therefore, since U_j is open, there exists a radius $r > 0$ such that $D(z,r) \subset U_j$. But now also $D(z,r) \subset U$. We have proven that U is open.

Let us then prove that the intersection $V := V_1 \cap \ldots \cap V_n$ is open. Fix $z \in V$. Then $z \in V_j$ for all $1 \le j \le n$, so for each of these indices there exists a radius $r_j > 0$ such that $D(z, r_j) \subset V_j$. Therefore, if we set $r := \min r_j$, we have $r > 0$, and

$$D(z,r) \subset D(z,r_1) \cap \ldots \cap D(z,r_n) \subset V_1 \cap \ldots V_n = V.$$

This shows that V is open. The finiteness of the family $V_1, \ldots, V_n$ was needed to ensure that the minimum $r = \min r_j$ stays (strictly) positive. □

Closed sets have exactly opposite behaviour:

COROLLARY 2.1.12. *Let $F_j \subset \mathbb{C}$, $j \in \mathcal{J}$, be an **arbitrary** collection of closed sets. The index set $\mathcal{J}$ need not be finite, or even countable! Then the intersection $\bigcap_{j\in\mathcal{J}} F_j$ is also closed.*

*Let $C_1, \ldots, C_n \subset \mathbb{C}$ be a **finite** family of closed sets. Then $C_1 \cup \ldots \cup C_n$ is closed.*

PROOF. Exercise. □

2.2. Sequences and limits

DEFINITION 2.2.1 (Limit of a sequence). Let $(z_n)_{n=1}^{\infty}$ be a sequence of complex numbers. We say that $z \in \mathbb{C}$ is the *limit* of the sequence (z_n), denoted $\lim_{n\to\infty} z_n = z$, if for every $\epsilon > 0$ there exists an index $n_0 \in \mathbb{N}$, depending on ϵ, such that

$$|z_n - z| < \epsilon \text{ for all } n \ge n_0.$$

This is equivalent to saying that $z_n \in D(z,\epsilon) \quad$ for all $n \ge n_0$. In such a case we say that (z_n) *converges* to z, and write $z_n \to z$ as $n \to \infty$. If $A \subset \mathbb{C}$, we also (slightly imprecisely) write $(z_n) \subset A$ to denote a sequence such that $z_n \in A$ for all $n \geq 1$.

REMARK 2.2.2. A sequence can have at most one limit: if $z_n \to z$ and $z_n \to w$ for some $z \neq w$, then taking $\epsilon = |z - w|/2 > 0$ would imply that $z_n \in D(z, \epsilon)$ and $z_n \in D(w, \epsilon)$ for all sufficiently large n, which is impossible. There are many sequences that do not have any limit, e.g. $z_n = n$ or $z_n = (-1)^n$.

PROPOSITION 2.2.3. *Let $(z_n)_{n=1}^{\infty}$ be a sequence of complex numbers, and $z \in \mathbb{C}$. Then $z_n \to z$ if and only if*

$$\mathrm{Re}(z_n) \to \mathrm{Re}(z) \quad \textit{and} \quad \mathrm{Im}(z_n) \to \mathrm{Im}(z).$$

PROOF. Write $z_n = x_n + iy_n$ and $z = x + iy$, where $x_n = \mathrm{Re}(z_n)$, $y_n = \mathrm{Im}(z_n)$, and so on. Then, note that by the triangle inequality

$$\max\{|x_n - x|, |y_n - y|\} \leq |z_n - z| \leq |x_n - x| + |y_n - y|.$$

These show that the conditions $|z_n - z| \to 0$ and $|x_n - x| + |y_n - y| \to 0$ are equivalent. $\square$

We will next show that a set of closed precisely when it contains the limits of all of its convergent sequences.

PROPOSITION 2.2.4. *A set $A \subset \mathbb{C}$ is closed if and only if for any sequence $(z_n) \subset A$ that converges to some $z \in \mathbb{C}$, one has $z \in A$.*

PROOF. "$\Longrightarrow$" Let $A \subset \mathbb{C}$ be closed, and let $(z_n) \subset A$ be a sequence that converges to some $z \in \mathbb{C}$. We argue by contradiction and assume that $z \in \mathbb{C} \setminus A$. Since A is closed, $\mathbb{C} \setminus A$ is open, and consequently there is some ball $B(z, r) \subset \mathbb{C} \setminus A$. But since $z_n \to z$, one would have $z_n \in B(z, r)$ for sufficiently large n, which contradicts the assumption that $(z_n) \subset A$.

"$\Longleftarrow$" Suppose that the limit of any convergent sequence $(z_n) \subset A$ is also in A. We need to show that A is closed, which is the same thing as showing that $\mathbb{C} \setminus A$ is open. We argue again by contradiction and assume that there is some $z \in \mathbb{C} \setminus A$ so that for all $n \geq 1$ there is a point $z_n \in D(z, 1/n) \cap A$. We thus obtain a sequence (z_n) converging to z. By assumption we would have $z \in A$, which is a contradiction. $\square$

The next results are related to the very important notion of *compactness*. We will not give a full development here, but we will rather give the minimal facts required later for the proofs of Cauchy's theorem and its applications.

DEFINITION 2.2.5 (Boundedness and compactness). We say that a sequence $(z_n) \subset \mathbb{C}$ is *bounded* if there is $M > 0$ so that

$$|z_n| \leq M \qquad \text{for all } n.$$

A set $A \subset \mathbb{C}$ is *bounded* if there is $M > 0$ such that $|z| \leq M$ for all $z \in A$. A set $A \subset \mathbb{C}$ is *compact* if it is closed and bounded.

There are many equivalent definitions of a compact set, but the one above works in $\mathbb{C}$. Next we recall the important Bolzano-Weierstrass theorem from *Vector analysis 1*, stating that any bounded sequence in $\mathbb{C} = \mathbb{R}^2$ has a convergent subsequence.

THEOREM 2.2.6 (Bolzano-Weierstrass). *If $(z_n)_{n=1}^{\infty} \subset \mathbb{C}$ is a bounded sequence, there is a subsequence $(z_{n_k})_{k=1}^{\infty}$ such that $z_{n_k} \to z$ for some $z \in \mathbb{C}$.*

The Bolzano-Weierstrass theorem yields another equivalent definition of a compact set.

PROPOSITION 2.2.7 (Sequential compactness). *A set $A \subset \mathbb{C}$ is compact if and only if any sequence $(z_n) \subset A$ has a subsequence converging to some point of A.*

PROOF. "$\Longrightarrow$" Let A be compact and $(z_n) \subset A$. Since A is bounded, by Bolzano-Weierstrass some subsequence (z_{n_k}) converges to some $z \in \mathbb{C}$. Since A is also closed, Proposition 2.2.4 shows that $z \in A$.

"$\Longleftarrow$" Suppose $A \subset \mathbb{C}$ is such that any sequence in A has a subsequence converging to some point of A. If $(z_n) \subset A$ converges to some $z \in \mathbb{C}$, some subsequence (z_{n_k}) converges to some $w \in A$, but since (z_{n_k}) also converges to z one must have $z = w \in A$ by Remark 2.2.2. Proposition 2.2.4 then shows that A is closed.

To show that A is bounded, we argue by contradiction and suppose that for any n there is $z_n \in A$ with $|z_n| > n$. By our assumption there is a subsequence with $z_{n_k} \to z$ for some $z \in A$. This in particular implies that (z_{n_k}) is bounded, which contradicts the fact that $|z_{n_k}| > n_k$ where $n_k \to \infty$. $\square$

The proof of Cauchy's theorem will be based on the following consequence of Bolzano-Weierstrass.

THEOREM 2.2.8 (Cantor's intersection theorem). *Let $K_1, K_2, K_3, \ldots$ be closed nonempty subsets of $\mathbb{C}$ such that K_1 is compact and*

$$K_1 \supset K_2 \supset K_3 \supset \ldots.$$

Then

$$\bigcap_{n=1}^{\infty} K_n \neq \emptyset.$$

PROOF. Since the sets K_n are nonempty, we can choose some point $z_n \in K_n$ for any n. The sets K_j are nested, which implies that $(z_n) \subset K_1$. The set K_1 was compact, so Proposition 2.2.7 ensures that there is a subsequence $(z_{n_k})_{k=1}^{\infty}$ converging to some $z \in K_1$.

Now for any $m \geq 1$, the sequence $(z_{n_k})_{k \geq m}$ is contained in K_m and it also converges to z. Since K_m was closed, Proposition 2.2.4 ensures that the limit z is also in K_m. This is true for any $m \geq 1$, so one must have $z \in \cap_{m=1}^{\infty} K_m$. □

2.3. Continuity

Next we discuss the notion of continuity for functions $f : X \to \mathbb{C}$ when $X \subset \mathbb{C}$.

DEFINITION 2.3.1 (Continuity). Let $X \subset \mathbb{C}$ be a set, and let $f\colon X \to \mathbb{C}$ be a map. We say that f is *continuous at* $z \in X$ if for every $\epsilon > 0$ there exists $\delta = \delta(\varepsilon, z) > 0$ such that

$$|f(w) - f(z)| < \epsilon \text{ for all } w \in X \text{ with } |w - z| < \delta.$$

This is equivalent to saying that $f(D(z, \delta) \cap X) \subset D(f(z), \epsilon)$.

If $A \subset X$, and f is continuous at every point $z \in A$, we say that f is *continuous in* A.

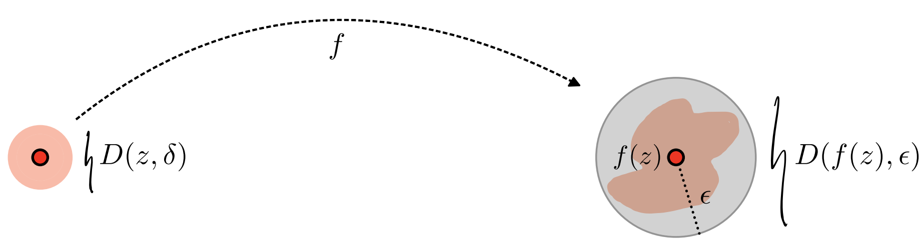


FIGURE 1. The continuity of f at $z \in X$.

WARNING 2.3.2. The continuity of $f\colon X \to \mathbb{C}$ at $z \in X$ (or in $A \subset X$) depends heavily on the domain of definition "X". For example, $\mathrm{Arg}\colon (-\infty, 0) \to \mathbb{R}$ (the restriction of the principal argument to the negative reals) is the constant function, and therefore continuous in $(-\infty, 0)$. However, $\mathrm{Arg}\colon \mathbb{C} \setminus \{0\} \to \mathbb{R}$ is not continuous at any point of $(-\infty, 0)$.

In other words, it is possible that $f\colon A \to \mathbb{C}$ is continuous in A, but $f\colon X \to \mathbb{C}$ is discontinuous at every point of A. The pair $f = \mathrm{Arg}$ and $A = (-\infty, 0)$ is an example.

The continuity of f at $z \in X$ is illustrated in Figure 1. The following fundamental theorem connects the notions of limits and continuity.

THEOREM 2.3.3 (Continuity via sequences). *Let $X \subset \mathbb{C}$ be a set, and let $f\colon X \to \mathbb{C}$ be a map. Then f is continuous at $z \in X$ if and only if for any sequence $(z_n) \subset X$ with $z_n \to z$, one has $f(z_n) \to f(z)$.*

PROOF OF THEOREM 2.3.3. "$\Longrightarrow$" Let us first assume that f is continuous at $z \in X$. Fix a sequence $(z_n) \subset X$ satisfying $z_n \to z$. We need to prove that $f(z_n) \to f(z)$. By Definition 2.2.1, this means demonstrating that for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$f(z_n) \in D(f(z), \epsilon) \quad \text{for all } n \geq n_0. \tag{2.3.1}$$

For this purpose, fix $\epsilon > 0$. Now, by the continuity assumption, there exists $\delta > 0$ such that

$$f(D(z, \delta) \cap X) \subset D(f(z), \epsilon). \tag{2.3.2}$$

On the other hand, since $(z_n) \subset X$ and $z_n \to z$, there exists an index $n_0 \in \mathbb{N}$, depending only on δ, such that $z_n \in D(z, \delta) \cap X$ for all $n \geq n_0$. Therefore, for $n \geq n_0$ we have

$$f(z_n) \in f(D(z, \delta) \cap X) \overset{(2.3.2)}{\subset} D(f(z), \epsilon).$$

This means that (2.3.1) is satisfied for all $n \geq n_0$. This is what we wanted.

"$\Longleftarrow$" Let us then assume that whenever $(z_n) \subset X$ with $z_n \to z$, one has $f(z_n) \to f(z)$. We claim that f is continuous at z. To prove this, we need to fix $\epsilon > 0$ and show that if $\delta > 0$ is small enough, depending only on ϵ and z, then the inclusion (2.3.2) holds. We argue by contradiction and suppose that no choice of $\delta = 1/n$ works for this purpose, for $n \in \mathbb{N}$. In other words

$$f(D(z, \tfrac{1}{n}) \cap X) \not\subset D(f(z), \epsilon), \qquad n \in \mathbb{N}.$$

In still other words, for every $n \in \mathbb{N}$ we may find a point $z_n \in D(z, \frac{1}{n}) \cap X$ such that $|f(z_n) - f(z)| \geq \epsilon$. Now (z_n) is clearly a sequence of points in X with $z_n \to z$. However, it is not possible that $f(z) = \lim_{n\to\infty} f(z_n)$, since $|f(z_n) - f(z)| \geq \epsilon$ for all $n \in \mathbb{N}$. This contradicts our assumption and completes the proof. □

The following proposition is analogous to Proposition 2.2.3:

PROPOSITION 2.3.4. *Let $X \subset \mathbb{C}$ be a set, and let $f\colon X \to \mathbb{C}$ be a function. Then f is continuous at $z \in X$ if and only if the real-valued functions*

$$\mathrm{Re}(f)\colon X \to \mathbb{R} \quad \textit{and} \quad \mathrm{Im}(f)\colon X \to \mathbb{R}$$

are continuous at z. These functions are defined by $(\mathrm{Re}(f))(w) := \mathrm{Re}(f(w))$ and $(\mathrm{Im}(f))(w) := \mathrm{Im}(f(w))$ for all $w \in X$.

PROOF. This can be deduced from Proposition 2.2.3 by using Theorem 2.3.3. We leave the details as an exercise. □

If you have already found a few continuous functions, you can create more continuous functions with (at least) the following operations:

PROPOSITION 2.3.5. *Assume that $f, g\colon X \to \mathbb{C}$ are continuous at $z \in X$, and $h\colon g(X) \to \mathbb{C}$ is continuous at $g(z) \in g(X)$. Then, the following functions are continuous at $z \in X$:*

$$f + g\colon X \to \mathbb{C}, \quad fg\colon X \to \mathbb{C}, \quad \text{and} \quad h \circ g\colon X \to \mathbb{C}.$$

Here $(fg)(w) = f(w)g(w)$ is the complex product of f and g, and $(h\circ g)(w) = h(g(w))$ is the composition of h and g.

If, in addition to the previous hypotheses, $g(z) \neq 0$, then f/g is continuous at $z \in X$, where $(f/g)(w) := f(w)(g(w))^{-1}$.

PROOF. This proof is virtually the same as the proof for a similar result for real-valued functions on $\mathbb{R}$, so we leave the details as a voluntary exercise. It is worth remarking that $(f/g)(w)$ is only well-defined for those $w \in X$ with $g(w) \neq 0$. However, by the continuity of g at z, and since $g(z) \neq 0$, it holds that $g(w) \neq 0$ for $D(z, \epsilon) \cap X$ for some $\epsilon > 0$. Therefore f/g is, at least, defined on $D(z, \epsilon) \cap X$, and continuous at z. □

COROLLARY 2.3.6. *Let $a_0, a_1, \ldots, a_n \in \mathbb{C}$. Then the polynomial*

$$z \mapsto p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0$$

is continuous on $\mathbb{C}$.

PROOF. The function p can be written as sum of (complex) products of continuous functions. Therefore the conclusion follows from Proposition 2.3.5. □

PROPOSITION 2.3.7. *Here are a few more important continuous functions:*

(1) *The functions $z \mapsto \bar{z}$ and $z \mapsto |z|$ are continuous in $\mathbb{C}$.*
(2) *The function $z \mapsto z^{-1}$ is continuous in $\mathbb{C} \setminus \{0\}$.*
(3) *The function $z \mapsto \operatorname{Arg}(z)$ is continuous in $\mathbb{C} \setminus (-\infty, 0]$.*
(4) *If $n \geq 2$, the function $z \mapsto \sqrt[n]{z}$ is continuous in $\mathbb{C} \setminus (-\infty, 0)$.*
(5) *The functions $e^z, \sin z, \cos z$ are continuous in $\mathbb{C}$.*
(6) *The function $\operatorname{Log} z$ is continuous in $\mathbb{C} \setminus (-\infty, 0]$.*

PROOF. Exercise. □

Next we discuss a basic mapping property of continuous functions. If $f : A \to \mathbb{C}$ is continuous and A is closed, or bounded, then $f(A)$ is **not** necessarily closed, or bounded. Examples are given by

- $f_1 : \mathbb{C} \to \mathbb{C}$, $f_1(z) = e^z$, where $\mathbb{C}$ is closed but $f_1(\mathbb{C}) = \mathbb{C} \setminus \{0\}$ is not closed; or

- $f_2 : D(0,1) \setminus \{0\} \to \mathbb{C}$, $f_2(z) = z^{-1}$, where $D(0,1) \setminus \{0\}$ is bounded but $f_2(D(0,1) \setminus \{0\}) = \mathbb{C} \setminus \bar{D}(0,1)$ is not bounded.

However, if A is *compact* then also $f(A)$ is compact.

PROPOSITION 2.3.8. *If $A \subset \mathbb{C}$ is compact and $f : A \to \mathbb{C}$ is continuous, then $f(A)$ is compact.*

PROOF. We will use Proposition 2.2.7. Let (w_n) be a sequence in $f(A)$. Then, for each n, $w_n = f(z_n)$ for some $z_n \in A$. Since A is compact, Proposition 2.2.7 ensures that there is a subsequence (z_{n_k}) converging to some $z \in A$. By Proposition 2.3.3, the sequence $f(z_{n_k})$ converges to $f(z) \in f(A)$. This proves that any sequence $(w_n) \subset f(A)$ has a subsequence converging to a point of $f(A)$, so $f(A)$ is compact by Proposition 2.2.7. □

Since compact sets are bounded, we have the following corollary that will be used later in the proof of the fundamental theorem of algebra (via Liouville's theorem):

COROLLARY 2.3.9. *If $A \subset \mathbb{C}$ is compact and $f : A \to \mathbb{C}$ is continuous, then there is $M > 0$ such that $|f(z)| \le M$ for all $z \in A$.*

Another, slightly stronger corollary will be used when proving the maximum modulus principle.

COROLLARY 2.3.10. *If $A \subset \mathbb{C}$ is compact and $f : A \to \mathbb{C}$ is continuous, then there is $z_0 \in A$ such that*

$$\max_{z \in A} |f(z)| = |f(z_0)|.$$

PROOF. Let $g(z) = |f(z)|$. Then g is continuous, and by Proposition 2.3.8 the set $g(A)$ is a compact subset of $\mathbb{R}$. Thus $g(A)$ is closed and bounded, and therefore $m := \sup g(A)$ is a finite real number. By the definition of supremum there is a sequence $(t_n) \subset g(A)$ with $t_n \to m$, and since $g(A)$ is closed one must have $m \in g(A)$ using Proposition 2.2.4. It follows that there is $z_0 \in A$ with $|f(z_0)| = m$. This proves the result. □

2.4. Connected sets and regions

During this course, we will discuss two important results for analytic functions that require the notion of a *connected set*:

- If $f : U \to \mathbb{C}$ is an analytic function with $f'(z) = 0$ for all $z \in U$, and if $U \subset \mathbb{C}$ is open and connected, then f is constant in U.
- (Maximum modulus principle) If $f : U \to \mathbb{C}$ is an analytic function where $U \subset \mathbb{C}$ is open and connected, and if $|f(z_0)| = \max_{z \in U} |f(z)|$ for some $z_0 \in U$, then f is constant in U.

The following is a general way of defining connectedness: a set $X \subset \mathbb{C}$ is called connected if it **cannot** be decomposed into two pieces as

$$X = (X \cap U) \cup (X \cap V), \tag{2.4.1}$$

where $U, V \subset \mathbb{C}$ are open, disjoint, and $X \cap U \neq \emptyset \neq X \cap V$. If X itself is open, this general notion turns out to be equivalent with another (more intuitive?) notion called *path connectedness*. In these lectures we only need to deal with open connected sets, so we prefer to take path connectedness as our main definition.

