Inverse Problems for Nonsmooth First Order Perturbations of the Laplacian

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Doctoral Thesis
University of Helsinki
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2004

Abstract

We consider inverse boundary value problems in $\mathbf{R}^n$, $n \geq 3$, for operators which may be written as first order perturbations of the Laplacian. The purpose is to obtain global uniqueness theorems for such problems when the coefficients are nonsmooth. We use complex geometrical optics solutions of Sylvester-Uhlmann type to achieve this. A main tool is an extension of the Nakamura-Uhlmann intertwining method to operators which have continuous coefficients.

For the inverse conductivity problem for a $C^{1+\varepsilon}$ conductivity, we construct complex geometrical optics solutions whose properties depend explicitly on ε. This implies the uniqueness result of Päivärinta-Panchenko-Uhlmann for $C^{3/2}$ conductivities. For the magnetic Schrödinger equation, the result is that the Dirichlet-to-Neumann map uniquely determines the magnetic field corresponding to a Dini continuous magnetic potential in $C^{1,1}$ domains. For the steady state heat equation with a convection term, we obtain global uniqueness of Lipschitz continuous convection terms in Lipschitz domains.

Acknowledgements

I would like to recognize several people who have helped in bringing this thesis to its completion.

My deepest gratitude is due to the two advisors of this thesis, Lassi Päivärinta and Gunther Uhlmann. Professor Päivärinta has provided me constant support during the writing of this thesis, and has created a stimulating environment in which to study inverse problems. Research for the thesis was started during my one year visit in Seattle, and I want to express my gratitude to Professor Uhlmann for his hospitality during that time, and for introducing me to the subject of the thesis.

I am grateful to the pre-examiners, Professors Allan Greenleaf and Erkki Somersalo, for reading the manuscript and making useful comments.

An exciting part of the work has been the possibility to learn from experts in different fields of mathematics. I would like to thank Professors Giovanni Alessandrini, Russell Brown, Carlos Kenig, Hart Smith, and Kjell-Ove Widman for valuable advice, and Kari Astala, Juha Kinnunen, Matti Lassas, and Petri Ola for insightful discussions.

For financial support I am indebted to the Graduate School in Mathematical Analysis and Its Applications and to the University of Helsinki.

Finally, I would like to thank family and friends (you know who you are) for your support and friendship, and God for making all this possible.

Helsinki, September 2004

Mikko Salo

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Chapter 1

Introduction

1.1 Inverse conductivity problem

The inverse conductivity problem has attracted a great deal of interest in the last 25 years, and both its theoretical and applied aspects have been under intense study. The problem forms the basis for an imaging method called electrical impedance tomography. Physically, the idea is to find the electrical conductivity of a body by making current and voltage measurements at the boundary. Possible applications include medical imaging, geophysical prospection, and nondestructive testing of mechanical parts. For references see the survey Borcea [6].

Mathematically, let $\Omega \subseteq \mathbf{R}^n$ be a bounded open set with Lipschitz boundary, and let $\sigma \in L^\infty(\Omega)$ be a positive function which represents the electrical conductivity of the body Ω. If there are no sources or sinks of current, the voltage potential u inside the body solves the Dirichlet problem for the conductivity equation,

$$\begin{cases} \operatorname{div}(\sigma \nabla u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

if the voltage at the boundary is f. This problem has a unique solution $u \in H^1(\Omega)$ for any $f \in H^{1/2}(\partial\Omega)$.

On the boundary, one can measure the outgoing current flux for a given boundary voltage. Thus the boundary measurements are given by the Dirichlet-to-Neumann map

$$\Lambda_\sigma : f \mapsto \sigma \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega}.$$

The map Λ_σ may be defined in a weak sense using the equation (1.1), so that it becomes a bounded linear map $H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$.

The inverse conductivity problem is to recover the electrical conductivity σ from the boundary measurements Λ_σ. To ensure the possibility of unique recovery, one should have a global uniqueness result stating that whenever σ_1, σ_2 are two conductivities with $\Lambda_{\sigma_1} = \Lambda_{\sigma_2}$, then necessarily $\sigma_1 = \sigma_2$.

Global uniqueness results have been obtained for different classes of conductivities using the complex geometrical optics solutions of Sylvester and Uhlmann [43]. These are solutions of the conductivity equation which have the form $e^{\rho \cdot x}(\sigma^{-1/2} + \omega)$, where $\rho \in \mathbf{C}^n$ is a complex parameter with $\rho \cdot \rho = 0$. Here ω is an error term which should be small when ρ is large, so when $|\rho| \to \infty$ the solution looks like a harmonic exponential multiplied by $\sigma^{-1/2}$. Inserting these solutions in a suitable integral identity and letting $|\rho| \to \infty$ gives the global uniqueness result. This applies in the case $n \geq 3$ which is the only case considered here.