For $a, b \in \mathbb{R}$ with $a < b$, we say that a continuous map $\gamma\colon [a,b] \to \mathbb{C}$ is a *path*.

Definition 2.4.1 (Connected set). Assume that $U \subset \mathbb{C}$ is open. Then we say that U is *connected* if for every $z, w \in U$ there exists a path $\gamma\colon [0,1] \to U$ such that

$$\gamma(0) = z \quad \text{and} \quad \gamma(1) = w.$$

In other words, every pair of points $z, w \in U$ can be *connected* by a path in U.

Intuitively, a set is connected if it cannot be decomposed in two separate disjoint pieces. This is illustrated by the following examples.

Example 2.4.2. The sets $\mathbb{C}$, $D(z,r)$, and $\mathbb{C} \setminus \{0\}$ are connected (why)? The sets $D(-2,1) \cup D(2,1)$ or $\mathbb{R} \setminus \{0\}$ are not connected (to see this, consider a path $\gamma(t) = x(t) + iy(t)$ that would connect two points lying in different "components", and derive a contradiction by using the intermediate value theorem for $x(t)$).

Remark 2.4.3. An open and connected set is often called a *region*. So, Definition 2.4.1 is actually the definition of a region.

This exercise explains the connection between different definitions of connectedness:

Exercise 2.1. Let $U \subset \mathbb{C}$ be open and connected in the general sense explained around (2.4.1). Show that U is path connected, i.e. connected in the sense of Definition 2.4.1.

Search for "topologist's sine curve" for a closed set which is connected but not path connected.

We record a useful proposition for future reference.

Proposition 2.4.4. *Let $U \subset \mathbb{C}$ be open and connected. Let $f\colon U \to \mathbb{C}$ be a continuous function which is* locally constant*: for every $z \in U$ there exists $r > 0$ such that f is constant in $D(z,r)$. Then f is constant in U.*

PROOF. Fix $z \in U$. We claim that $f(w) = f(z)$ for all $w \in U$. Fix $w \in U$. Since U is connected, there is a path $\gamma\colon [0,1] \to U$ with $\gamma(0) = z$ and $\gamma(1) = w$. Then $f \circ \gamma\colon [0,1] \to U$ is locally constant. In particular, if $u = \mathrm{Re}(f)$, the function $u(\gamma(t))$ is continuous $[0,1] \to \mathbb{R}$ and locally constant. It follows that $u(\gamma(t))$ is constant on $[0,1]$ (exercise). Similarly, if $v = \mathrm{Im}(f)$, then $v(\gamma(t))$ is constant. Thus $f(w) = (f \circ \gamma)(1) = (f \circ \gamma)(0) = f(z)$, as claimed. $\square$

Summary of Chapter 2. Here is a list of key topics from this chapter:

- Distance, open and closed discs.
- Open and closed sets, their unions and intersections.
- Limits of sequences.
- Compact sets, Bolzano-Weierstrass and Cantor's intersection theorem.
- Continuity of functions, characterisation of continuity via sequences (Theorem 2.3.3).
- Producing new continuous functions from existing ones (via addition, multiplication, inverses, and composition).
- A continuous function on a compact set is bounded.
- Connected open sets (also known as regions).

CHAPTER 3

Analytic functions

The course so far has been preparation to discuss the theory of *analytic functions* on (open subsets of) $\mathbb{C}$. In English literature, these functions are often called *holomorphic*, but in Finland one typically talks about analytic functions. We follow this tradition.

We have already seen some analytic functions, e.g.

$$z^n, \qquad \sqrt[n]{z}, \qquad e^z, \qquad \sin z, \qquad \operatorname{Log} z, \qquad \ldots$$

However, there are many more. Analytic functions f can be characterized in many ways, e.g. the by following equivalent conditions:

- f has a complex derivative.
- $f = u + iv$ satisfies the Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$.
- f solves the equation $\overline{\partial} f = 0$.
- f can be written as a convergent power series $\sum_{n=0}^{\infty} a_n z^n$.
- f is *conformal*, i.e. it preserves (infinitesimally) angles and orientations.

The first three conditions will be discussed more precisely in this chapter. The last two conditions will appear in *Complex analysis 2*.

3.1. Complex derivative

Definition 3.1.1 (Complex derivative). Let $U \subset \mathbb{C}$ be open, and $z \in U$. A function $f : U \to \mathbb{C}$ is *complex differentiable at z* if the limit

$$\lim_{\substack{w \to z \\ w \in U \setminus \{z\}}} \frac{f(w) - f(z)}{w - z} \tag{3.1.1}$$

exists. In this case, we denote

$$f'(z) := \lim_{\substack{w \to z \\ w \in U \setminus \{z\}}} \frac{f(w) - f(z)}{w - z}.$$

This complex number is called the *complex derivative of f at z*.

Remark 3.1.2. The limit

$$\lim_{\substack{w \to z \\ w \in U \setminus \{z\}}} \frac{f(w) - f(z)}{w - z} = f'(z)$$

means that for **any** sequence $(w_n) \subset U \setminus \{z\}$, the quotient $\frac{f(w_n)-f(z)}{w_n-z}$ converges to the same complex number $f'(z)$ which is independent of the sequence (w_n). It is rather cumbersome to write this all the time, so we will abbreviate

$$\lim_{w \to z} \frac{f(w)-f(z)}{w-z} := \lim_{\substack{w \to z \\ w \in U \setminus \{z\}}} \frac{f(w)-f(z)}{w-z}$$

in the sequel.

Remark 3.1.3. On this course, we mostly talk about **complex** differentiable functions and their **complex** derivatives. That is why we will abbreviate our terminology in the sequel, and will simply talk about differentiable functions and their derivatives. But there is a real risk of confusion. You might be familiar with vector-valued functions $f = (u, v)\colon \mathbb{R}^2 \to \mathbb{R}^2$ and their (total) derivative $Df(z)$ which is the matrix

$$Df(z) = \begin{bmatrix} \partial_1 u(z) & \partial_2 u(z) \\ \partial_1 v(z) & \partial_2 v(z) \end{bmatrix}.$$

On first sight, this notion has nothing to do with the complex derivative $f'(z)$! In fact, it has a lot to do with $f'(z)$, and this connection will be fully explained in Section 3.2.

Definition 3.1.4 (Analytic function). Let $U \subset \mathbb{C}$ be open, and let $f\colon U \to \mathbb{C}$ be a function which is complex differentiable at every point in U. Then f is called *analytic* in U.

3.1.1. First properties and some counterexamples. As a general heuristic principle, everything you know about the differentiability of real-valued functions on $\mathbb{R}$ will also work for complex differentiable functions on $\mathbb{C}$, but the converse is far from true! Complex differentiability is an incredibly strong assumption. This is due to the fact that the limit in (3.1.2) must exist for **any** sequence in $U \setminus \{z\}$ (since $\mathbb{C}$ is two-dimensional there are many different such sequences) and the limit must be **independent** of the choice of sequence. Note also that the limit involves complex multiplication (i.e. division by $w - z$).

For example, it will turn out that complex analytic functions are automatically **infinitely** differentiable, which is of course not the case for differentiable functions on $\mathbb{R}$.

Proposition 3.1.5 (Analytic functions are continuous). *Assume that $U \subset \mathbb{C}$ is open, and $f\colon U \to \mathbb{C}$ is differentiable at $z \in U$. Then f is continuous at z.*

PROOF. The proof is the same as for differentiable functions on $\mathbb{R}$. Note that if $w \neq z$, then

$$|f(w) - f(z)| = |w - z|\frac{|f(w) - f(z)|}{|w - z|}.$$

As $w \to z$, the factor $|f(w) - f(z)|/|w - z|$ tends to $|f'(z)| \in \mathbb{R}$, and of course $|w - z| \to 0$. Therefore $|f(w) - f(z)| \to 0$ as $w \to z$, and this implies the continuity of f at z. $\square$

Here is a useful (and probably familiar) characterisation of differentiability:

PROPOSITION 3.1.6. *Assume that $U \subset \mathbb{C}$ is open, $f\colon U \to \mathbb{C}$, and $z \in U$.*

- *Assume that f is differentiable at z. Then there exists a function $\epsilon\colon U \to \mathbb{C}$ such that $\epsilon(w) \to 0$ as $w \to z$, and*

$$f(w) - f(z) = f'(z)(w - z) + \epsilon(w)(w - z), \qquad w \in U. \tag{3.1.2}$$

- *Assume that there exists a number $\alpha \in \mathbb{C}$, and a function $\epsilon\colon U \to \mathbb{C}$ such that $\epsilon(w) \to 0$ as $w \to z$, and*

$$f(w) - f(z) = \alpha(w - z) + \epsilon(w)(w - z), \qquad w \in U \setminus \{z\}. \tag{3.1.3}$$

  Then f is differentiable at z, and $f'(z) = \alpha$.

PROOF. Assume first that f is differentiable at z, and set $\alpha := f'(z)$. Then, define

$$\epsilon(w) := \begin{cases} \frac{f(w)-f(z)}{w-z} - \alpha, & w \in U \setminus \{z\}, \\ 0, & w = z. \end{cases}$$

Now it is easy to check that $\epsilon(w) \to 0$ as $w \to z$, and clearly (3.1.2) holds.

Conversely, if (3.1.3) holds, then

$$\left|\frac{f(w) - f(z)}{w - z} - \alpha\right| = |\epsilon(w)|, \qquad w \in U \setminus \{z\}.$$

Since $\epsilon(w) \to 0$ as $w \to z$, this shows that f is differentiable at z, and $f'(z) = \alpha$. $\square$

PROPOSITION 3.1.7 (Differentiation rules). *Let $U \subset \mathbb{C}$ be open, and let $f, g\colon U \to \mathbb{C}$ be differentiable at $z \in U$. Let also $\lambda \in \mathbb{C}$. Then the functions λf, $f + g$, and fg are all differentiable at z. If $g(z) \neq 0$, also the ratio f/g is differentiable at z.*

Moreover, the derivatives have the following explicit expressions:

$$(\lambda f)'(z) = \lambda f'(z), \quad (f+g)'(z) = f'(z)+g'(z), \quad (fg)'(z) = f'(z)g(z)+f(z)g'(z),$$

and

$$\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{(g(z))^2} \quad \textit{(assuming } g(z) \neq 0\textit{)}.$$

PROOF. The proofs are exactly the same as in the case of differentiable functions on $\mathbb{R}$. $\square$

COROLLARY 3.1.8. *Let $n \in \{1, 2, \ldots\}$. Then the function $p_n(z) = z^n$ is analytic in $\mathbb{C}$, and the derivative is $p_n'(z) = nz^{n-1}$.*

PROOF. First consider the case $n = 1$. Then,

$$p_1'(z) = \lim_{w \to z} \frac{w - z}{w - z} = 1 = 1 \cdot z^0, \qquad z \in \mathbb{C},$$

as desired. The cases $n \geq 2$ can be established by induction and using Proposition 3.1.7:

$$\begin{aligned} p_n'(z) = (p_{n-1}p)'(z) &= p_{n-1}'(z)p_1(z) + p_{n-1}(z)p_1'(z) \\ &= (n-1)z^{n-2}z + z^{n-1} \cdot 1 = nz^{n-1}. \end{aligned}$$

This completes the proof. $\square$

REMARK 3.1.9. As a consequence of the previous corollary, every polynomial function $p(z) = a_n z^n + \ldots + a_1 z + a_0$ is analytic in $\mathbb{C}$ and its derivative is $p'(z) = na_n z^{n-1} + \ldots + 2a_2 z + a_1$.

Functions like $\sqrt[n]{z}$, e^z, $\operatorname{Log} z$ are also analytic in the right domain of definition. This will be proved a bit later, after we have developed some theory that will make the proofs easier.

Here comes the first indication that complex differentiability is something very special:

EXAMPLE 3.1.10. The function $f(z) = \bar{z}$ is **not** complex differentiable at any point $z \in \mathbb{C}$. This will be an exercise. This example is remarkable, because when expressed in coordinates, the map $z \mapsto \bar{z}$ is just $(x, y) \mapsto (x, -y)$. The coordinate functions are $u(x, y) = x$ and $v(x, y) = -y$, so in particular the map is infinitely differentiable as a map $\mathbb{R}^2 \to \mathbb{R}^2$.

Another dramatic example is the function $z \mapsto |z|^2 = x^2 + y^2$. If this function was complex differentiable at a point $z \neq 0$, then the same would also be true for $z \mapsto \bar{z} = |z|^2/z$. But we know that $z \mapsto \bar{z}$ is not differentiable anywhere, so $z \mapsto |z|^2$ also cannot be differentiable outside $z = 0$.

REMARK 3.1.11. We have now seen that all polynomials of the variable "z" are analytic on $\mathbb{C}$, but even the simplest polynomial of the variable "$\bar{z}$" fails to be analytic even at a single point. This is an indication of a general principle: every analytic function can be (locally) expressed as a convergent power series of the form $f(z) = \sum_{n \geq 0} a_n z^n$. This fact will be established on *Complex Analysis 2*.

3.1.2. Derivatives of composed and inverse functions. We continue with more results which (hopefully) look familiar from the theory of real-valued functions on $\mathbb{R}$.

THEOREM 3.1.12 (Chain rule). *Let $U, V \subset \mathbb{C}$ be open sets, and let $z \in U$. Assume that $f : U \to \mathbb{C}$ is a function with the properties that f is differentiable at z, and $f(U) \subset V$. Assume further $g : V \to \mathbb{C}$ is differentiable at $f(z)$.*

Then the composed map $g \circ f : U \to \mathbb{C}$ is differentiable at z, and

$$(g \circ f)'(z) = g'(f(z))f'(z).$$

PROOF. The following argument is easy to remember, but not entirely accurate:

$$\lim_{w \to z} \frac{(g \circ f)(w) - (g \circ f)(z)}{w - z} = \lim_{w \to z} \frac{g(f(w)) - g(f(z))}{f(w) - f(z)} \frac{f(w) - f(z)}{w - z} = g'(f(z))f'(z).$$

The problem here is that it may happen that $f(w) = f(z)$ (even if $w \neq z$), and then we are multiplying and dividing by 0. The rigorous argument uses Proposition 3.1.6, which allows us to express both $f(w)$ and $g(f(w))$ in the form

$$\begin{cases} f(w) - f(z) = (f'(z) + \epsilon_f(w))(w - z), \\ g(f(w)) - g(f(z)) = [g'(f(z)) + \epsilon_g(f(w))](f(w) - f(z)) \end{cases}$$

where $\epsilon_f(w) \to 0$ as $v \to z$ and $\epsilon_g(v) \to 0$ as $v \to f(z)$. Then, for $w \neq z$, we have

$$\begin{aligned} \frac{g(f(w)) - g(f(z))}{w - z} &= \frac{[g'(f(z)) + \epsilon_g(f(w))](f(w) - f(z))}{w - z} \\ &= [g'(f(z)) + \epsilon_g(f(w))][f'(z) + \epsilon_f(w)]. \end{aligned} \tag{3.1.4}$$

Here $\epsilon_f(w) \to 0$ as $w \to 0$ by definition, but also $\epsilon_g(f(w)) \to 0$ as $w \to z$, since $f(w) \to f(z)$ by the continuity of f at z (Proposition 3.1.5). Therefore the expression on line (3.1.4) tends to $g'(f(z))f'(z)$ as $w \to z$, as desired. □

We next discuss inverse functions.

PROPOSITION 3.1.13 (Derivative of inverse function). *Let $U \subset \mathbb{C}$ be open, and let $f : U \to \mathbb{C}$ be a map (not necessarily injective). Further, assume that $V \subset \mathbb{C}$ is open, and $g : V \to U$ is a map which is continuous at a point $w \in V$ and satisfies*

$$f(g(v)) = v \text{ for all } v \in V. \tag{3.1.5}$$

Then $f|_{g(V)} : g(V) \to V$ is bijective. Moreover, if f is differentiable at $g(w)$, and $f'(g(w)) \neq 0$, then g is differentiable at w, and

$$g'(w) = \frac{1}{f'(g(w))}.$$

REMARK 3.1.14. It is not assumed in Proposition 3.1.13 that f is injective, and the result will indeed be applied to situations where f is not injective on U – one example will be $f(z) = z^2$ defined on $\mathbb{C}$, see Section 3.1.3. We also note that the result is true without the assumption $f'(g(w)) \neq 0$, see [**Pa90**, Theorem VIII.1.8]. However, proving this requires results which go beyond this course.

PROOF OF PROPOSITION 3.1.13. We first show that $f|_{g(V)}: g(V) \to V$ is a bijection. Indeed, (3.1.5) says that $f \circ g = \mathrm{Id}_V$, so it suffices to show that $g \circ f|_{g(V)} = \mathrm{Id}_{g(V)}$. To see this, fix $g(w) \in g(V)$ (with $w \in V$), and note that

$$g(f(g(w))) \overset{(3.1.5)}{=} g(w).$$

Thus $g \circ f|_{g(V)} = \mathrm{Id}_{g(V)}$, as claimed.

Next assume that f is differentiable at $g(w)$, and $f'(g(w)) \neq 0$. We need to show that

$$\lim_{\substack{v \to w \\ v \in V \setminus \{w\}}} \frac{g(v) - g(w)}{v - w} = \frac{1}{f'(g(w))}. \tag{3.1.6}$$

To prove this, fix $v \in V \setminus \{w\}$, and note that automatically $g(w) \neq g(v)$, because otherwise $w = f(g(w)) = f(g(v)) = v$ by (3.1.5). This observation allows us to modify the left hand side of (3.1.6):

$$\frac{g(v) - g(w)}{v - w} = \frac{g(v) - g(w)}{f(g(v)) - f(g(w))} = \left(\frac{f(g(v)) - f(g(w))}{g(v) - g(w)} \right)^{-1}. \tag{3.1.7}$$

Now, as $v \to w$, we have $g(v) \to g(w)$ by the assumed continuity of g at w. Therefore

$$\lim_{v \to w} \frac{f(g(v)) - f(g(w))}{g(v) - g(w)} = f'(g(w)) \neq 0,$$

which can be plugged back into (3.1.7) to obtain (3.1.6). □

3.1.3. Analytic branches of inverse functions. Proposition 3.1.13 gives a method for checking that the inverse of an analytic function is also analytic. However, many analytic functions are not injective in their "natural" domain of definition, so they do not have global inverses. A fundamental example is $z \mapsto z^n$. Recall from Section 1.4 that the equation $z^n = w$ always has n distinct solutions of the form $z = \sqrt[n]{|w|} e^{i(\mathrm{Arg}(w) + 2\pi k)/n}$ for $k = 0, 1, \ldots, n-1$.

This issue is simple to fix: we just need to restrict $z \mapsto z^n$ to some smaller (open) set $U \subset \mathbb{C}$ where it is injective. Then, writing $V := \{z^n : z \in U\}$, we can define an inverse $g: V \to U$. The only problem is that now g depends on the choice of U, and different choices of U give rise to different inverses. The various choices are called *branches of the inverse*. See also our previous discussion in Remark 1.3.15. We formalise this in the following (slightly non-obvious!) way:

Definition 3.1.15 (Branch of f^{-1}). Let $U \subset \mathbb{C}$ be a set, and let $f\colon U \to \mathbb{C}$ be a map (not necessarily injective). Assume that $V \subset f(U)$ is another set, and $g\colon V \to U$ is a map which is continuous in V and satisfies

$$f(g(v)) = v, \qquad v \in V. \tag{3.1.8}$$

Then g is called a *branch of f^{-1} in V*.

Let us clarify the definition with an example.

Example 3.1.16. Fix $n \geq 2$, and recall the principal n^{th} square root function $g_n = \sqrt[n]{\cdot}\colon \mathbb{C} \to \mathbb{C}$ defined by

$$g_n(w) := \sqrt[n]{w} := \begin{cases} 0, & w = 0, \\ \sqrt[n]{|w|}e^{i\mathrm{Arg}(w)/n}, & w \neq 0. \end{cases}$$

Let $f_n(z) := z^n$. Now

$$f_n(g_n(v)) = (\sqrt[n]{v})^n = v, \qquad v \in \mathbb{C},$$

so the formula (3.1.8) holds for all $v \in \mathbb{C}$. However, g_n is **not** a branch of f_n^{-1} in $\mathbb{C}$, because g_n is not continuous in $\mathbb{C}$. However, g_n is continuous in $\mathbb{C} \setminus (-\infty, 0]$, and therefore a branch of f_n^{-1} in $V := \mathbb{C} \setminus (-\infty, 0]$.

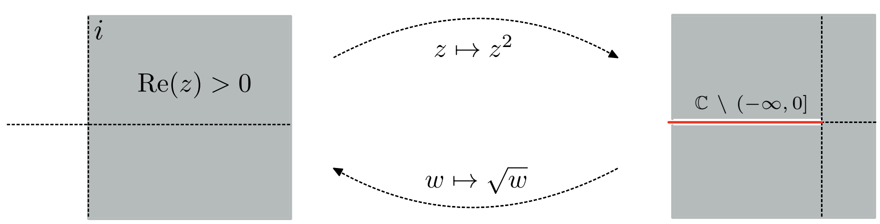


Figure 1. The mapping properties of $z \mapsto z^2$ and $w \mapsto \sqrt{w}$.

What does $g_n(V)$ look like? Figure 1 shows the image of the principal square root $\sqrt{\cdot}$ restricted to $V = \mathbb{C} \setminus (-\infty, 0]$: the image is the half-plane $\{\mathrm{Re}(z) > 0\}$. More generally, the image of $\sqrt[n]{\cdot}$ of the domain V is the sector $S_n = \{re^{i\theta} : r > 0 \text{ and } -\frac{\pi}{n} < \theta < \frac{\pi}{n}\}$.

The following result is a restatement of Proposition 3.1.13. Again, the result is true without the assumption $f'(g(w)) \neq 0$ for $w \in V$, but this goes beyond this course:

Theorem 3.1.17 (Analyticity of branches of the inverse). *Let $U \subset \mathbb{C}$ be open and $f\colon U \to \mathbb{C}$ analytic, let $V \subset f(U)$, and let $g\colon V \to U$ be a branch* *of f^{-1} in V. If additionally $f'(g(w)) \neq 0$ for all $w \in V$, then g is analytic in V and*

$$g'(w) = \frac{1}{f'(g(w))}, \qquad w \in V. \tag{3.1.9}$$

The following corollary is immediate:

Corollary 3.1.18 (Analyticity of principal roots). *The principal n^{th} root*

$$\sqrt[n]{\cdot}\colon \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$$

is analytic in $\mathbb{C} \setminus (-\infty, 0]$, and the image of $\mathbb{C} \setminus (-\infty, 0]$ is the sector $S_n = \{re^{i\theta} : r > 0 \text{ and } -\frac{\pi}{n} < \theta < \frac{\pi}{n}\}$. The derivative is given by

$$(\sqrt[n]{\cdot})'(w) = \frac{1}{n(\sqrt[n]{w})^{n-1}}, \qquad w \in \mathbb{C} \setminus (0, \infty]. \tag{3.1.10}$$

Proof. We observed in Example 3.1.16 that $g_n(w) := \sqrt[n]{w}$ is a branch of $f_n(z) = z^n$ in $\mathbb{C} \setminus (-\infty, 0]$. Since $\sqrt[n]{w} \neq 0$ for $w \neq 0$, we also verify that

$$f_n'(g_n(w)) = ng_n(w)^{n-1} \neq 0, \qquad w \in C \setminus (-\infty, 0].$$

Therefore g_n is analytic in $\mathbb{C} \setminus (-\infty, 0]$ by Theorem 3.1.17. Recalling that the derivative of $z \mapsto z^n$ is nz^{n-1}, the formula (3.1.10) follows from (3.1.9). □

Remark 3.1.19. It is a common question in complex analysis to ask: "Does the square root have a branch in an open set $V \subset \mathbb{C}$?" or "Does the logarithm have a branch in an open set $V \subset \mathbb{C}$?" When you see these questions, think of Definition 3.1.15 – this is precisely what the questions mean!

For example, the "square root has a branch in U" if there exists a continuous map $g\colon V \to \mathbb{C}$ such that $g(w)^2 = w$ for all $w \in V$. The analyticity of g follows from Theorem 3.1.17 if one can verify that $f'(g(w)) \neq 0$ for $w \in V$.

3.2. Cauchy-Riemann equations

We investigate the following questions:

- What is the difference between complex differentiability and the "usual" or "real" differentiability of maps $f\colon \mathbb{R}^2 \to \mathbb{R}^2$?
- Given a "real" differentiable function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, is there a "test" to see whether it is also complex differentiable, i.e. analytic?
- How to compute $f'(z)$ (if it exists) starting from the real derivative $Df(z)$?

Before proceeding, let us recap from *Vector analysis 1* the definition of real differentiability for maps $\mathbb{R}^m \to \mathbb{R}^n$.

DEFINITION 3.2.1 (Real differentiability). Let $U \subset \mathbb{R}^m$ be open, and let $f\colon U \to \mathbb{R}^n$ be a map. Then f is called *real differentiable* at a point $p \in U$ if there exists a linear map $L\colon \mathbb{R}^m \to \mathbb{R}^n$ (called the *derivative of f at p* and often denoted $L = Df(p)$) such that

$$f(q) - f(p) = L(q-p) + \epsilon(q)|q-p|, \tag{3.2.1}$$

Here $\epsilon = \epsilon_p\colon U \to \mathbb{R}^n$ is a map which satisfies $\epsilon(q) \to 0$ as $q \to p$.

We will only need the above definition when $m = 2$ and $n \in \{1, 2\}$. If $U \subset \mathbb{C} = \mathbb{R}^2$ and $u : U \to \mathbb{R}$ is differentiable, it is customary to denote the partial derivatives of $u(x, y)$ by

$$u_x(x,y) = \frac{\partial u}{\partial x}(x,y), \qquad u_y(x,y) = \frac{\partial u}{\partial y}(x,y).$$

We also recall the following proposition, which explains how the linear map L can be expressed as a matrix of partial derivatives:

PROPOSITION 3.2.2. *Let $U \subset \mathbb{R}^2$ be open, and let $f = (u, v)\colon U \to \mathbb{R}^2$ be a map. Assume that f is differentiable at $z \in U$. Then both u and v are also differentiable at z, their partial derivatives exist at z, and the linear map L can be represented in terms of the partial derivatives as follows:*

$$L(x,y) = \begin{bmatrix} u_x(z) & u_y(z) \\ v_x(z) & v_y(z) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \qquad (x,y) \in \mathbb{R}^2.$$

We then return to complex differentiability. The formula (3.2.1) looks very similar to the characterisation of complex differentiability we have seen in Proposition 3.1.6: as a reminder, that formula looked like

$$f(w) - f(z) = f'(z)(w-z) + \epsilon(w)(w-z), \tag{3.2.2}$$

where $\epsilon\colon U \to \mathbb{C}$ and $\epsilon(w) \to 0$ as $w \to z$. If desired, this can be brought precisely to the form (3.2.1) by defining $\tilde{\epsilon}(w) = (w-v)\epsilon(w)/|w-z|$: then $\tilde{\epsilon}\colon U \to \mathbb{C}$, it still holds that $\tilde{\epsilon}(w) \to 0$ as $w \to z$, and (3.2.2) can be rewritten as

$$f(w) - f(z) = f'(z)(w-z) + \tilde{\epsilon}(w)|w-z|. \tag{3.2.3}$$

The bigger difference between (3.2.1) and (3.2.2) (or (3.2.3)) is that in (3.2.2) the linear map L has a special form, namely $L(\zeta) = f'(z)\zeta$. To understand this further, let us show that $\zeta \mapsto f'(z)\zeta$ can be written as a matrix multiplication.

EXAMPLE 3.2.3. Let $z \in \mathbb{C}$. Then the map $w \mapsto zw$ is linear as a map $\mathbb{R}^2 \to \mathbb{R}^2$. Indeed, if we write $z = a + ib$ and $w = x + iy$, then

$$zw = M_z(x,y) \quad \text{where} \quad M_z := M_{a+ib} := \begin{bmatrix} a & -b \\ b & a \end{bmatrix}. \tag{3.2.4}$$

Thus, complex multiplication by z defines a (real) linear map $M_z\colon \mathbb{R}^2 \to \mathbb{R}^2$.

The following corollary is worth stating separately:

COROLLARY 3.2.4. *Let $U \subset \mathbb{C}$ be open, and let $f = (u, v)\colon U \to \mathbb{C}$ be complex differentiable at $z \in U$. Then f is also real differentiable at z, as a map $U \to \mathbb{R}^2$.*

PROOF. This follows immediately from the formula (3.2.2), and the fact that $L(\zeta) := f'(z)\zeta$ defines a (real) linear map $\mathbb{R}^2 \to \mathbb{R}^2$. $\square$

Now we can put some pieces together. If $f = (u, v) = u + iv$ is complex differentiable at $z \in U$, then f is also real differentiable, and its derivative is the linear map given by $L(\zeta) = f'(z)\zeta$. Here $f'(z) = a + ib$ for some $a, b \in \mathbb{R}$. On the other hand, Proposition 3.2.2 says that the linear map L can be also written in terms of the partial derivatives of u and v. Thus, we get the following equation between matrices:

$$\begin{bmatrix} u_x(z) & u_y(z) \\ v_x(z) & v_y(z) \end{bmatrix} = L = M_{f'(z)} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}. \tag{3.2.5}$$

Two matrices only agree if their components agree, so we can read off the following *Cauchy-Riemann equations*:

$$\begin{cases} u_x(z) = v_y(z), \\ u_y(z) = -v_x(z). \end{cases} \tag{3.2.6}$$

We have arrived at the following theorem:

THEOREM 3.2.5 (Cauchy-Riemann equations: necessity). *Let $U \subset \mathbb{C}$ be open, and assume that $f = u+iv\colon U \to \mathbb{C}$ is complex differentiable at $z \in U$. Then the real and imaginary parts $u, v\colon U \to \mathbb{R}$ are real differentiable at z, and their partial derivatives satisfy the Cauchy-Riemann equations (3.2.6).*

Theorem 3.2.5 says that the equations (3.2.6) are necessarily satisfied for a complex differentiable function $f = u + iv$. In fact, the equations are also sufficient, so they give a useful "test" for complex differentiability:

THEOREM 3.2.6 (Cauchy-Riemann equations: sufficiency). *Let $U \subset \mathbb{R}^2$ be open, and let $f = (u, v)\colon U \to \mathbb{R}^2$ be real differentiable at a point $z \in U$. Assume that the component functions u, v satisfy the Cauchy-Riemann equations (3.2.6). Then f is complex differentiable at z.*

PROOF. Since f is real differentiable at z, Proposition 3.2.2 tells us that the derivative $L = Df(z)$ is represented by the matrix

$$L = \begin{bmatrix} u_x(z) & u_y(z) \\ v_x(z) & v_y(z) \end{bmatrix}.$$

Further, applying the Cauchy-Riemann equations we find that

$$L = \begin{bmatrix} u_x(z) & u_y(z) \\ v_x(z) & v_y(z) \end{bmatrix} =: \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = M_{a+ib},$$

where $a = u_x(z)$ and $b = v_x(z)$, and we use the M_{a+ib}-notation familiar from (3.2.4). Now, by the definition of real differentiability (Definition 3.2.1) and (3.2.4) we deduce that

$$\begin{aligned} f(w) - f(z) &\overset{\text{def.}}{=} M_{a+ib}(w-z) + \epsilon(w)|w-z| \\ &\overset{(3.2.4)}{=} (a+ib)(w-z) + \tilde{\epsilon}(w)(w-z), \end{aligned}$$

where $\epsilon(w) \to 0$ as $w \to z$, and $\tilde{\epsilon}(w) := |w-z|\epsilon(w)/(w-z)$ has the same properties. But the final equation is exactly the same as our characterisation of complex differentiability, recall either (3.2.2) or Proposition 3.1.6. Thus f is complex differentiable at z, and

$$f'(z) = a + ib = u_x(z) + iv_x(z). \tag{3.2.7}$$

This completes the proof. $\square$

The formula (3.2.7) gives an explicit way of relating $f'(z)$ to the partial derivatives of u and v. This is worth recording separately:

Corollary 3.2.7. *Let $U \subset \mathbb{C}$ be open, and assume that $f = u + iv$ is complex differentiable at $z \in U$. Then,*

$$f'(z) = u_x(z) + iv_x(z) = v_y(z) - iu_y(z).$$

Proof. The first equation is (3.2.7), and the second equation is a restatement of the Cauchy-Riemann equations (3.2.6). $\square$

The next result shows how one can use the Cauchy-Riemann equations to verify analyticity and compute complex derivatives: everything reduces to computing real derivatives.

Theorem 3.2.8 (e^z and $\operatorname{Log} z$ are analytic). *The functions $f : \mathbb{C} \to \mathbb{C}$, $f(z) = e^z$ and $g : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$, $g(z) = \operatorname{Log} z$ are analytic. Their derivatives are given by*

$$\begin{aligned} (e^z)' &= e^z, & z &\in \mathbb{C}, \\ (\operatorname{Log} z)' &= \tfrac{1}{z}, & z &\in \mathbb{C} \setminus (-\infty, 0]. \end{aligned}$$

Proof. If $z = x + iy$, the definition $e^z = e^x(\cos y + i \sin y)$ gives that the real and imaginary parts of $f(z)$ are

$$u(x,y) = e^x \cos y, \qquad v(x,y) = e^x \sin y.$$

We can check that the Cauchy-Riemann equations hold by taking real derivatives:

$$u_x = e^x \cos y = v_y,$$
$$u_y = -e^x \sin y = -v_x.$$

Thus f is analytic by Proposition 3.2.6. Corollary 3.2.7 shows that

$$f'(z) = u_x + iv_x = e^x(\cos y + i \sin y) = e^z.$$

Let $\mathcal{S} := \{ -\pi < \operatorname{Im}(z) < \pi\}$. By Remark 1.6.3 the map

$$g : \mathbb{C} \setminus (-\infty, 0] \to \mathcal{S},\ g(w) = \operatorname{Log} w \tag{3.2.8}$$

is the inverse of $f : \mathcal{S} \to \mathbb{C} \setminus (-\infty, 0]$, $f(z) = e^z$. The map g is also continuous, and one has

$$f'(g(w)) = e^{g(w)} \neq 0, \qquad w \in \mathbb{C} \setminus (-\infty, 0].$$

Since e^z is analytic, it follows from Theorem 3.1.17 that Log is analytic on $\mathbb{C} \setminus (-\infty, 0]$ and that

$$(\operatorname{Log} z)' = \frac{1}{z}, \qquad z \in \mathbb{C} \setminus (-\infty, 0].$$ □

We next show that one can rewrite the Cauchy-Riemann equations more compactly, in terms of the *Wirtinger derivatives*

$$\overline{\partial} f(z) := \frac{1}{2}(f_x + if_y),$$
$$\partial f(z) := \frac{1}{2}(f_x - if_y).$$

Here, if $f = u + iv$, we write $f_x = u_x + iv_x$.

PROPOSITION 3.2.9. *Let $U \subset \mathbb{R}^2$ be open, and let $f = (u, v)\colon U \to \mathbb{R}^2$ be real differentiable at a point $z \in U$. Then f is complex differentiable at z if and only if*

$$\overline{\partial} f(z) = 0.$$

Moreover, if this holds, then $f'(z) = \partial f(z)$.

PROOF. We observe that

$$2\overline{\partial} f(z) = f_x + if_y = u_x + iv_x + i(u_y + iv_y) = u_x - v_y + i(u_y + v_x).$$

Thus $\overline{\partial} f(z) = 0$ if and only if the Cauchy-Riemann equations (3.2.6) hold. The first part of the result then follows from Theorem 3.2.6. Moreover, if this holds, then

$$\partial f(z) = \frac{1}{2}(f_x - if_y) = \frac{1}{2}(u_x + iv_x - i(u_y + iv_y)) = \frac{1}{2}(u_x + v_y + i(v_x - u_y)) = u_x + iv_x.$$

Corollary 3.2.7 gives $f'(z) = \partial f(z)$. □

WARNING 3.2.10. In Theorem 3.2.6 we assumed that the map $f\colon U \to \mathbb{R}^2$ is real differentiable at $z \in U$. This hypothesis **cannot** be weakened to *both partial derivatives of u, v exist at $z \in U$ and satisfy the Cauchy-Riemann equations* (3.2.6). Indeed, this weaker hypothesis does not even guarantee that f is continuous z! For example, consider $f\colon \mathbb{C} \to \mathbb{C}$ which satisfies

$$f(r, 0) = (0, 0) = f(0, s), \qquad r, s \in \mathbb{R},$$

or in other words f vanishes on the real and imaginary axes $\mathbb{R}$ and $i\mathbb{R}$. Then

$$\partial_1 u(0) = \partial_2 u(0) = \partial_1 v(0) = \partial_2 v(0) = 0,$$

no matter how we define f outside the "cross" $\mathbb{R} \cup i\mathbb{R}$. Now, we can for example define $f \equiv (10^{10}, 10^{10})$ outside $\mathbb{R} \cup i\mathbb{R}$. Then f is not continuous at 0, even though it satisfies the Cauchy-Riemann equations at 0.

The previous warning demonstrates that the existence of partial derivatives at a single point implies basically nothing (see Example 3.2.12 for an even scarier situation). However, the following result from *Vector Analysis 1* says that assuming a bit more saves the day.

THEOREM 3.2.11. *Let $U \subset \mathbb{R}^2$ be open, let $f = (u, v)\colon U \to \mathbb{R}^2$ be a map, and let $z \in U$. Assume that the partial derivatives u_x, u_y, v_x, v_y exist everywhere on U and are continuous at z. Then f is (real) differentiable at z. If, additionally, the Cauchy-Riemann equations* (3.2.6) *are satisfied, then f is also complex differentiable at z.*

These results still leave open the question: what if the partial derivatives exist everywhere on U and satisfy the Cauchy-Riemann equations, but are **not** assumed to be continuous? The following example shows that even in this case f need not be continuous, let alone differentiable:

EXAMPLE 3.2.12. Consider the map

$$f(z) = \begin{cases} \exp(-z^{-4}), & z \neq 0, \\ 0, & z = 0. \end{cases}$$

The complex exponential e^z was defined in Definition 1.5.1. Then f has the following properties:

- f is analytic in $\mathbb{C} \setminus \{0\}$.
- The real and imaginary parts of f have partial derivatives everywhere, and they satisfy the Cauchy-Riemann equations everywhere – even at the origin!
- f is **not** continuous at 0.

The analyticity of f in $\mathbb{C} \setminus \{0\}$ follows from the analyticity of $z \mapsto e^z$ and $z \mapsto -z^{-4}$ in $\mathbb{C} \setminus \{0\}$. To study continuity at 0, note that for $r \in \mathbb{R}$, we have

$$f(r) = e^{-r^{-4}} \to 0 \quad \text{and} \quad f(ir) = e^{-(ir)^{-4}} = e^{-r^{-4}} \to 0$$

as $r \to 0$. So, f decays to zero along both $\mathbb{R}$ and $i\mathbb{R}$. In fact, the decay is so fast that the partial derivatives of f exist and equal "0" at the origin (in particular the Cauchy-Riemann equations are valid at "0"). On the other hand, f has very different behaviour along the line $\operatorname{span}(e^{-i\pi/4})$:

$$f(re^{-i\pi/4}) = \exp(-(re^{-i\pi/4})^{-4}) = \exp(-r^{-4}e^{i\pi}) = \exp(r^{-4}) \xrightarrow{r\to 0} \infty.$$

Thus f "blows up" when approaching the origin along the line $\operatorname{span}(e^{-i\pi/4})$, and is certainly not continuous at the origin.

3.3. A few applications

We record a few easy consequences of the Cauchy-Riemann equations. Recall the definition of connected open sets from Definition 2.4.1.

THEOREM 3.3.1. *Assume that $U \subset \mathbb{R}^2$ is open and connected, and $f\colon U \to \mathbb{C}$ is an analytic function such that $f'(z) = 0$ for all $z \in U$. Then f is constant on U.*

PROOF. Let $f = u + iv$. From Corollary 3.2.7 we see that the partial derivatives of u and v vanish identically U. Therefore the gradients $\nabla u = (u_x, u_y)$ and $\nabla v = (v_x, v_y)$ vanish identically on U. This is all the information we will need, and after this point, the proof has nothing to do with complex analysis.

To show that f is constant, it suffices to show that both u and v are constant. Let us concentrate on u, for example. If we manage to show that u is locally constant, then it follows from Proposition 2.4.4 and the connectedness of U that u is constant on U. So, let us show that u is locally constant.

Fix $z \in U$. Since U is open, there exists a radius $r > 0$ such that $D := D(z, r) \subset U$. We will show that $u|_D \equiv u(z)$. If $w \in D$, the line segment between z and w is contained in $D \subset U$. This segment can be parametrised by the path $\gamma_w(t) = tw + (1 - t)z$, for $t \in [0, 1]$. Now, the composition $u \circ \gamma_w \colon [0, 1] \to \mathbb{R}$ is (real) differentiable, and by the chain rule in several variables (see *Vector analysis 1*)

$$(u \circ \gamma_w)'(t) = \nabla u(\gamma_w(t)) \cdot \gamma_w'(t) = 0, \qquad t \in [0, 1].$$

Therefore $u \circ \gamma_w$ is constant, and $u(z) = u(\gamma_w(0)) = u(\gamma_w(1)) = u(w)$. Since $w \in D$ was arbitrary, we see that $u|_D \equiv u(z)$, as desired. $\square$

We close the section with a closely related result that will be needed in the proof of the maximum modulus principle.

THEOREM 3.3.2. *Let $U \subset \mathbb{C}$ be open and connected, and assume that $f\colon U \to \mathbb{C}$ is analytic. Assume that one of the following three functions is constant on U:*

$$u = \mathrm{Re}(f), \quad v = \mathrm{Im}(f), \quad \textit{or} \quad |f|.$$

Then f is constant on U. The same conclusion is also true if $f\colon U \to \mathbb{C} \setminus \{0\}$, and $\mathrm{Arg}(f)$ is constant on U.

PROOF. Exercise. $\square$

Summary of Chapter 3. Here is a list of key topics from this chapter:

- Complex derivatives and analytic functions.
- e^z, $\mathrm{Log}\, z$ and polynomials of z are analytic, but $z \mapsto \bar{z}$ is not.
- Product rule, chain rule, derivatives of inverses.
- Branches of f^{-1}.
- Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$. Comparison between complex differentiability and real differentiability for maps $\mathbb{R}^2 \to \mathbb{R}^2$.
- Analytic functions with $f'(z) = 0$ are constant on connected sets.

CHAPTER 4

Complex integration

4.1. Paths

We defined *paths* in Section 2.4, but let us recap and extend the terminology:

Definition 4.1.1 (Path). A *path* is a continuous map $\gamma\colon [a,b] \to \mathbb{C}$, where $a, b \in \mathbb{R}$ with $a < b$. The *trace* of γ is the set

$$\gamma^* := \gamma([a,b]) = \{\gamma(t) : t \in [a,b]\}.$$

We say that γ *is a path in a set* $X \subset \mathbb{C}$ if $\gamma^* \subset X$. Finally, γ is a *closed path* (also called *periodic path*) if $\gamma(a) = \gamma(b)$.

Definition 4.1.2 (C^1-path). A map $\gamma\colon [a,b] \to \mathbb{C}$ is a C^1-*path* if γ is differentiable on $[a,b]$ (with one-sided derivatives at the endpoints) and the derivative $\gamma'\colon [a,b] \to \mathbb{C}$ is continuous.

We say that $\gamma\colon [a,b] \to \mathbb{C}$ is a *piecewise* C^1-*path* if there exist numbers $a = t_0 < t_1 < \ldots < t_n = b$ such that $\gamma|_{[t_{j-1},t_j]}\colon [t_{j-1}, t_j] \to \mathbb{C}$ is a C^1-path for all $1 \leq j \leq n$.

Remark 4.1.3. The differentiability of γ on $[a,b]$ means, by definition, that if $\gamma = \alpha + i\beta$ with $\alpha = \operatorname{Re}\gamma$ and $\beta = \operatorname{Im}\gamma$, then the components $\alpha, \beta\colon [a,b] \to \mathbb{R}$ are differentiable on $[a,b]$ (with one-sided derivatives at the endpoints) and both derivatives α' and β' are continuous on $[a,b]$. Note that

$$\gamma'(t) = \alpha'(t) + i\beta'(t) \in \mathbb{C}.$$

A second remark is that if γ is (only) piecewise C^1, then there may be a finite set $X \subset [a,b]$ such that $\gamma'(t)$ does not exist for $t \in X$.

The following further notation will be useful:

Notation (Segment). Let $z, w \in \mathbb{C}$ with $z \neq w$. Then $[z,w]\colon [0,1] \to \mathbb{C}$ the path parametrised by

$$[z,w](t) := tw + (1-t)z, \qquad t \in [0,1].$$

Note that $[z,w]$ is a C^1-path, and its image $[z,w]^*$ is the line segment $\{tw + (1-t)z : t \in [0,1]\}$ connecting z to w.

EXAMPLE 4.1.4 (Circle path). Let $n \in \mathbb{Z}$, $z \in \mathbb{C}$, and $r > 0$. Consider the path $\gamma\colon [0, 2\pi] \to \mathbb{C}$ defined by

$$\gamma(t) := z + re^{int}, \qquad t \in [0, 2\pi].$$

Then γ is a closed C^1-path, and its image $\gamma^* = S(z, r)$ is the circle centred at z with radius r. Note that γ "goes around" $S(z, r)$ exactly $|n|$ times. If $n > 0$, then γ "travels" counterclockwise, and if $n < 0$, then γ "travels" clockwise. If $n = 0$, then $\gamma \equiv z + r$ is a constant path.

The notion of "going around $|n|$ times" will be formalised later when we talk about winding numbers.

From existing paths, one can construct new ones using the following notions:

DEFINITION 4.1.5 (Reverse and composite paths). Let $\gamma\colon [a, b] \to \mathbb{C}$ be a path. We write $\overleftarrow{\gamma}\colon [a, b] \to \mathbb{C}$ for the *reverse* path

$$\overleftarrow{\gamma}(t) := \gamma(a + b - t), \qquad t \in [a, b].$$

Note that $\overleftarrow{\gamma}(a) = \gamma(b)$ and $\overleftarrow{\gamma}(b) = \gamma(a)$, and $(\overleftarrow{\gamma})^* = \gamma^*$.

Let $\gamma\colon [a, b] \to \mathbb{C}$ and $\eta\colon [c, d] \to \mathbb{C}$ be paths satisfying $\gamma(b) = \eta(c)$. We define the *composite path* $\gamma \star \eta\colon [a, b + d - c] \to \mathbb{C}$ with the formula

$$(\gamma \star \eta)(t) := \begin{cases} \gamma(t), & t \in [a, b], \\ \eta(t - b + c), & t \in [b, b + d - c]. \end{cases}$$

EXERCISE 4.1. Show that if $\gamma, \eta\colon [0, 1] \to \mathbb{C}$ are piecewise C^1-paths, then $\gamma \star \eta\colon [0, 2] \to \mathbb{C}$ is also a piecewise C^1-path, and $(\gamma \star \eta)^* = \gamma^* \cup \eta^*$.

In the next section, we will be studying *complex path integrals*, denoted $\int_\gamma f(z)\,dz$. It will turn out that the value of this integral depends not only on the trace γ^*, but also on the exact parametrisation of γ^*. For example, the paths γ and $\overleftarrow{\gamma}$ have the same trace, but the associated path integrals have opposite signs (Proposition 4.2.15). For this reason, it is interesting to ask: if two paths γ, η have the same trace, when can we guarantee that $\int_\gamma f(z)\,dz = \int_\eta f(z)\,dz$? The next definition will give a sufficient condition (see Proposition 4.2.15).

DEFINITION 4.1.6 (Reparametrisation). Let $\gamma\colon [a, b] \to \mathbb{C}$ and $\eta\colon [c, d] \to \mathbb{C}$ be paths. We say that η is *reparametrisation* of γ if there exists a continuous bijection $\rho\colon [c, d] \to [a, b]$ such that $\rho(c) = a$, $\rho(d) = b$, and $\eta = \gamma \circ \rho$.

If ρ is piecewise C^1, then we say that η is a piecewise C^1-reparametrisation of γ.

Remark 4.1.7. If $\gamma\colon [a,b] \to \mathbb{C}$ and $\eta\colon [c,d] \to \mathbb{C}$ are piecewise C^1-paths and $\eta = \gamma \circ \rho$ is a piecewise C^1-reparametrisation of γ, then the chain rule implies that

$$\eta'(t) = \gamma'(\rho(t))\rho'(t) \tag{4.1.1}$$

for all $t \in [c,d]$ outside a finite set.

Example 4.1.8. Let $\gamma\colon [a,b] \to \mathbb{C}$ be a non-closed path. Then the reverse path $\overleftarrow{\gamma}\colon [a,b] \to \mathbb{C}$ is **never** a reparametrisation of γ. Indeed, if $\eta = \gamma \circ \rho\colon [a,b] \to \mathbb{C}$ is any reparametrisation of γ, then $\eta(a) = \gamma(\rho(a)) = \gamma(a)$. However, $\overleftarrow{\gamma}(a) = \gamma(b) \neq \gamma(a)$.

This discussion leaves open the question of closed paths, for example $\gamma(t) = e^{2\pi i t}$, but see the next exercise for a negative answer.

Exercise 4.2. Let $\gamma\colon [0,1] \to \mathbb{C}$ be a path. Assume there are two distinct points $z_1, z_2 \in \gamma^*$ such that

$$\operatorname{card} \gamma^{-1}\{z_j\} = 1, \qquad j \in \{1,2\}. \tag{4.1.2}$$

Show that $\overleftarrow{\gamma}$ is **not** a reparametrisation of γ. Show by example that one point is not enough to guarantee the same conclusion.

If $\gamma(t) = e^{2\pi i t}$, then every point $z \in \gamma^* \setminus \{1\} = S(0,1) \setminus \{1\}$ satisfies (4.1.2). So, we can easily find two distinct points $z_1, z_2 \in \gamma^*$ satisfying (4.1.2), and therefore $\overleftarrow{\gamma}$ is not a reparametrisation of γ.

4.2. Complex path integral

We first define integrals of $\mathbb{C}$-valued functions:

Definition 4.2.1. Let $a < b$, and let $f\colon [a,b] \to \mathbb{C}$ be a function such that both $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ are Riemann-integrable. Then, we define

$$\int_a^b f(t)\,dt := \int_a^b u(t)\,dt + i \int_a^b v(t)\,dt.$$

The following is one of the most central definitions on the course.

Definition 4.2.2 (Complex path integral). Let $\gamma\colon [a,b] \to \mathbb{C}$ be a piecewise C^1-path, and let $f\colon \gamma^* \to \mathbb{C}$ be continuous. Then the *complex path integral of f over γ* is

$$\begin{aligned}\int_\gamma f(z)\,dz &:= \int_a^b f(\gamma(t))\gamma'(t)\,dt \\ &= \int_a^b \operatorname{Re}\bigl(f(\gamma(t))\gamma'(t)\bigr)\,dt + i\int_a^b \operatorname{Im}\bigl(f(\gamma(t))\gamma'(t)\bigr)\,dt.\end{aligned}$$

Let us clarify and contemplate the definition with a few remarks.

REMARK 4.2.3. The functions $\operatorname{Re}\big(f(\gamma(t))\gamma'(t)\big)$ and $\operatorname{Im}\big(f(\gamma(t))\gamma'(t)\big)$ are piecewise continuous and hence Riemann-integrable. The derivative $\gamma'(t)$ is only defined for $t \in [a,b] \setminus X$, where $X \subset [a,b]$ is a finite set. However, we may define $\gamma'(t)$ arbitrarily for $t \in X$, and then check that the values of the Riemann integrals above are independent of the choice. These values are the precise meaning of the integrals above.

REMARK 4.2.4. Let us spell out completely explicitly the integrals appearing in the definition of $\int_\gamma f(z)\,dz$. If $f = f_1 + if_2$ and $\gamma = \gamma_1 + i\gamma_2$, then

$$f(\gamma(t))\gamma'(t) = \big(f_1(\gamma(t))\gamma_1'(t) - f_2(\gamma(t))\gamma_2'(t)\big) + i\big(f_1(\gamma(t))\gamma_2'(t) + f_2(\gamma(t))\gamma_1'(t)\big),$$

so $\int_\gamma f(z)\,dz$ is explicitly given by

$$\begin{aligned}\int_\gamma f(z)\,dz = &\int_a^b \left[f_1(\gamma(t))\gamma_1'(t) - f_2(\gamma(t))\gamma_2'(t)\right]\,dt \\ &+ i\int_a^b \left[f_1(\gamma(t))\gamma_2'(t) + f_2(\gamma(t))\gamma_1'(t)\right]\,dt.\end{aligned} \tag{4.2.1}$$

REMARK 4.2.5. There is a standard definition for the **real** path integral of a continuous map $F\colon \gamma^* \to \mathbb{R}^2$ over a path $\gamma\colon [a,b] \to \mathbb{R}^2$. It looks like

$$\int_\gamma F(s)\,ds := \int_a^b F(\gamma(t)) \cdot \gamma'(t)\,dt,$$

where $F(\gamma(t)) \cdot \gamma'(t)$ refers to the dot product between the vectors $F(\gamma(t))$ and $\gamma'(t) = (\gamma_1'(t), \gamma_2'(t))$. In this course we will never deal with real path integrals, so the notation $\int_\gamma f(z)\,dz$ will **always** refer to the complex path integral in Definition 4.2.2.

The following example turns out to be extremely important:

EXAMPLE 4.2.6. Consider the map $f\colon \mathbb{C} \setminus \{0\} \to \mathbb{C}$, $f(z) = 1/z$, and the "circle path" $\gamma\colon [0, 2\pi] \to \mathbb{C}$ defined by $\gamma(t) := e^{it}$. Then $\gamma'(t) = ie^{it}$, so

$$\int_\gamma f(z)\,dz = \int_0^{2\pi} f(\gamma(t))\gamma'(t)\,dt = i\int_0^{2\pi} \frac{e^{it}\,dt}{e^{it}} = i\int_0^{2\pi} dt = 2\pi i. \tag{4.2.2}$$

More generally, if $n \in \mathbb{Z}$, $z_0 \in \mathbb{C}$, and $\gamma_n(t) := z_0 + re^{int}$, then $\gamma_n'(t) = inre^{int}$, and hence

$$\frac{1}{2\pi i}\int_{\gamma_n} \frac{dz}{z - z_0} = n, \qquad n \in \mathbb{Z}. \tag{4.2.3}$$

The right hand side corresponds to our intuition of "how many times the path γ_n winds around z_0". We will return to this!

WARNING 4.2.7. In previous analysis courses, you have perhaps calculated integrals on $\mathbb{R}$ with the rule

$$\int_a^b f(t)\,dt = F(b) - F(a), \tag{4.2.4}$$

where $F' = f$ on $[a,b]$. This is **not** as simple in for complex path integrals and one needs to be careful. There is a correct version of (4.2.4) for complex path integrals, but as discussed in Theorem 4.3.4 it can only be applied if the primitive F is analytic in an open set containing γ^*.

We also introduce the following variant of the path integral:

DEFINITION 4.2.8 (Arc length integral). Let $\gamma\colon [a,b] \to \mathbb{C}$ be piecewise C^1, and let $f = f_1 + if_2\colon \gamma^* \to \mathbb{C}$ be continuous. The *arc length integral of f over γ* is the complex number

$$\int_\gamma f(z)\,|dz| := \int_a^b f(\gamma(t))|\gamma'(t)|\,dt := \int_a^b f_1(\gamma(t))|\gamma'(t)|\,dt + i\int_a^b f_2(\gamma(t))|\gamma'(t)|\,dt,$$

where $|\gamma'(t)| = \sqrt{\gamma_1'(t)^2 + \gamma_2'(t)^2}$ is the modulus of $\gamma'(t) \in \mathbb{C}$.

EXAMPLE 4.2.9. Let us continue with $f(z) = 1/z$ and $\gamma(t) := e^{it}$ for $t \in [0, 2\pi]$. Since $|\gamma'(t)| = 1$, the arc length integral is

$$\int_\gamma f(z)\,|dz| = \int_0^{2\pi} \frac{dt}{e^{it}} = \int_0^{2\pi} e^{-it}\,dt = \int_0^{2\pi} \cos(t)\,dt - i\int_0^{2\pi} \sin(t)\,dt = 0.$$

In particular, $\int_\gamma f(z)\,|dz| = 0 \neq 2\pi i = \int_\gamma f(z)\,dz$.

The arc length integral is useful because it gives an upper bound for the complex path integral (and the upper bound is often easier to compute):

PROPOSITION 4.2.10. *Let $\gamma\colon [a,b] \to \mathbb{C}$ be piecewise C^1, and let $f\colon \gamma^* \to \mathbb{C}$ be continuous. Then,*

$$\left|\int_\gamma f(z)\,dz\right| \le \int_\gamma |f(z)|\,|dz|.$$

The proof will be easier when we have first recorded the following basic facts:

PROPOSITION 4.2.11. *The complex path integral is (complex) linear: if $\gamma\colon [a,b] \to \mathbb{C}$ is a piecewise C^1-path, $f, g\colon \gamma^* \to \mathbb{C}$ are continuous, and $\alpha, \beta \in \mathbb{C}$, then*

$$\int_\gamma (\alpha f + \beta g)(z)\,dz = \alpha\int_\gamma f(z)\,dz + \beta\int_\gamma g(z)\,dz.$$

Proof. This is "intuitively obvious" by the linearity of the Riemann integral, but you may feel a little less secure when you recall the proper meaning of $\int_\gamma f(z)\,dz$ from (4.2.1). Checking the computations is an exercise. $\square$

We are now ready to prove Proposition 4.2.10:

Proof of Proposition 4.2.10. Let $w := \int_\gamma f(z)\,dz$. If $w = 0$, there is nothing to prove. Otherwise, we may write the value of the integral in polar coordinates:

$$w = |w|e^{i\theta}, \qquad \theta \in \arg(w).$$

Now, by the linearity of the complex path integral, we may write

$$\left|\int_\gamma f(z)\,dz\right| = |w| = e^{-i\theta}w = \int_\gamma e^{-i\theta} f(z)\,dz.$$

The left hand side is clearly a real number, so also the right hand side is a real number. On the other hand, by the definition of the complex path integral,

$$\int_\gamma e^{-i\theta} f(z)\,dz = \int_a^b \operatorname{Re}\big(e^{-i\theta} f(\gamma(t))\gamma'(t)\big)\,dt + i\int_a^b \operatorname{Im}\big(e^{-i\theta} f(\gamma(t))\gamma'(t)\big)\,dt.$$

Since the left hand side is a real number, also the right hand side must be. Therefore

$$\int_a^b \operatorname{Im}\big(e^{-i\theta} f(\gamma(t))\gamma'(t)\big)\,dt = 0,$$

and we have now shown that

$$\left|\int_\gamma f(z)\,dz\right| = \int_a^b \operatorname{Re}\big(e^{-i\theta} f(\gamma(t))\gamma'(t)\big)\,dt.$$

The right hand side is the Riemann integral of $t \mapsto \operatorname{Re}\big(e^{-i\theta} f(\gamma(t))\gamma'(t)\big)$, so by the basics of real analysis,

$$\int_a^b \operatorname{Re}\big(e^{-i\theta} f(\gamma(t))\gamma'(t)\big)\,dt \le \int_a^b \big|\operatorname{Re}\big(e^{-i\theta} f(\gamma(t))\gamma'(t)\big)\big|\,dt.$$

To conclude the proof, it remains to note that

$$\big|\operatorname{Re}\big(e^{-i\theta} f(\gamma(t))\gamma'(t)\big)\big| \le \big|e^{-i\theta} f(\gamma(t))\gamma'(t)\big| \le |f(\gamma(t))||\gamma'(t)|,$$

so

$$\int_a^b \big|\operatorname{Re}\big(e^{-i\theta} f(\gamma(t))\gamma'(t)\big)\big|\,dt \le \int_a^b |f(\gamma(t))||\gamma'(t)|\,dt \overset{\text{def.}}{=} \int_\gamma |f(z)|\,|dz|.$$

This completes the proof. $\square$

We record the following corollary:

COROLLARY 4.2.12. *Let $\gamma\colon [a,b] \to \mathbb{C}$ be piecewise C^1, and let $f\colon \gamma^* \to \mathbb{C}$ be continuous. Then,*

$$\left| \int_\gamma f(z)\,dz \right| \le \|f\|_\infty \mathrm{length}(\gamma),$$

where $\|f\|_\infty := \sup\{|f(z)| : z \in \gamma^*\}$, and

$$\mathrm{length}(\gamma) := \int_a^b |\gamma'(t)|\,dt.$$

PROOF. We simply note that

$$\left| \int_\gamma f(z)\,dz \right| \overset{\text{Cor. 4.2.10}}{\le} \int_\gamma |f(z)|\,|dz| \overset{\text{def.}}{=} \int_a^b |f(\gamma(t))||\gamma'(t)|\,dt \le \|f\|_\infty \int_a^b |\gamma'(t)|\,dt.$$

This completes the proof. $\square$

Note that $\mathrm{length}(\gamma) < \infty$ for every C^1-path $\gamma\colon [a,b] \to \mathbb{C}$, since $t \mapsto |\gamma'(t)|$ is continuous, and thus uniformly bounded on $[a,b]$. In fact, $\mathrm{length}(\gamma) < \infty$ also remains true for piecewise C^1-paths $\gamma\colon [a,b] \to \mathbb{C}$, since γ can be written as $\gamma = \gamma_1 \star \cdots \star \gamma_n$, where each γ_j is C^1, and

$$\mathrm{length}(\gamma) = \mathrm{length}(\gamma_1) + \cdots + \mathrm{length}(\gamma_n) < \infty.$$

Our definition of "length(γ)" agrees with other notions of length the reader may possibly have seen, for example the 1-dimensional Hausdorff measure $\mathcal{H}^1(\gamma^*)$. This will not be needed explicitly, so we will simply give the following example as justification:

EXAMPLE 4.2.13. Let $z, w \in \mathbb{C}$ with $z \neq w$. Consider the segment

$$[z,w](t) = tw + (1-t)z, \qquad t \in [0,1].$$

Then $[z,w]'(t) = w - z$ for all $t \in [0,1]$, and consequently

$$\mathrm{length}([z,w]) \overset{\text{def.}}{=} \int_0^1 |[z,w]'(t)|\,dt = \int_0^1 |w-z|\,dt = |w-z|.$$

So, our definition of "length" coincides with the expected result for (at least) segments.

The following consequence of Corollary 4.2.12 is often useful:

COROLLARY 4.2.14. *Let $\gamma\colon [a,b] \to \mathbb{C}$ be piecewise C^1, and let $f_k, f\colon \gamma^* \to \mathbb{C}$ be continuous functions such that $f_k \to f$ uniformly on γ^*, as $k \to \infty$. Then,*

$$\lim_{k\to\infty} \int_\gamma f_k(z)\,dz = \int_\gamma f(z)\,dz.$$

Proof. By Corollary 4.2.12, we have

$$\left|\int_\gamma f_k(z)\,dz - \int_\gamma f(z)\,dz\right| = \left|\int_\gamma (f_k - f)(z)\,dz\right| \le \|f_k - f\|_{L^\infty(\gamma^*)} \cdot \mathrm{length}(\gamma).$$

Here $\mathrm{length}(\gamma) < \infty$, and $\|f_k - f\|_{L^\infty(\gamma^*)} \to 0$ by assumption, as $k \to \infty$. $\square$

Here are a few more basic properties of complex path integrals:

Proposition 4.2.15. *Let $\gamma\colon [a,b] \to \mathbb{C}$ be piecewise C^1, and let $f\colon \gamma^* \to \mathbb{C}$ be continuous. Then,*

$$\int_{\overleftarrow{\gamma}} f(z)\,dz = -\int_\gamma f(z)\,dz. \tag{4.2.5}$$

Let $\eta\colon [c,d] \to \mathbb{C}$ be a piecewise C^1-reparametrisation of γ. Then,

$$\int_\eta f(z)\,dz = \int_\gamma f(z)\,dz.$$

Assume that $\alpha\colon [a,b] \to \mathbb{C}$ and $\beta\colon [c,d] \to \mathbb{C}$ are piecewise C^1-paths, and $\alpha(b) = \beta(c)$, so the composite path $\alpha \star \beta\colon [a,b] \to \mathbb{C}$ is defined. Assume that $f\colon (\alpha \star \beta)^* \to \mathbb{C}$ is continuous. Then,

$$\int_{\alpha\star\beta} f(z)\,dz = \int_\alpha f(z)\,dz + \int_\beta f(z)\,dz. \tag{4.2.6}$$

Regarding (4.2.5), it is noteworthy that the paths γ and $\overleftarrow{\gamma}$ have the same trace γ^*. So, the value of the complex path integral over γ is not determined by γ^* alone!

Proof of Proposition 4.2.15. We prove the claim about reparametrisations, and leave the other statements as exercises. We assumed that η is a piecewise C^1-reparametrisation of γ. This means that there exists a piecewise C^1 bijection $\rho\colon [c,d] \to [a,b]$ with the properties $\rho(c) = (a)$, $\rho(d) = b$, and $\eta = \gamma \circ \rho$. With this information in mind, we use the definition of the complex path integral:

$$\int_\eta f(z)\,dz \overset{\text{def.}}{=} \int_c^d f(\eta(t))\eta'(t)\,dt = \int_c^d f(\gamma(\rho(t)))\gamma'(\rho(t))\rho'(t)\,dt. \tag{4.2.7}$$

Here we plugged in the formula $\eta'(t) = \gamma'(\rho(t))\rho'(t)$ recorded in (4.1.1). Next, we use the following change-of-variables formula familiar from the theory of Riemann integrals:

$$\int_c^d g(\rho(t))\rho'(t)\,dt = \int_a^b g(s)\,ds,$$

where $g\colon [a,b] \to \mathbb{R}$ is Riemann integrable. We apply this formula to the (piecewise) continuous function $g(s) := f(\gamma(s))\gamma'(s)$, or to be more precise its real and imaginary parts separately. The conclusion is that

$$(4.2.7) = \int_a^b f(\gamma(s))\gamma'(s)\,ds \overset{\text{def.}}{=} \int_\gamma f(z)\,dz.$$

this is what we claimed. □

4.3. Primitives

Recall the *fundamental theorem of calculus*: if $g\colon [a,b] \to \mathbb{R}$ is Riemann integrable, and $G\colon [a,b] \to \mathbb{R}$ is a differentiable function satisfying $G'(t) = g(t)$ for all $t \in [a,b]$, then

$$G(b) - G(a) = \int_a^b g(t)\,dt = \int_a^b G'(t)\,dt. \tag{4.3.1}$$

We now aim for a path integral version of that theorem.

Definition 4.3.1. Let $U \subset \mathbb{C}$ be open, and let $f\colon U \to \mathbb{C}$ be a map. We say that $F\colon U \to \mathbb{C}$ is *a primitive of f in U* if F is analytic in U, and $F'(z) = f(z)$ for all $z \in U$.

If F is a primitive of $f\colon U \to \mathbb{C}$, and $c \in \mathbb{C}$, then $F + c$ is also clearly a primitive of f. The converse is true if U is connected:

Proposition 4.3.2. *Let $U \subset \mathbb{C}$ be open and connected, and let $f\colon U \to \mathbb{C}$ be a map. Assume that $F_1, F_2\colon U \to \mathbb{C}$ are both primitives of f. Then there exists a constant $c \in \mathbb{C}$ such that $F_1 = F_2 + c$.*

Proof. Note that $G := F_1 - F_2\colon U \to \mathbb{C}$ is an analytic function with $G'(z) = 0$ for all $z \in U$. Therefore G is a constant on U by Corollary 3.3.1. □

Example 4.3.3. All the primitives of $z \mapsto e^z$ in $\mathbb{C}$ are of the form $F(z) = e^z + c$, $c \in \mathbb{C}$. All the primitives of $w \mapsto 1/w$ in the set $\mathbb{C} \setminus (-\infty, 0]$ are of the form $G(w) = \operatorname{Log} w + c$, $c \in \mathbb{C}$. These facts follow from recalling that $z \mapsto e^z$ and $w \mapsto \operatorname{Log} w$ are examples of primitives (by Theorem 3.2.8), and on the other hand the open sets $\mathbb{C}$ and $\mathbb{C} \setminus (-\infty, 0]$ are connected.

We then prove the path integral version of (4.3.1), which could be called the "fundamental theorem of calculus for complex path integrals" :

Theorem 4.3.4 (Path integrals via primitives). *Let $U \subset \mathbb{C}$ be open, and assume that $f\colon U \to \mathbb{C}$ is continuous. Let $\gamma\colon [a,b] \to U$ be a piecewise C^1-path. If F is a primitive of f in U, then*

$$\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a)).$$

In particular, if γ is a closed path, then $\gamma(a) = \gamma(b)$, so

$$\int_\gamma f(z)\,dz = 0.$$

REMARK 4.3.5. Recall that the principal logarithm Log is a primitive of $z \mapsto 1/z$ in $\mathbb{C} \setminus (-\infty, 0]$. Therefore, if $\gamma\colon [a,b] \to \mathbb{C} \setminus (-\infty, 0]$ is a closed piecewise C^1-path, we may deduce from Theorem 4.3.4 that

$$\int_\gamma \frac{1}{z}\,dz = \operatorname{Log}\gamma(b) - \operatorname{Log}\gamma(a) = 0.$$

On the other hand, this does **not** apply to the path $\gamma(t) = e^{it}$, for $t \in [0, 2\pi]$, because γ^* is not contained in $\mathbb{C} \setminus (-\infty, 0]$. In fact for this path $\gamma(t)$ we computed

$$\int_\gamma \frac{1}{z}\,dz = 2\pi i \neq 0 = \operatorname{Log}\gamma(2\pi) - \operatorname{Log}\gamma(0).$$

On the other hand, there is nothing special about $(-\infty, 0]$. By the same argument, we have $\int_\gamma dz/z = 0$ for every closed piecewise C^1-path γ with $\gamma^* \subset \mathbb{C} \setminus \ell_v$ (for arbitrary $v \in \mathbb{C} \setminus \{0\}$), where $\ell_v = \{tv : t \geq 0\}$. Reason: there exists a branch of the logarithm in $\mathbb{C} \setminus \ell_v$ (exercise).

The following lemma is needed in the proof of Theorem 4.3.4:

LEMMA 4.3.6. *Let $F\colon U \to \mathbb{C}$ be analytic, and let $\gamma\colon [a,b] \to \mathbb{C}$ be a C^1-path. Then,*

$$(F \circ \gamma)'(t) = F'(\gamma(t))\gamma'(t), \qquad t \in [a,b],$$

where the left hand side refers to the real derivative of the composed map $F \circ \gamma\colon [a,b] \to \mathbb{R}^2$.

PROOF. To calculate $(F \circ \gamma)'(t)$, we apply the real-variable chain rule:

$$(F \circ \gamma)'(t) = DF(\gamma(t))\gamma'(t),$$

using here that complex differentiability implies real differentiability (so $DF(z)$ makes sense for $z \in U$). On the other hand, since F is complex differentiable, the matrix $DF(\gamma(t))$ equals the matrix $M_{F'(\gamma(t))}$ corresponding to complex multiplication by $F'(\gamma(t))$, as recorded in (3.2.5). Therefore $(F \circ \gamma)'(t) = M_{F'(\gamma(t))}\gamma'(t) = F'(\gamma(t))\gamma'(t)$, as claimed. $\square$

We are then ready to prove Theorem 4.3.4.

PROOF OF THEOREM 4.3.4. Assume first that $\gamma\colon [a,b] \to \mathbb{C}$ is a C^1-path, and not just piecewise C^1. The primitive $F\colon U \to \mathbb{C}$ is analytic, so the previous lemma implies that

$$(F \circ \gamma)'(t) = F'(\gamma(t))\gamma'(t) = f(\gamma(t))\gamma'(t), \qquad t \in [a,b].$$

Consequently,

$$\int_\gamma f(z)\,dz \overset{\text{def.}}{=} \int_a^b f(\gamma(t))\gamma'(t)\,dt = \int_a^b (F\circ\gamma)'(t)\,dt = F(\gamma(b)) - F(\gamma(a)),$$

where the last equation applied the fundamental theorem of calculus (4.3.1) to the Riemann integrable (even continuous) function $g := (F\circ\gamma)' \colon [a,b] \to \mathbb{C}$ and its real primitive $G := F\circ\gamma$ (or to be precise the real and imaginary parts of these functions separately).

Finally, let us deduce the case of piecewise C^1-paths. If $\gamma\colon [a,b]\to\mathbb{C}$ is piecewise C^1, we may write $\gamma = \gamma_1 \star\cdots\star\gamma_n$ where each $\gamma_j\colon [t_{j-1}, t_j]\to\mathbb{C}$ is a C^1-path, $a = t_0 < \ldots < t_n = b$, and $\gamma_j(t_{j-1}) = \gamma_{j-1}(t_j)$ for $1 \le j \le n$. Consequently,

$$\int_\gamma f(z)\,dz \overset{(4.2.6)}{=} \sum_{j=1}^n \int_{\gamma_j} f(z)\,dz = \sum_{j=1}^n [F(\gamma_j(t_j)) - F(\gamma_j(t_{j-1}))],$$

by the first part of the proof applied separately to the paths γ_j. The sum on the right hand side "telescopes" and its value is

$$F(\gamma_n(t_n)) - F(\gamma_1(t_0)) = F(\gamma(b)) - F(\gamma(a)).$$

This completes the proof. $\square$

We record the following corollary:

Corollary 4.3.7. *Let $U \subset \mathbb{C}$ be open and $f\colon U\to\mathbb{C}$ be analytic with f' continuous. Then, if $z, w \in U$, and also the segment connecting z, w is contained in U, we have*

$$f(w) - f(z) = \int_{[z,w]} f'(\zeta)\,d\zeta.$$

Here $[z,w](t) := tw + (1-t)z$ for $t\in[0,1]$.

Proof. The function f is a primitive of f', so Theorem 4.3.4 yields

$$\int_{[z,w]} f'(\zeta)\,d\zeta = f([z,w](1)) - f([z,w](0)) = f(w) - f(z).$$

This is what we claimed. $\square$

Theorem 4.3.4 can be used to show that certain maps do **not** have primitives:

Example 4.3.8. The map $f\colon \mathbb{C}\setminus\{0\}\to\mathbb{C}$ defined by $f(z) = 1/z$ does **not** have a primitive $F\colon\mathbb{C}\setminus\{0\}\to\mathbb{C}$. To see this, let $\gamma(t) := e^{it}$, $t\in[0,2\pi]$. If the primitive F existed, then our calculation (4.2.2) and Theorem 4.3.4 would contradict each other:

$$2\pi i = \int_\gamma f(z)\,dz = F(\gamma(0)) - F(\gamma(2\pi)) = 0.$$

Let us emphasise, however, that the existence of primitives is highly dependent on the choice of the domain: for example, $z \mapsto 1/z$ **does** have a primitive in every slit domain of the form $\mathbb{C} \setminus \ell_v$ (recall Remark 4.3.5). Indeed, the primitive is given by a(ny) branch of the logarithm in $\mathbb{C} \setminus \ell_v$.

EXERCISE 4.3. The function $z \mapsto \bar{z}$ has no primitive in any open set $U \subset \mathbb{C}$.

The following corollary of Theorem 4.3.4 will be needed in the next section:

COROLLARY 4.3.9. *Let $\gamma\colon [a,b] \to \mathbb{C}$ be a closed piecewise C^1-path, and let $p(z) = a_n z^n + \ldots + a_1 z + a_0$ be a polynomial. Then,*

$$\int_\gamma p(z)\,dz = 0.$$

PROOF. Define the function

$$P(z) := \tfrac{a_n}{n+1} z^{n+1} + \tfrac{a_{n-1}}{n} z^n + \ldots + \tfrac{a_1}{2} z^2 + a_0 z$$

By Corollary 3.1.8 one has

$$P'(z) = p(z).$$

Thus the polynomial p has the primitive P in the whole plane $\mathbb{C}$, so the conclusion follows immediately from Theorem 4.3.4. □

Finally, we record the following corollary of Theorem 4.3.4:

PROPOSITION 4.3.10 (Integration by parts). *Let $U \subset \mathbb{C}$ be open, and let $\gamma\colon [a,b] \to U$ be a piecewise C^1-path. Assume that $f, g\colon U \to \mathbb{C}$ are analytic. Assume additionally that f', g' are continuous. Then,*

$$\int_\gamma f(z) g'(z)\,dz = [f(\gamma(b))g(\gamma(b)) - f(\gamma(a))g(\gamma(a))] - \int_\gamma g(z) f'(z)\,dz.$$

PROOF. Exercise. □

Summary of Section 4. Here is a list of key topics from this section:

- Piecewise C^1-paths.
- The reverse path $\overleftarrow{\gamma}$ and the composite path $\gamma \star \eta$.
- Reparametrisation of paths.
- The definitions of the complex path integral and arc length integral.
- Example 4.2.6: the path integral over the circle path of the function $1/z$.
- Path integrals over $\overleftarrow{\gamma}$, $\gamma \star \eta$, and reparametrisations of γ.
- Bounding the path integral from above by the arc length integral.
- Primitives and the "fundamental theorem of calculus for complex path integrals" (Theorem 4.3.4).

CHAPTER 5

Cauchy's theorem and applications

The rest of the course will reveal a sequence of increasingly breathtaking results concerning analytic functions. They are all based on a fairly innocent-looking statement, known as Cauchy's theorem (in fact we will see several different versions of Cauchy's theorem). It will state (see Theorem 5.1.4) that if $U \subset \mathbb{C}$ is a convex open set, and $f\colon U \to \mathbb{C}$ is analytic, then

$$\int_\gamma f(z)\,dz = 0 \tag{5.0.1}$$

for every closed piecewise C^1-path in U. With modest effort, this fundamental fact will imply – among other things – that every analytic function defined on an arbitrary open set is infinitely differentiable! In fact, here is a schematic picture of the rest of the course:

$$\text{Cauchy's theorem} \implies \text{Cauchy's integral formula}$$

$$\implies \begin{cases} \text{Derivatives of analytic functions are analytic} \\ \text{Cauchy's integral formula for derivatives (CIFD)} \\ \text{Mean value and maximum modulus principles} \implies \text{Schwarz lemma} \end{cases}$$

$$\text{(CIFD)} \implies \begin{cases} \text{Morera's theorem} \implies \text{Analytic continuation to a point} \\ \text{Cauchy's estimates} \implies \text{Liouville's theorem} \implies \begin{cases} \text{Fundamental theorem} \\ \text{of algebra.} \end{cases} \end{cases}$$

When revising for the course, it is advisable to return to this schematic picture and see if you can summarise these results – and implications – to yourself! There will be no further "summary sections", since the schematic above contains them all.

5.1. Cauchy's theorem for convex sets

Before proving Cauchy's theorem (5.0.1) in general, we will first a version of it where $\gamma = \partial\triangle$ is a *triangle*, namely a path of the following form:

$$\partial\triangle := [a,b] \star [b,c] \star [c,a]. \tag{5.1.1}$$

Here $[z, w](t) := tw + (1-t)z$ was the "segment path" connecting z, w. Note that

$$\int_{\partial\triangle} f(z)\,dz \overset{(4.2.6)}{=} \int_{[a,b]} f(z)\,dz + \int_{[b,c]} f(z)\,dz + \int_{[c,a]} f(z)\,dz.$$

We will also use the notation $\triangle := \triangle(a, b, c) \subset \mathbb{C}$ for the "solid triangle" (not just the boundary) spanned by $\{a, b, c\}$. Thus, the trace $(\partial\triangle)^*$ equals the boundary of $\triangle$, and $\partial\triangle$ "parametrises" this boundary.

THEOREM 5.1.1 (Cauchy's theorem for triangles). *Let $a, b, c \in \mathbb{C}$, and let $\partial\triangle := \partial\,\triangle\,(a, b, c)$ be the path introduced in (5.1.1). Assume that $U \subset \mathbb{C}$ is an open set with $\triangle \subset U$, and let $w_0 \in U$. Assume that f is continuous in U and analytic in $U \setminus \{w_0\}$. Then,*

$$\int_{\partial\triangle} f(z)\,dz = 0.$$

REMARK 5.1.2. The "special point" $w_0 \in U$ may seem like an unnecessary technicality, but it will be indispensable when proving the *Cauchy integral formula* (Theorem 5.2.13).

PROOF OF THEOREM 5.1.1. We will assume that the triangle $\triangle$ is non-degenerate, that is, the points $\{a, b, c\}$ are not contained on a common line. The statement also remains true in this case, and is much easier to prove; think this through yourself!

We will start with the special case where $w_0 \notin \triangle$. Let a', b', c' be the midpoints of the segments $[a, b]$, $[b, c]$, and $[c, a]$, see Figure 1. These give rise to 4 new triangles $\triangle_1, \triangle_2, \triangle_3, \triangle_4 \subset \triangle$. Moreover,

$$\int_{\partial\triangle} f(z)\,dz = \sum_{j=1}^{4} \int_{\partial\triangle_j} f(z)\,dz. \tag{5.1.2}$$

This is because if $[z, w]$ is an edge on one of the smaller triangles which meets the interior of $\triangle$ (a red path in Figure 1), then also $\overleftarrow{[z, w]}$ is an edge on another small triangle, and the integrals over these two edges cancel each other out. So, only the edges on the boundary of $\triangle$ are not cancelled out by other edges, and their sum gives back the integral over $\partial\triangle$.

It now follows from (5.1.2) that

$$\left|\int_{\partial\triangle} f(z)\,dz\right| \le 4 \max_{1\le j\le 4} \left|\int_{\partial\triangle_j} f(z)\,dz\right|.$$

Let $\triangle^1 \in \{\triangle_1, \ldots, \triangle_4\}$ be the triangle which attains the maximum. We focus attention on $\triangle^1$, and repeat the "subdivision" trick inside $\triangle^1$: we

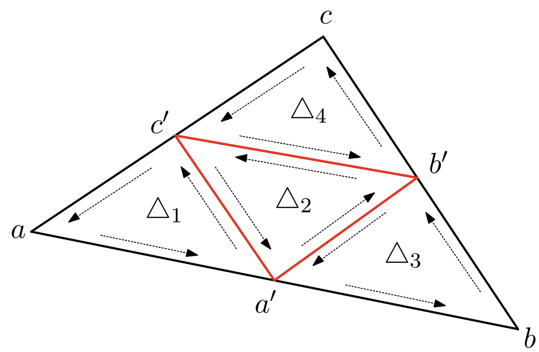


FIGURE 1. Dividing the triangle $\triangle$ into 4 smaller triangles $\triangle_1, \dots, \triangle_4$, and how the "interior" segments of $\partial\triangle_1, \dots, \partial\triangle_4$ cancel each other out.

divide $\triangle^1$ it into 4 smaller triangles $\triangle^1_1, \dots, \triangle^1_4$. Repeating the reasoning above, we can find one of them satisfying

$$\left|\int_{\partial\triangle} f(z)\,dz\right| \le 4\left|\int_{\partial\triangle^1} f(z)\,dz\right| \le 16\left|\int_{\partial\triangle^1_j} f(z)\,dz\right|.$$

Now, continuing inductively, we can find a sequence of triangles $\triangle =: \triangle^0 \supset \triangle^1 \supset \dots$ satisfying

$$\left|\int_{\partial\triangle} f(z)\,dz\right| \le 4^n\left|\int_{\partial\triangle^n} f(z)\,dz\right|, \qquad n \ge 0. \tag{5.1.3}$$

As $n \to \infty$, the triangles $\triangle_n$ get smaller and smaller: in fact

$$\mathrm{length}(\partial\triangle^n) = 2^{-n}\mathrm{length}(\partial\triangle), \qquad n \ge 0. \tag{5.1.4}$$

Moreover, the triangles $\triangle^n$ "converge" to a unique point $z_0 \in \triangle$. More precisely, by Cantor's intersection theorem (Theorem 2.2.8)

$$\bigcap_{n\ge0} \triangle^n = \{z_0\} \subset \triangle.$$

The map f is differentiable at z_0, so given $\epsilon > 0$, there exists a radius $r > 0$ such that

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| \le \epsilon|z - z_0|, \qquad z \in D(z_0, r). \tag{5.1.5}$$

In particular, the estimate (5.1.5) holds for all $z \in \triangle^n$ when $n \geq n_\epsilon$ is sufficiently large. For $n \geq n_\epsilon$, we therefore have

$$\begin{aligned}\left|\int_{\partial\triangle^n} f(z)\,dz\right| &\overset{\text{Cor. 4.3.9}}{=} \left|\int_{\partial\triangle^n} [f(z) - f(z_0) - f'(z_0)(z-z_0)]\,dz\right| \\ &\overset{\text{Prop. 4.2.10}}{\leq} \int_{\partial\triangle^n} \epsilon|z-z_0|\,|dz| \leq \epsilon(\mathrm{length}(\partial\triangle^n))^2 \\ &\overset{(5.1.4)}{=} \epsilon 4^{-n}(\mathrm{length}(\partial\triangle))^2.\end{aligned}$$

The first step used the fact that the path integral over $\partial\triangle^n$ of the polynomial $z \mapsto f(z_0)+f'(z_0)(z-z_0)$ vanishes, whereas the second step used the estimate of the path integral from above by the arc length integral. The third step used c

$$|z - z_0| \leq \mathrm{diam}(\triangle^n) \leq \mathrm{length}(\partial\triangle^n), \qquad z \in \triangle^n.$$

When the estimate above is combined with (5.1.3), we obtain

$$\left|\int_{\partial\triangle} f(z)\,dz\right| \leq 4^n \left|\int_{\partial\triangle^n} f(z)\,dz\right| \leq \epsilon \cdot (\mathrm{length}(\partial\triangle))^2.$$

Letting $\epsilon \to 0$ completes the proof in the special case $w_0 \notin \triangle$.

Let us finally consider the case $w_0 \in \triangle$. There are two distinct cases to consider, both shown in Figure 2: either w_0 is a corner of $\triangle$, or then it is not. If w_0 is a corner, we "isolate" it to a small triangle $\triangle_1 \subset \triangle$. Then, we

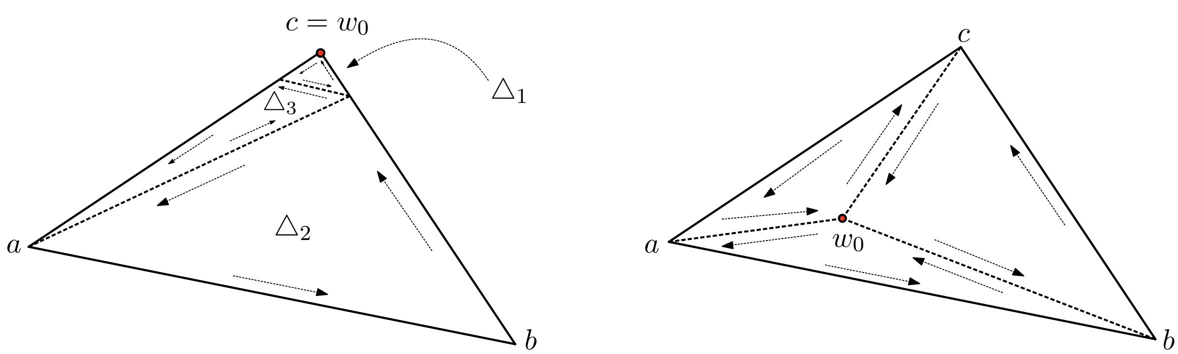


FIGURE 2. The special cases where (i) w_0 is a corner of $\triangle$, and (ii) where $w_0 \in \triangle$ but w_0 is not a corner.

also form two further triangles $\triangle_2, \triangle_3 \subset \triangle$ in such a way that

$$\int_{\partial\triangle} f(z)\,dz = \int_{\partial\triangle_1} f(z)\,dz + \int_{\partial\triangle_2} f(z)\,dz + \int_{\partial\triangle_3} f(z)\,dz. \tag{5.1.6}$$

Note that the triangles $\triangle_2, \triangle_3$ have no "special points": f is analytic on a neighbourhood of $\triangle_2 \cup \triangle_3$. Therefore the two latter terms in (5.1.6) are zero by the first part of the proof. For the first term, we estimate the path integral from above by the arc length integral:

$$\left|\int_{\partial\triangle_1} f(z)\,dz\right| \le \|f\|_{L^\infty(\triangle)}\mathrm{length}(\partial\triangle_1).$$

Since f is continuous on $\triangle$, we have $\|f\|_{L^\infty(\triangle)} < \infty$. On the other hand the length of $\partial\triangle_1$ can be made as small as we like. This implies that $\int_{\partial\triangle} f(z)\,dz = 0$.

Finally, consider the case where $w_0 \in \triangle$ is not a corner. We form new triangles $\triangle_1, \triangle_2, \triangle_3 \subset \triangle$ with the properties that w_0 is a common corner of $\triangle_1, \triangle_2, \triangle_3$, and (5.1.6) holds (see Figure 2, and think what would happen if w_0 lies on an edge of $\triangle$). Now, by the "corner case" we just handled above, all the three terms on the right hand side of (5.1.6) are zero. Therefore $\int_{\partial\triangle} f(z)\,dz = 0$ also in this case, and the proof is complete. □

We may immediately generalise the previous theorem to all closed piecewise C^1-paths γ, but only under the assumption that f is analytic in a **convex open set** containing γ^*.

Definition 5.1.3. A set $A \subset \mathbb{C}$ is *convex* if for any $z, w \in A$, the line segment $[z,w]^* = \{\ (1-t)z + tw\ :\ t \in [0,1]\}$ is contained in A.

Theorem 5.1.4 (Cauchy's theorem in a convex set). *Assume that $U \subset \mathbb{C}$ is a* ***convex*** *open set, $w_0 \in U$, f is continuous in U, and analytic in $U \setminus \{w_0\}$. Then f has a primitive in U.*

As a consequence (and by Theorem 4.3.4), if $\gamma\colon [a,b] \to U$ is a closed piecewise C^1-path, then

$$\int_\gamma f(z)\,dz = 0.$$

Proof. We will define the primitive $F\colon U \to \mathbb{C}$ with the following explicit formula. Fix $a \in U$ arbitrary, and set

$$F(z) := \int_{[a,z]} f(\zeta)\,d\zeta, \qquad z \in U.$$

The integral is well-defined, because $[a,z]^* \subset U$ by the convexity of U. We also note that

$$F(z) = -\int_{\overleftarrow{[a,z]}} f(\zeta)\,d\zeta = -\int_{[z,a]} f(\zeta)\,d\zeta, \qquad z \in U. \tag{5.1.7}$$

We then claim that F is complex differentiable for all $z \in U$ and that $F'(z) = f(z)$. Fix distinct points $z, w \in U$, and consider the triangle

$$\partial\triangle := [a,w] \star [w,z] \star [z,a],$$

whose trace is contained in U by convexity. In fact the entire solid triangle $\triangle$ is also contained in U – as required to apply Theorem 5.1.1. Then,

$$F(w)-F(z) \overset{(5.1.7)}{=} \int_{[a,w]} f(\zeta)\,d\zeta+\int_{[z,a]} f(\zeta)\,d\zeta = \int_{\partial\triangle} f(\zeta)\,d\zeta-\int_{[w,z]} f(\zeta)\,d\zeta.$$

The first term on the right vanishes by Theorem 5.1.1, so we may deduce that

$$\frac{F(w)-F(z)}{w-z} = -\frac{1}{w-z}\int_{[w,z]} f(\zeta)\,d\zeta = \frac{1}{z-w}\int_{[w,z]} f(\zeta)\,d\zeta.$$

On the other hand, it is a simple computation to check that

$$\frac{1}{z-w}\int_{[w,z]} c\,d\zeta = c, \qquad c\in\mathbb{C},$$

so in particular with $c := f(z)$ we have

$$\frac{F(w)-F(z)}{w-z} - f(z) = \frac{1}{z-w}\int_{[w,z]} [f(\zeta)-f(z)]\,d\zeta.$$

Since f is continuous at z, for every $\epsilon > 0$ there exists $\delta > 0$ such that $|f(\zeta) - f(z)| < \epsilon$ as soon as $|\zeta - z| < \delta$. In particular, this holds for all $\zeta \in [w, z]^*$ if $|w-z| < \delta$. Consequently, for $|w-z| < \delta$ we have the estimate

$$\left|\frac{F(w)-F(z)}{w-z} - f(z)\right| \le \frac{1}{|z-w|}\int_{[w,z]} |f(\zeta)-f(z)|\,|d\zeta| \le \|f-f(z)\|_{L^\infty([w,z])} \le \epsilon.$$

Since $\epsilon > 0$ was arbitrary, this shows that

$$\lim_{w\to z}\frac{F(w)-F(z)}{w-z} = f(z).$$

This completes the proof. $\square$

Remark 5.1.5. Is convexity necessary in Theorem 5.1.4? The answer is "yes and no". The answer is "no" in the sense that there is a more general theorem, the *global Cauchy theorem* which replaces the convexity assumption with strictly weaker hypotheses. This theorem will be covered in *Complex Analysis 2*.

The answer is "yes" in the sense that the convexity assumption cannot be simply removed. For example, consider the non-convex domain $U = \mathbb{C}\setminus\{0\}$, and let $\partial\triangle \subset \mathbb{C}\setminus\{0\}$ be a triangular path which surrounds the origin: thus the "solid triangle" $\triangle$ is **not** contained in $\mathbb{C}\setminus\{0\}$. Then, it turns out that

$$\int_{\partial\triangle}\frac{dz}{z} = 2\pi i,$$

even though $z \mapsto 1/z$ is analytic in $\mathbb{C}\setminus\{0\}$. It is an important exercise to think, why this fact does not contradict Theorem 5.1.1, and why the proof of Theorem 5.1.1 does not work in this situation.

We close this section by recording the following corollary of the proof of Theorem 5.1.4:

COROLLARY 5.1.6. *Let $U \subset \mathbb{C}$ be open, and assume that $f\colon U \to \mathbb{C}$ is a continuous function satisfying*

$$\int_{\partial\triangle} f(z)\,dz = 0, \qquad \triangle \subset U. \tag{5.1.8}$$

Then, f has a primitive in every open disc $D \subset U$ (or more generally in every convex open set $V \subset U$).

PROOF. Let $D \subset U$ be a disc. Then D is a convex open set, and (5.1.8) holds for all triangles $\triangle \subset D$. These were all the properties we needed in the proof of Theorem 5.1.4 to conclude that f has a primitive in D. □

REMARK 5.1.7. Let $f\colon U \to \mathbb{C}$ be continuous, where $U \subset \mathbb{C}$ is convex and open. Consider the following three properties:
(1) f is analytic in U.
(2) $\int_{\partial\triangle} f(z)\,dz = 0$ for all triangles $\triangle \subset U$.
(3) f has a primitive in U.
What are the implications between (1), (2), and (3)? Theorem 5.1.1 shows that (1) $\Longrightarrow$ (2), and Corollary 5.1.6 shows that (2) $\Longrightarrow$ (3). It turns out that also (3) $\Longrightarrow$ (1), see Theorem 5.3.1 (applied to the primitive). So, (1) – (3) are equivalent for convex open sets.

The convexity of U was used in both implications (1) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (3), but the converse implications are true for all open $U \subset \mathbb{C}$: namely, (3) $\Longrightarrow$ (2) by Theorem 4.3.4, and the implication (2) $\Longrightarrow$ (1) is known as *Morera's theorem*, which will also be proved soon.

5.2. Cauchy's integral formula

Our first application of Cauchy's theorem (Theorem 5.1.4) will be the following "representation formula" for analytic functions defined on a convex open set $U \subset \mathbb{C}$:

$$f(z) = \frac{1}{2\pi i}\int_{\partial D(z,r)} \frac{f(\zeta)}{\zeta - z}\,d\zeta, \qquad z \in U,$$

where $r > 0$ is so small that $D(z,r) \subset U$, and $\partial D(z,r)$ refers to the path $\gamma(t) = z + re^{it}$, $t \in [0, 2\pi]$. This formula, and a more general version of it recorded in Theorem 5.2.13, is known as *Cauchy's integral formula*. It will open the door (in Section 5.3) to establishing even stronger properties of analytic functions.

To prove Cauchy's integral formula in suitable generality, we begin with the definition of *winding numbers*.

DEFINITION 5.2.1 (Winding number). Let $\gamma\colon [a,b] \to \mathbb{C}$ be a closed piecewise C^1-path, and let $z \in \mathbb{C} \setminus \gamma^*$. The *winding number of γ around z* is defined by

$$n_\gamma(z) := \frac{1}{2\pi i} \int_\gamma \frac{d\zeta}{\zeta - z}.$$

EXAMPLE 5.2.2. Let $\gamma_k\colon [0, 2\pi] \to \mathbb{C}$ be the circle path $\gamma_k(t) := z_0 + re^{ikt}$, where $k \in \mathbb{Z}$. In Example 4.2.6 we computed that

$$n_{\gamma_k}(z_0) = \frac{1}{2\pi i} \int_{\gamma_k} \frac{dz}{z - z_0} = k. \tag{5.2.1}$$

Thus, the winding number $n_{\gamma_k}(z_0)$ captures the intuition that γ_k "winds $|k|$ times around z_0". The sign of the winding number tells us whether the "winding" happens clockwise (case $k < 0$) of counterclockwise (case $k > 0$).

The following property of winding numbers is elementary but useful:

PROPOSITION 5.2.3. *Let $\gamma\colon [a,b] \to \mathbb{C}$ and $\eta\colon [c,d] \to \mathbb{C}$ be closed piecewise C^1-paths such that $\gamma(b) = \eta(c)$. Then,*

$$n_{\gamma \star \eta}(z) = n_\gamma(z) + n_\eta(z) \quad \textit{and} \quad n_\gamma(z) = -n_{\overleftarrow{\gamma}}(z)$$

for all $z \in \mathbb{C} \setminus [\gamma^* \cup \eta^*]$ (just $z \in \mathbb{C} \setminus \gamma^*$ for the second claim).

PROOF. Exercise. □

Amazingly, it turns out that the winding number is always an integer:

THEOREM 5.2.4. *Let $\gamma\colon [a,b] \to \mathbb{C}$ be a closed piecewise C^1-path. Then the map $z \mapsto n_\gamma(z)$, defined on $\mathbb{C} \setminus \gamma^*$, is continuous and $\mathbb{Z}$-valued: $n_\gamma(z) \in \mathbb{Z}$ for all $z \in \mathbb{C} \setminus \gamma^*$.*

We need the following lemma in the proof:

LEMMA 5.2.5. *Let $\eta\colon [a,b] \to \mathbb{R}$ be a continuous function which is differentiable at all points $t \in [a,b] \setminus X$, where $X \subset [a,b]$ is finite. If $\eta'(t) = 0$ for all $t \in [a,b] \setminus X$, then η is a constant, and in particular $\eta(b) = \eta(a)$.*

PROOF. Since X is finite, we may write $(a,b) \setminus X$ as a finite union of disjoint open intervals $I_1, \ldots, I_n$. It follows from the assumptions that η is a constant on each interval individually, say $\eta \equiv c_j$ on I_j. By continuity, if I_j, I_{j+1} are adjacent intervals with $t \in \overline{I}_j \cap \overline{I}_{j+1}$, then $c_j = \eta(t) = c_{j+1}$. It follows that all the constants c_j are actually the same, and by another appeal to continuity they equal $c_1 = \eta(a)$. □

Proof of Theorem 5.2.4. The continuity of n_γ is fairly clear from the definition: if $(z_k) \subset \mathbb{C} \setminus \gamma^*$ is a sequence of points converging to a point $z \in \mathbb{C} \setminus \gamma^*$, then

$$\lim_{k\to\infty} n_\gamma(z_k) = \lim_{k\to\infty} \frac{1}{2\pi i} \int_\gamma \frac{d\zeta}{\zeta - z_k} = \frac{1}{2\pi i} \int_\gamma \frac{d\zeta}{\zeta - z} = n_\gamma(z). \tag{5.2.2}$$

Thus n_γ satisfies the criterion for continuity in terms of sequences. It takes a little effort to show that the exchange of limits and integration in (5.2.2) is legitimate: one has to check that the functions $\zeta \mapsto f_k(\zeta) := (\zeta - z_k)^{-1}$ converge uniformly on γ^* to $\zeta \mapsto f(\zeta) := (\zeta - z)^{-1}$, and then apply Corollary 4.2.14. To check the uniform convergence on γ^*, note that $\mathrm{dist}(z, \gamma^*) =: r > 0$, so also $\mathrm{dist}(z_k, \gamma^*) \geq r/2$ for all $k \in \mathbb{N}$ sufficiently large.

We then show that n_γ is $\mathbb{Z}$-valued. Recall from Proposition 1.5.7 that $e^\alpha = 1$ if and and only if $\alpha \in 2\pi i\mathbb{Z}$. Thus, to prove that $n_\gamma(z) \in \mathbb{Z}$, it suffices to show that

$$e^{2\pi i n_\gamma(z)} = 1, \qquad z \in \mathbb{C} \setminus \gamma^*.$$

Furthermore, if we write down the definition of the path integral appearing in $n_\gamma(z)$, this claim is equivalent to $\varphi(b) = 1$, where

$$\varphi(t) := \exp\left(\int_a^t \frac{\gamma'(s)}{\gamma(s) - z}\, ds\right), \qquad t \in [a, b]. \tag{5.2.3}$$

(We are suppressing the point "z" from the notation.) To prove (5.2.3), start by noting that

$$\varphi'(t) = \frac{\gamma'(t)}{\gamma(t) - z} \exp\left(\int_a^t \frac{\gamma'(s)}{\gamma(s) - z}\, dz\right) = \frac{\gamma'(t)}{\gamma(t) - z}\varphi(t),$$

or equivalently

$$\varphi'(t)(\gamma(t) - z) - \gamma'(t)\varphi(t) = 0 \tag{5.2.4}$$

for all $t \in [a, b]$ where γ is differentiable. Since γ is piecewise C^1, this is true for all $t \in [a, b] \setminus X$, where $X \subset [a, b]$ is a finite set. Now, formula (5.2.4) shows that $\eta(t) := \varphi(t)/(\gamma(t) - z)$ defines a continuous map $[a, b] \to \mathbb{C}$, differentiable outside X, and satisfying

$$\eta'(t) = \frac{\varphi'(t)(\gamma(t) - z) - \gamma'(t)\varphi(t)}{(\gamma(t) - z)^2} = 0, \qquad t \in [a, b] \setminus X.$$

By Lemma 5.2.5 (applied separately to the real and imaginary parts of η), this shows that

$$\frac{\varphi(b)}{\gamma(b) - z} = \eta(b) = \eta(a) = \frac{\varphi(a)}{\gamma(a) - z} = \frac{1}{\gamma(a) - z},$$

noting that $\varphi(a) = e^0 = 1$ by the definition (5.2.3). Now, it remains to use the assumption that γ is a closed path, so $\gamma(b) = \gamma(a)$:

$$\varphi(b) = \frac{\gamma(b) - z}{\gamma(a) - z} = \frac{\gamma(a) - z}{\gamma(a) - z} = 1,$$

as claimed. The proof is complete. $\square$

REMARK 5.2.6. The proof of Theorem 5.2.4 was a little magical. The following heuristics may make it more transparent. Assume (for the sake of the discussion) that $z = 0$, and there exists a branch of the logarithm "g" (in other words: a primitive of $\zeta \mapsto 1/\zeta$) in some open set U containing γ^*. In this case,

$$\int_a^t \frac{\gamma'(s)}{\gamma(s) - z}\,ds = \int_{\gamma|_{[a,t]}} \frac{d\zeta}{\zeta} \overset{\text{Thm. 4.3.4}}{=} g(\gamma(t)) - g(\gamma(a)). \tag{5.2.5}$$

Consequently, the map "φ" appearing in the proof has the simple expression

$$\varphi(t) = e^{g(\gamma(t)) - g(\gamma(a))} = \frac{e^{g(\gamma(t))}}{e^{g(\gamma(a))}} = \frac{\gamma(t)}{\gamma(a)}.$$

In particular, since γ is a closed path, we have $\varphi(b) = 1$, as desired.

The assumption $z = 0$ is innocent, but the existence of "g" is not: indeed, the existence of "g" would actually show that $\int_\gamma d\zeta/\zeta = 0$ (by (5.2.5)), which is generally false. The point of the actual proof is that even though "g" may not exist, the map φ still behaves in the same manner as in the "cheat" proof above.

What do continuous $\mathbb{Z}$-valued functions actually look like?

PROPOSITION 5.2.7. *Let $U \subset \mathbb{C}$ be open and connected, and let $g \colon U \to \mathbb{R}$ be continuous such that $g(U) \subset \mathbb{Z}$. Then g is constant.*

PROOF. Exercise. $\square$

COROLLARY 5.2.8. *Let $\gamma \colon [a, b] \to \mathbb{C}$ be a closed piecewise C^1-path, and let $U \subset \mathbb{C} \setminus \gamma^*$ be open and connected. Then n_γ is constant on U. If U is unbounded, this constant is 0.*

PROOF. By Theorem 5.2.4, the map $z \mapsto g(z) = n_\gamma(z)$ is continuous and $\mathbb{Z}$-valued in U, so the constancy on U follows immediately from the previous proposition.

Assume then that U is unbounded, and let $(z_k) \subset U$ be a sequence with $|z_k| \to \infty$ as $k \to \infty$. Let $m \in \mathbb{Z}$ be the constant value of n_ζ on U. Then,

$$m = \lim_{k \to \infty} n_\gamma(z_k) = \lim_{k \to \infty} \frac{1}{2\pi i} \int_\gamma \frac{d\zeta}{\zeta - z_k} = 0,$$

because the functions $\zeta \mapsto (\zeta - z_k)^{-1}$ converge uniformly to 0 on γ^*, as $z_k \to \infty$. $\square$

REMARK 5.2.9. In general, it can be tricky to find out the constant value of $z \mapsto n_\gamma(z)$ in a given (bounded, open, connected) set $U \subset \mathbb{C} \setminus \gamma^*$. Tools for this problem will be presented on the course *Complex Analysis 2*.

EXAMPLE 5.2.10 (Winding numbers of circle paths). Recall from (5.2.1) that

$$n_{\gamma_k}(z_0) = \frac{1}{2\pi i}\int_{\gamma_k} \frac{dz}{z - z_0} = k,$$

where $z_0 \in \mathbb{C}$, and $\gamma_k(t) = z_0 + re^{ikt}$. From this fact and Corollary 5.2.8, we may deduce that

$$n_{\gamma_k}(w) = \frac{1}{2\pi i}\int_{\gamma_k} \frac{dz}{z - w} = k, \qquad w \in D(z_0, r),$$

since $D(z_0, r) \subset \mathbb{C} \setminus \gamma_k^*$ is connected. Thus, we have rigorously proven the "intuitively clear" fact that the path γ_k winds the same number of times around z_0, and every point $w \in D(z_0, r)$.

For $w \in \mathbb{C} \setminus \bar{D}(z_0, r)$, we have $n_{\gamma_k}(w) = 0$ by Corollary 5.2.8, because $\mathbb{C} \setminus \bar{D}(z_0, r)$ is an unbounded connected set contained in $\mathbb{C} \setminus \gamma_k^*$. We repeat and record these important conclusions:

$$\eta_{\gamma_k}(w) = \begin{cases} k, & w \in D(z_0, r), \\ 0, & w \in \mathbb{C} \setminus \bar{D}(z_0, r). \end{cases} \tag{5.2.6}$$

With the techniques of this course, we are not able to compute the winding numbers of very general closed paths γ. However, the following simple proposition is surprisingly useful:

PROPOSITION 5.2.11. *Let $\gamma, \eta\colon [0,1] \to \mathbb{C}$ be closed piecewise C^1-paths such that $\gamma(0) = \eta(0)$. Let $z \in \mathbb{C} \setminus [\gamma^* \cup \eta^*]$, and assume that z lies in an unbounded connected open set $U \subset \mathbb{C} \setminus \eta^*$. Then,*

$$n_{\gamma\star\eta}(z) = n_\gamma(z).$$

PROOF. We have $n_{\gamma\star\eta}(z) = n_\gamma(z) + n_\eta(z)$ by Proposition 5.2.3, and further $n_\eta(z) = 0$ by Corollary 5.2.8. $\square$

The proposition can for example be applied as follows:

EXAMPLE 5.2.12. Let $\gamma := [-1,1] \star \sigma^+$ be the "upper semicircle", where $\sigma^+(t) = e^{it}$ for $t \in [0, \pi]$. It is intuitively clear that γ winds exactly once around the point $z = i/2$, thus $n_\gamma(i/2) = 1$. We can deduce this from Proposition 5.2.11 as follows.

Let $\eta := \sigma^- \star \overleftarrow{[-1,1]}$ be the "lower semicircle", where $\sigma^-(t) = e^{it}$ for $t \in [\pi, 2\pi]$. Then $\gamma \star \eta$ is the standard circle path $\partial D(0,1)$ (to be precise: a reparametrisation of of $\partial D(0,1)$). Thus,

$$n_{\gamma\star\eta}(i/2) = 1.$$

On the other hand, $i/2$ clearly lies in some unbounded connected set $U \subset \mathbb{C} \setminus \eta^*$, for example $U := \{\mathrm{Im}\,(z) > 0\}$. Thus $n_\gamma(i/2) = n_{\gamma\star\eta}(i/2) = 1$ by Proposition 5.2.11.

We then arrive at the key result of this section:

THEOREM 5.2.13 (Cauchy's integral formula in a convex set). *Let $U \subset \mathbb{C}$ be a convex open set, let $\gamma\colon [a,b] \to U$ be a closed piecewise C^1-path, and let $f\colon U \to \mathbb{C}$ be analytic. Then,*

$$f(z) \cdot n_\gamma(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta, \qquad z \in U \setminus \gamma^*.$$

PROOF. Fix $z \in U \setminus \gamma^*$, and consider the function $g\colon U \to \mathbb{C}$ defined by

$$g(\zeta) := \begin{cases} \frac{f(\zeta)-f(z)}{\zeta - z}, & \zeta \in U \setminus \{z\}, \\ f'(z), & \zeta = z. \end{cases}$$

Clearly g is analytic in $U \setminus \{z\}$, and also continuous in U, because f is differentiable at z. Consequently, g satisfies the hypotheses of Cauchy's theorem in a convex set, Theorem 5.1.4. The conclusion is that

$$0 = \frac{1}{2\pi i}\int_\gamma g(\zeta)\,d\zeta = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)-f(z)}{\zeta - z}\,d\zeta. \tag{5.2.7}$$

Now, it suffices to note that

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{\zeta - z}\,d\zeta = f(z)\cdot n_\gamma(z). \tag{5.2.8}$$

This completes the proof. It is worth remarking that the hypothesis $z \in U \setminus \gamma^*$ was not needed for (5.2.7), but it was used when passing to (5.2.8) (to make sure that the integrals are individually well-defined). □

Let us explicitly combine the theorem with the computation in (5.2.6):

COROLLARY 5.2.14 (Cauchy's integral formula in a disc). *Let $U \subset \mathbb{C}$ be open, and assume that $\bar{D}(z_0, r) \subset U$. Then,*

$$f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)}{\zeta - z}\,d\zeta, \qquad z \in D(z_0, r), \tag{5.2.9}$$

where ∂D is an abbreviation for the path $\gamma(t) = z_0 + re^{it}$, $t \in [0, 2\pi]$.

PROOF. It is easy to check that there exists a convex open set $V \subset \mathbb{C}$ satisfying $\bar{D}(z_0, r) \subset V \subset U$ (e.g. $V = D(z_0, r + \epsilon)$). Therefore, (5.2.9) is an immediate consequence of Theorem 5.2.13 applied to V – recalling also from (5.2.6) that $n_{\partial D}(z) = 1$ for all $z \in D(z_0, r)$. □

5.2.1. Computing integrals with Cauchy's formula. Cauchy's formula yields a powerful method for calculating complicated integrals – even ones which apparently have nothing to do with complex numbers. This method will be treated more systematically in the *calculus of residues*, see *Complex Analysis 2*, but we give two illustrative examples.

EXAMPLE 5.2.15. Let us compute the integral

$$\int_{\partial D(0,2)} \frac{e^z}{z^2-1}\, dz. \tag{5.2.10}$$

This integral is not directly of the form recognisable from Cauchy's integral formula, but it can be easily brought into such a form. We begin by observing that $z^2-1=(z-1)(z+1)$. Once this has been noted, it is well-known that we always have a decomposition

$$\frac{1}{z^2-1} = \frac{A}{z-1} + \frac{B}{z+1} \tag{5.2.11}$$

for suitable coefficients $A, B \in \mathbb{C}$. The way to find these coefficients is to write

$$\frac{A}{z-1} + \frac{B}{z+1} = \frac{(z+1)A+(z-1)B}{(z-1)(z+1)} = \frac{(z+1)A+(z-1)B}{z^2-1}.$$

After this, it is evident that (5.2.11) holds if and only if

$$(A-B)+z(A+B) = (z+1)A+(z-1)B = 1 \quad \iff \quad \begin{cases} A-B=1, \\ A+B=0. \end{cases}$$

Thus, (5.2.11) is true for $A=\frac{1}{2}$ and $B=-\frac{1}{2}$. After this, we may decompose the integral (5.2.10) as

$$\int_{\partial D(0,2)} \frac{e^z}{z^2-1}\, dz = \tfrac{1}{2}\int_{\partial D(0,2)} \frac{e^z}{z-1}\, dz - \tfrac{1}{2}\int_{\partial D(0,2)} \frac{e^z}{z+1}\, dz. \tag{5.2.12}$$

Since

$$n_{\partial D(0,2)}(1) = n_{\partial D(0,2)}(-1),$$

it follows from Cauchy's integral formula, Theorem 5.2.13, that

$$\frac{1}{2\pi i}\int_{\partial D(0,2)} \frac{e^z}{z-1}\, dz = e^1 = e \quad \text{and} \quad \frac{1}{2\pi i}\int_{\partial D(0,2)} \frac{e^z}{z-1}\, dz = e^{-1}.$$

Comparing this with (5.2.12), we find the solution:

$$\int_{\partial D(0,2)} \frac{e^z}{z^2-1}\, dz = \pi i(e-e^{-1}).$$

In the next example, we compute an indefinite integral on the real line:

EXAMPLE 5.2.16. Let us compute the integral

$$\int_{\mathbb{R}} \frac{dt}{t^2+1} := \lim_{r\to\infty} \int_{-r}^{r} \frac{dt}{t^2+1}. \tag{5.2.13}$$

The correct answer π can also be obtained with "real-variable" methods, but now we will see how to use the Cauchy integral formula. Note that $g(z) := (z^2+1)^{-1}$ defines an analytic function $g\colon \mathbb{C}\setminus\{-i,i\} \to \mathbb{C}$. Moreover, this function can be factorised as

$$g(z) = \frac{f(z)}{z-i}, \qquad f(z) = \frac{1}{z+i},$$

and $f\colon \mathbb{C}\setminus\{-i\} \to \mathbb{C}$ is analytic. Let $U := \{\operatorname{Im} z > -1\}$. Then U is a convex open set which contains $\mathbb{R}$, the point i, and with the property that f is analytic in U. Now, if $\gamma\colon [a,b] \to U$ is a closed piecewise C^1-path, we may deduce from Cauchy's theorem that

$$\frac{n_\gamma(i)}{2i} = f(i)\cdot n_\gamma(i) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-i}\,dz = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z^2+1}. \tag{5.2.14}$$

This looks quite promising with (5.2.13) in mind. To wrap up, we need to choose γ suitably.

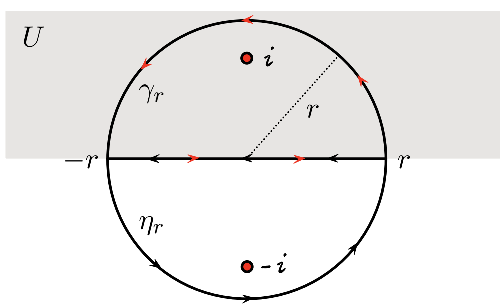


FIGURE 3. The paths γ_r (red arrows) and η_r (black arrows).

Since we are aiming for (5.2.13), the path $\gamma = \gamma_r$ should at least parametrise the interval $[-r,r]$. However, since γ_r needs to be a closed path, we need to decide some way of "closing" $[-r,r]$. The formula (5.2.14) suggests that we might wish to choose γ_r in such a way that $n_{\gamma_r}(i) = 1$ (although you are welcome to try other possibilities). A standard choice is the path

$$\gamma_r := [-r,r] \star \sigma_r,$$

where σ_r parametrises the large semi-circle $S(0,r)\cap\{\operatorname{Im} z \geq 0\}$ connecting r to $-r$ in the upper half-plane, see Figure 3. Explicitly, $\sigma_r(t) = re^{it}$ for $r \in [0,\pi]$. Clearly $\gamma^* \subset U$, so

$$\frac{1}{2\pi i}\int_{-r}^{r} \frac{dt}{t^2+1} = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z^2+1} - \frac{1}{2\pi i}\int_{\sigma_r} \frac{dz}{z^2+1} \overset{(5.2.14)}{=} \frac{n_\gamma(i)}{2i} - \frac{1}{2\pi i}\int_{\sigma_r} \frac{dz}{z^2+1}.$$

It is intuitively clear that $n_{\gamma_r}(i) = 1$ for $r > 1$, and this can be rigorously justified by the trick shown in Example 5.2.12, using Proposition 5.2.11.

Now, we claim that

$$\lim_{r\to\infty} \int_{\sigma_r} \frac{dz}{z^2+1} = 0. \tag{5.2.15}$$

Once this has been established, we find that

$$\int_{\mathbb{R}} \frac{dt}{t^2+1} = \lim_{r\to\infty} \int_{-r}^{r} \frac{dt}{t^2+1} = \lim_{r\to\infty} 2\pi i \cdot \frac{n_{\gamma_r}(i)}{2i} = \pi. \tag{5.2.16}$$

The equation (5.2.15) is obtained with "brute force", using the comparison between path integrals and arc length integrals in Proposition 4.2.10:

$$\left|\int_{\sigma_r} \frac{dz}{z^2+1}\right| \le \int_0^{\pi} \frac{|\sigma_r'(t)|}{|\sigma_r(t)^2+1|}\, dt = \int_0^{\pi} \frac{r\, dt}{|\sigma_r(t)^2+1|}\, dt.$$

Now, note that $\sigma_r(t)$ ranges in the circle $S(0,r)$, so $|\sigma_r(t)^2+1| \ge |\sigma_r(t)|^2 - 1 \ge r^2/2$ for every sufficiently large $r > 1$. Therefore,

$$\lim_{r\to\infty} \left|\int_{\sigma_r} \frac{dz}{z^2+1}\right| \le \lim_{r\to\infty} \int_0^{\pi} \frac{2\, dt}{r} = \lim_{r\to\infty} \frac{2\pi}{r} = 0.$$

This completes the justification of (5.2.16).

5.3. Applications

5.3.1. Derivatives of analytic functions. Our first application of Theorem 5.2.13 shows that if f is analytic in an open set U, then f is infinitely differentiable in U.

THEOREM 5.3.1. *Let $U \subset \mathbb{C}$ be open, and let $f\colon U \to \mathbb{C}$ be analytic. Then also $f'\colon U \to \mathbb{C}$ is analytic. As a consequence, the n^{th} complex derivative $f^{(n)}$ exists and is analytic for all $n \ge 0$.*

PROOF. We first claim the following. Let $z \in U$, and let $r > 0$ be so small that $\bar{D}(z,r) \subset U$. Then,

$$f'(w) = \frac{1}{2\pi i} \int_{\partial D(z,r)} \frac{f(\zeta)}{(\zeta - w)^2}\, d\zeta, \qquad w \in D(z,r). \tag{5.3.1}$$

Here $\partial D(z,r)$ refers to the circle path $\gamma(t) = z + re^{it}$, $t \in [0,2\pi]$. We start by applying the Cauchy integral formula in a disc, Corollary 5.2.14, to write

$$f(w) = \frac{1}{2\pi i} \int_{\partial D(z,r)} \frac{f(\zeta)}{\zeta - w}\, d\zeta, \qquad w \in D(z,r).$$

Now, morally the formula (5.3.1) follows by taking ∂_w-derivatives on both sides of the formula above:

$$f'(w) \overset{?}{=} \frac{1}{2\pi i} \int_{\partial D(z,r)} \partial_w \left(\frac{1}{\zeta - w}\right) f(\zeta)\, d\zeta = \frac{1}{2\pi i} \int_{\partial D(z,r)} \frac{f(\zeta)}{(\zeta - w)^2}\, d\zeta.$$

We placed a question mark on the first equation, since one needs to make sure that one is allowed to "differentiate under the integral sign": one needs to show that the function $F\colon D(z,r) \to \mathbb{C}$ defined by

$$F(w) := \int_{\partial D(z,r)} \frac{f(\zeta)}{\zeta - w} d\zeta, \qquad w \in D(z,r),$$

is analytic, and its ∂_w-derivative is given by

$$F'(w) = \int_{\partial D(z,r)} \frac{f(\zeta)}{(\zeta - w)^2}\, d\zeta. \tag{5.3.2}$$

We prove this using the definition of complex differentiability. Abbreviate $\partial D(z,r) =: \partial D$, fix $w \in D$, and start by writing

$$\frac{F(v) - F(w)}{v - w} = \frac{1}{v-w} \int_{\partial D} \left[\frac{1}{\zeta - v} - \frac{1}{\zeta - w}\right] f(\zeta)\, d\zeta = \int_{\partial D} \frac{f(\zeta)\, d\zeta}{(\zeta - v)(\zeta - w)},$$

for $v \in D(z,r)$. Consequently,

$$\begin{aligned} \frac{F(v) - F(w)}{v - w} - \int_{\partial D} \frac{f(\zeta)}{(\zeta - w)^2}\, d\zeta &= \int_{\partial D} \left[\frac{1}{(\zeta - v)(\zeta - w)} - \frac{1}{(\zeta - w)^2}\right] f(\zeta)\, d\zeta \\ &= (v - w) \int_{\partial D} \frac{f(\zeta)\, d\zeta}{(\zeta - v)(\zeta - w)^2}. \end{aligned}$$

Finally, recall that $w \in D(z,r)$, so in particular $\epsilon := \operatorname{dist}(w, \partial D) > 0$. Now, $v \in D(w, \frac{\epsilon}{2})$, we have $|\zeta - v| \geq \epsilon/2$ for all $\zeta \in \partial D$. Consequently,

$$\left| \frac{F(v) - F(w)}{v - w} - \int_{\partial D} \frac{f(\zeta)}{(\zeta - w)^2}\, d\zeta \right| \leq |v - w| \int_{\partial D} \frac{\|f\|_{L^\infty(D)}\, |d\zeta|}{|\zeta - v||\zeta - w|^2} \leq C_{D,f,\epsilon}|v - w|.$$

Here $C_{D,f,\epsilon} > 0$ is a constant depending only on D, f, ϵ. An explicit choice which works is $C_{D,f,r} = 2\epsilon^{-3}\|f\|_{L^\infty(D)}\operatorname{length}(\partial D)$. Letting $v \to z$ proves (5.3.2) at $w = z$.

This is not quite the end of the story: we have now proved the nice representation formula (5.3.2) for f', but this does not immediately say that f' is analytic. However, one can now show that

$$G(w) := \int_{\partial D} \frac{f(\zeta)}{(\zeta - w)^2}\, d\zeta$$

defines an analytic function in $D(z,r)$, and

$$f''(w) = \frac{1}{2\pi i} G'(w) = \frac{1}{\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta - w)^3}\, d\zeta. \tag{5.3.3}$$

Thus, the analyticity of f' follows by explicitly differentiating $G = 2\pi i f'$. Heuristically, the formula (5.3.3) for G' follows (again) from differentiation under the integral sign, and the careful justification requires calculations very similar to those we have just seen above. We leave the details as a voluntary exercise. $\square$

The formula (5.3.1) is "Cauchy's integral formula for f' in a disc". We next record a more general version of this formula for closed piecewise C^1-paths, and also for higher-order derivatives. This generalisation is very useful (see the schematic at the beginning of Section 5.1).

THEOREM 5.3.2 (Cauchy's integral formula for derivatives). *Let $U \subset \mathbb{C}$ be a convex open set, and let $f\colon U \to \mathbb{C}$ be analytic. Let $\gamma\colon [a,b] \to U$ be a closed piecewise C^1-path. Then,*

$$f^{(n)}(z) \cdot n_\gamma(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta, \qquad z \in U \setminus \gamma^*,\ n \geq 0. \tag{5.3.4}$$

PROOF. This theorem could be established by a "brute force" approach, using the Cauchy integral formula, and differentiating n times under the integral sign. Ignoring all technical difficulties, the proof can be compressed to the following line:

$$f^{(n)}(z) \cdot n_\gamma(z) = \frac{1}{2\pi i} \int_\gamma \partial_w^{(n)} \left(\frac{1}{\zeta - w} \right) \Big|_{w=z} f(\zeta)\, d\zeta = \frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - w)^{n+1}}\, d\zeta,$$

because $\partial_w^{(n)}(\zeta - w)^{-1} = n!(\zeta - w)^{-n-1}$. The main challenge of this approach would to justify the differentiation under the integral sign. This would be a little tedious, but fortunately there is a more elegant path.

We prove the claim by induction on n, the case $n = 0$ being Cauchy's integral formula, Theorem 5.2.13. Assume, then, that the formula (5.3.4) has been established for all analytic functions in U, for some fixed $n \geq 0$, and all $z \in U \setminus \gamma^*$. By Theorem 5.3.1, the derivative $f'\colon U \to \mathbb{C}$ is analytic, so in particular our induction hypothesis applies to f':

$$f^{(n+1)}(z) \cdot n_\gamma(z) = (f')^{(n)}(z) \cdot n_\gamma(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f'(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta, \qquad z \in U \setminus \gamma^*.$$

The left hand side looks good, but the right hand side still requires processing. We claim that

$$\int_\gamma \frac{f'(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta = (n+1) \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+2}}\, d\zeta, \qquad z \in U \setminus \gamma^*. \tag{5.3.5}$$

To prove this, fix $z \in U \setminus \gamma^*$, and consider the function $g\colon U \setminus \{z\} \to \mathbb{C}$, defined by

$$g(\zeta) := \frac{f(\zeta)}{(\zeta - z)^{n+1}}, \qquad \zeta \in U \setminus \{z\}.$$

Then g is clearly analytic in $U \setminus \{z\}$, and

$$g'(\zeta) = \frac{f'(\zeta)}{(\zeta - z)^{n+1}} - \frac{(n+1) f(\zeta)}{(\zeta - z)^{n+2}}, \qquad \zeta \in U \setminus \{z\}.$$

Now g' is continuous in $U \setminus \{z\}$ and has a primitive in $U \setminus \{z\}$ (namely g), so Theorem 4.3.4 implies that

$$0 = \int_\gamma g'(\zeta)\, d\zeta = \int_\gamma \frac{f'(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta - (n+1) \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+2}}\, d\zeta.$$

This is equivalent to (5.3.5), so the proof is complete. □

5.3.2. Cauchy's estimates and Liouville's theorem. Note that the formula (5.3.4) allows (in principle) you to compute $f^{(n)}$ by integrating f. After seeing this, it is hardly surprising that the size of $f^{(n)}$ can also be estimated by the size of f:

COROLLARY 5.3.3 (Cauchy's estimates). *Let $D = D(z, r) \subset \mathbb{C}$ be a disc, and let $f\colon D \to \mathbb{C}$ be analytic. Then,*

$$|f^{(n)}(w)| \le \frac{n! \cdot r \cdot \|f\|_{L^\infty(D)}}{(r - |w - z|)^{n+1}}, \qquad w \in D,\ n \ge 0. \tag{5.3.6}$$

In particular, $|f^{(n)}(z)| \le n! \|f\|_{L^\infty(D)}/r^n$.

PROOF. Let $0 < s < r$, and let ∂D_s be the usual circle path parametrising the boundary of $D_s := D(z, s)$. Then f is analytic in the convex open set $D = D(z, r)$ containing the trace of ∂D_s, so we may infer from (5.3.4) that

$$|f^{(n)}(w)| = \frac{n!}{2\pi} \left| \int_{\partial D_s} \frac{f(\zeta)\, d\zeta}{(\zeta - w)^{n+1}} \right| \le \frac{n! \cdot \|f\|_{L^\infty(D)}}{2\pi} \int_{\partial D_s} \frac{|d\zeta|}{|\zeta - w|^{n+1}}.$$

Now, note that $|\zeta - w| \ge |\zeta - z| - |w - z| = s - |w - z|$ for all $\zeta \in \partial D_s$, so

$$|f^{(n)}(w)| \le \frac{n! \cdot \|f\|_{L^\infty(D)} \cdot \mathrm{length}(\partial D_s)}{2\pi (s - |w - z|)^{n+1}}.$$

Since $\mathrm{length}(\partial D_s) = 2\pi s$, the estimate (5.3.6) now follows by letting $s \nearrow r$. □

Sometimes a weaker theorem may appear more surprising than a stronger one. If Cauchy's estimates failed to impress you, perhaps the next corollary of them does:

COROLLARY 5.3.4 (Liouville's theorem). *Let $f\colon \mathbb{C} \to \mathbb{C}$ be analytic and bounded. Then f is constant.*

PROOF. Fix $z \in \mathbb{C}$, and apply Cauchy's estimates in a disc $D(z, r) \subset \mathbb{C}$:

$$|f'(z)| \le \frac{\|f\|_{L^\infty(\mathbb{C})}}{r}, \qquad r > 0.$$

Letting $r \to \infty$ shows that $f'(z) = 0$, and therefore $f' \equiv 0$. Since $\mathbb{C}$ is connected, f is constant by Corollary 3.3.1. □

Remark 5.3.5. Liouville's theorem is deep and surprising, but an even stronger result is true: if $f\colon \mathbb{C} \to \mathbb{C}$ is analytic, then either f is constant, or then f takes all the values in $\mathbb{C}$, except for possibly one. This result is known as *(little) Picard's theorem*. The example $f(z) = e^z$ shows that Picard's theorem is is sharp, since $e^z \neq 0$ for all $z \in \mathbb{C}$.

5.3.3. The fundamental theorem of algebra. We saw early on in the course that the equation $z^n - w = 0$ has a solution $z \in \mathbb{C}$ (indeed n solutions) for all $w \in \mathbb{C}$. The same remains true for all non-constant polynomial equations $p(z) = 0$, and this result is known as the *fundamental theorem of algebra*. Rather unexpectedly, its proof is based Liouville's theorem!

Corollary 5.3.6 (Fundamental theorem of algebra). *Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$ be a polynomial, where $n \geq 1$ and $a_n \neq 0$ (so that p is non-constant). Then there exists $z \in \mathbb{C}$ such that $p(z) = 0$.*

Proof. We argue by contradiction and suppose that $p(z) \neq 0$ for all $z \in \mathbb{C}$. Then $f(z) := p(z)^{-1}$ defines an analytic function on $\mathbb{C}$ by Proposition 3.1.7. We claim that f is bounded. To see this, note that

$$|p(z)| = |z|^n \left| a_n + \frac{a_{n-1}}{z} + \ldots + \frac{a_0}{z^n} \right| \geq |z|^n \left(|a_n| - \frac{|a_{n-1}|}{|z|} - \ldots - \frac{|a_0|}{|z|} \right) \to \infty,$$

as $|z| \to \infty$, using the assumption $a_n \neq 0$. In particular, there exists $R > 0$ (depending on $a_0, \ldots, a_n$) such that we have $|p(z)| \geq 1$ for $|z| \geq R$. Thus

$$|f(z)| \leq 1, \qquad |z| \leq R.$$

On the other hand, since f is continuous and $\bar{D}(0, R)$ is compact, Corollary 2.3.9 shows that there is $M > 0$ such that

$$|f(z)| \leq M, \qquad |z| \leq R.$$

Putting these bounds together, we see that

$$|f(z)| \leq \max\{M, 1\}, \qquad z \in \mathbb{C}.$$

By Liouville's theorem (Corollary 5.3.4), we may now deduce that f is constant. This constant is non-zero, since $f(z) = p(z)^{-1} \neq 0$ for all $z \in \mathbb{C}$. So, $f \equiv \alpha \neq 0$. But then $p \equiv \alpha^{-1}$, which contradicts the fact that $|p(z)| \to \infty$ as $|z| \to \infty$. $\square$

Remark 5.3.7. Corollary 5.3.6 only appears to give us **one** point $z \in \mathbb{C}$ with $p(z) = 0$. This may seem disappointing, since we already knew that $z^n - w$ has n distinct solutions if $w \neq 0$. In fact, Corollary 5.3.6 yields a similar conclusion, as we now explain.

Assume that p is a polynomial of degree $n \geq 1$. Thus, there exists $z_1 \in \mathbb{C}$ such that $p(z_1) = 0$. Now, it is a fact from abstract algebra we can factorise

$p(z) = p_1(z)(z - z_1)$, where p_1 is a polynomial of degree $n - 1$. If $n - 1 \geq 1$, we may reapply Corollary 5.3.6 to p_1. Then $p(z) = (z_1 - z)(z_2 - z)p_2(z)$, where p_2 is a polynomial of degree $n - 2$.

We may repeat the same argument precisely n times. This way we eventually find a factorisation $p(z) = c(z - z_1) \cdots (z - z_n)$, where $p(z_j) = 0$ for all $1 \leq j \leq n$, and $c \in \mathbb{C} \setminus \{0\}$. Thus, Corollary 5.3.6 allows us to find not only one zero for p, but precisely n zeros.

However, it is entirely possible that $z_i = z_j$ for some $i \neq j$, or perhaps even $z_i = z_j$ for **all** $1 \leq i, j \leq n$. In this case $p(z) = c(z - z_1)^n$, and we say that p has a *zero of order n at z_1*. In general, a degree n polynomial p can have any number $k \in \{1, \ldots, n\}$ of **distinct** zeroes in $z_1, \ldots, z_k \in \mathbb{C}$, but the sum of their orders always equals n.

5.3.4. Morera's theorem and analytic continuation to a point. Before reading the next corollary, it is advisable to recap Corollary 5.1.6.

Corollary 5.3.8 (Morera's theorem). *Let $U \subset \mathbb{C}$ be open, and let $f \colon U \to \mathbb{C}$ be a continuous function satisfying*

$$\int_{\partial \triangle} f(z)\, dz = 0 \tag{5.3.7}$$

for all triangles $\triangle \subset U$. Then f is analytic in U.

Proof. According to Corollary 5.1.6, the function f has a primitive in every open disc $D \subset U$. In other words, for every $D \subset U$ there exists an analytic function $F \colon D \to \mathbb{C}$ such that $F'(z) = f(z)$ for all $z \in D$. Now it follows from Theorem 5.3.1 applied to F that $F' = f$ is analytic in D. Since $D \subset U$ was arbitrary, it follows that f is analytic in U. $\square$

Morera's theorem has a further corollary worth recording:

Corollary 5.3.9 (Analytic continuation to a point). *Let $U \subset \mathbb{C}$ be open, and let $w_0 \in U$. Assume that $f \colon U \to \mathbb{C}$ is continuous, and analytic in $U \setminus \{z_0\}$. Then f is analytic in U.*

Proof. It follows from Cauchy's theorem for triangles (which allowed for one "special point", recall Theorem 5.1.1), that (5.3.7) holds for all triangles $\triangle \subset U$. The conclusion now follows immediately from Morera's theorem, Corollary 5.3.8. $\square$

Remark 5.3.10. Not impressed by Corollary 5.3.9? To get impressed, consider how terribly a similar result fails in $\mathbb{R}$. For example, the function $f(t) := |t|$ is continuous on $\mathbb{R}$ and infinitely differentiable on $(-\infty, 0) \cup (0, \infty)$, but not differentiable at $t = 0$.

5.3.5. Mean value and maximum modulus principles. We start with the following corollary of Cauchy's integral formula in a disc, which has independent interest. It states that if f is analytic, then $f(z)$ is the average of the values $f(z+re^{it})$ for $t \in [0, 2\pi)$:

COROLLARY 5.3.11 (Mean value principle). *Let $U \subset \mathbb{C}$ be open, and let $f\colon U \to \mathbb{C}$ be analytic. If $\bar{D}(z,r) \subset U$, then*

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z+re^{it})\,dt.$$

PROOF. This is immediate from Cauchy's integral formula in a disc (Corollary 5.2.14), and recalling that $\partial D(z,r)$ refers to the circle path $\gamma(t) = z + re^{it}$, $t \in [0, 2\pi]$:

$$f(z) = \frac{1}{2\pi i}\int_{\partial D(z,r)} \frac{f(\zeta)\,d\zeta}{\zeta - z} = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(\gamma(t))ire^{it}}{re^{it}}\,dt = \frac{1}{2\pi}\int_0^{2\pi} f(z+re^{it})\,dt.$$

This completes the proof. □

We then move to the main result of the section. The *maximum modulus principle* has at least two distinct and useful formulations, and we will present them one after the other.

THEOREM 5.3.12 (Maximum modulus principle I). *Let $U \subset \mathbb{C}$ be open and connected, and let $f\colon U \to \mathbb{C}$ be analytic. If $|f|$ reaches its maximum in U, then f is constant. More precisely, if there exists $z_0 \in U$ such that $|f(z)| \le |f(z_0)|$ for all $z \in U$, then f is constant.*

REMARK 5.3.13. One cannot swap "maximum" to "minimum" in Theorem 5.3.12. E.g., $z \mapsto z$ is analytic in $D(0,1)$ and $|z|$ reaches its minimum at 0. However, if $|f|$ has a minimum $|f(z_0)| > 0$ in U, then f is constant. This follows by applying Theorem 5.3.12 to $1/f$.

PROOF OF THEOREM 5.3.12. In order to show that (an analytic function) f is constant in the connected set U, it suffices to show that $|f|$ is constant in U – we observed this in Corollary 3.3.2. Assume that the point $z_0 \in U$ exists, as in the hypothesis of the theorem, and let

$$M := |f(z_0)| = \max\{|f(z)| : z \in U\}.$$

It suffices to show that the set $V := \{z \in U : |f(z)| = M\}$ is all of U.

We will first verify that V is open. Let $z \in V$, and let $r_0 > 0$ be a disc such that $\bar{D}(z, r_0) \subset U$. We claim that actually $D(z, r_0) \subset V$. To see this, fix $r \in (0, r_0)$ arbitrary, and apply the mean value principle (Theorem 5.3.11) as follows:

$$M = |f(z)| = \left|\frac{1}{2\pi}\int_0^{2\pi} f(z+re^{it})\,dt\right| \le \frac{1}{2\pi}\int_0^{2\pi} |f(z+re^{it})|\,dt. \tag{5.3.8}$$

Here $z + re^{it} \in U$ for all $t \in [0, 2\pi]$, so $|f(z + re^{it})| \leq M$ by assumption. Since $t \mapsto |f(z + re^{it})|$ is continuous, (5.3.8) then forces $|f(z + re^{it})| = M$ for all $t \in [0, 2\pi]$. In other words $\partial D(z, r) \subset V$. But since $r \in (0, r_0)$ was arbitrary, we have established that $D(z, r_0) \subset V$, and thus V is open.

We then proceed with the proof that $V = U$. Let $z \in U$ be arbitrary: we claim that $z \in U$, that is, $|f(z)| = M$. Let $\gamma \colon [0, 1] \to U$ be a path with $\gamma(0) = z_0$ and $\gamma(1) = z$. Note that $|f(\gamma(0))| = M$. Let

$$t_0 := \sup\{t \in [0, 1] : |f(\gamma(t))| = M\}.$$

The continuity of $|f| \circ \gamma$ implies $|f(\gamma(t_0))| = M$. We claim that $t_0 = 1$: then $|f(z)| = |f(\gamma(1))| = M$, as desired. If this failed, and $t_0 < 1$, then by the openness of V and the continuity of γ, we could find an interval $I := (t_0 - \epsilon, t_0 + \epsilon)$ such that $\gamma(t) \in V$ for all $t \in I$. In other words $|f(\gamma(t))| = M$ for all $t \in I$, which would evidently contradict the "sup" definition of t_0. Thus $t_0 = 1$, and $V = U$. $\square$

Corollary 5.3.14 (Maximum modulus principle II). *Let $U \subset \mathbb{C}$ be a open, connected, <u>and bounded</u>. If $f \colon \bar{U} \to \mathbb{C}$ is continuous and analytic in U, then*

$$\max_{z \in \bar{U}} |f(z)| = \max_{z \in \partial U} |f(z)|. \tag{5.3.9}$$

Here ∂U refers to the boundary of U.

Proof. Since U is bounded and $|f| \colon \bar{U} \to \mathbb{R}$ is continuous, there exists $z_0 \in \bar{U}$ such that $|f|$ attains its maximum at z_0:

$$|f(z_0)| = \max_{z \in \bar{U}} |f(z)|.$$

This follows from Corollary 2.3.10. If $z_0 \in \partial U$, we are done. Otherwise $z_0 \in U$. Then the hypotheses of Theorem 5.3.12 are satisfied, so f is constant on U. By continuity f is also the same constant on $\bar{U}$, so (5.3.9) is also valid in this situation. $\square$

Warning 5.3.15. The boundedness assumption on U is necessary. For example, consider the open and connected (but unbounded) set $U = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$. Now, the analytic function $z \mapsto e^z$ satisfies

$$|e^z| = 1, \qquad z \in \partial U = \{z \in \mathbb{C} : \mathrm{Re}(z) = 0\},$$

but $z \mapsto |e^z|$ is unbounded in U. There are, however, variants of Corollary 5.3.14 for unbounded domains, where a true statement is obtained by adding hypotheses to the function f. Results of this type often go by the name *Phragmen-Lindelöf principle*.

We finish by recording the following corollary of the maximum modulus principle:

COROLLARY 5.3.16 (Schwarz lemma). *Assume that* $f\colon \mathbb{D} := D(0,1) \to \mathbb{C}$ *is analytic,* $f(0) = 0$*, and* $|f(z)| \le 1$ *for all* $z \in \mathbb{D}$*. Then,*

$$|f'(0)| \le 1 \quad \textit{and} \quad |f(z)| \le |z| \textit{ for all } z \in \mathbb{D}.$$

Moreover, if there exists $z_0 \in \mathbb{D} \setminus \{0\}$ *such that* $|f(z_0)| = |z_0|$*, then there exists* $\lambda \in \mathbb{C}$ *such that* $|\lambda| = 1$*, and* $f(z) = \lambda z$ *for all* $z \in \mathbb{D}$*.*

PROOF. Exercise. (*Hint:* Apply the maximum modulus principle to the function $g\colon \mathbb{D} \to \mathbb{C}$ defined by

$$g(z) := \begin{cases} f(z)/z, & z \neq 0, \\ f'(0), & z = 0. \end{cases}$$

Start by explaining why g is analytic in $\mathbb{D}$.) □

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