The contribution of this work to the inverse conductivity problem is Theorem 1.2 below, which is a slight improvement of a result in Päivärinta-Panchenko-Uhlmann [34]. The theorem shows that complex geometrical optics solutions to the conductivity equation exist and that their behaviour is explicitly controlled by the regularity of the conductivity. The proof is based on estimates for the inhomogeneous problem for a related operator, which are important enough to be stated as Theorem 1.1.

We need some notation before stating the theorems. If $k \in \mathbf{N}$ then $C^k(\mathbf{R}^n)$ is the space of k times continuously differentiable functions on $\mathbf{R}^n$, and if $s = k + \gamma$ with $0 < \gamma < 1$ then $C^s(\mathbf{R}^n)$ consists of those functions in C^k whose kth partial derivatives are Hölder continuous with exponent γ. The space C_c^s means the functions in C^s which have compact support. We denote by $L^2_\delta(\mathbf{R}^n)$ where $\delta \in \mathbf{R}$ the weighted L^2 space with norm

$$\|f\|_{L^2_\delta} = \Big(\int (1 + |x|^2)^\delta |f(x)|^2 \, dx \Big)^{1/2}.$$

Then H^k_δ, $k \in \mathbf{N}$, is the space of functions in L^2_δ whose derivatives up to order k are in L^2_δ. The norm is $\|f\|_{H^k_\delta} = \sum_{|\alpha| \leq k} \|\partial^\alpha f\|_{L^2_\delta}$. If $s \geq 0$ then H^s_δ is defined by real interpolation (Bergh-Löfström [5]) using the spaces H^k_δ.

We will also use the operators $\Delta_\rho = \Delta + 2\rho \cdot \nabla$ and $\nabla_\rho = \nabla + \rho$, where $\rho \in \mathbf{C}^n$ satisfies $\rho \cdot \rho = 0$. These operators arise naturally in the construction of complex geometrical optics solutions. One may define the inverse of Δ_ρ on the Fourier side as $\Delta_\rho^{-1} f = \mathscr{F}^{-1}\{\frac{1}{-|\xi|^2 + 2i\rho \cdot \xi} \hat{f}(\xi)\}$. The following norm estimates are fundamental.

Proposition 1.1. [43], [8] Let $-1 < \delta < 0$. The operator Δ_ρ^{-1} is a bounded map from $L^2_{\delta+1}$ to H^1_δ and satisfies

$$\|\Delta_\rho^{-1}\|_{L^2_{\delta+1} \to L^2_\delta} \leq \frac{C_0}{|\rho|},$$
$$\|\Delta_\rho^{-1}\|_{L^2_{\delta+1} \to H^1_\delta} \leq C_0$$

where $C_0 = C_0(n, \delta)$.

We now state our results concerning the inverse conductivity problem. The first one is a general norm estimate, the second ensures the existence of complex geometrical optics solutions, and the third is a uniqueness result.

Theorem 1.1. Suppose $a \in C_c^1(\mathbf{R}^n)$ and let $-1 < \delta < 0$. If $\rho \in \mathbf{C}^n$ with $\rho \cdot \rho = 0$ and $|\rho|$ is large enough, then for any $f \in L^2_{\delta+1}(\mathbf{R}^n)$ the equation

$$(\Delta_\rho + \nabla a \cdot \nabla_\rho)u = f$$

has a unique solution $u \in \Delta_\rho^{-1} L^2_{\delta+1}(\mathbf{R}^n)$. The solution satisfies

$$\|u\|_{L^2_\delta} \leq \frac{C}{|\rho|} \|f\|_{L^2_{\delta+1}},$$
$$\|u\|_{H^1_\delta} \leq C \|f\|_{L^2_{\delta+1}}$$

where C is independent of ρ and f.

Theorem 1.2. Let $\sigma \in C^{1+\varepsilon}(\mathbf{R}^n)$ with $0 \leq \varepsilon \leq 1$, so that $\sigma > 0$ in $\mathbf{R}^n$ and $\sigma = 1$ outside a large ball. Let $\rho \in \mathbf{C}^n$ with $\rho \cdot \rho = 0$ and let $|\rho|$ be sufficiently large. Then the equation

$$\mathrm{div}(\sigma \nabla u) = 0$$

has a solution $u = u(x, \rho)$ of the form

$$u = e^{\rho \cdot x}(\sigma^{-1/2} + \omega),$$

where $\omega = \omega(x, \rho) \in H^1_\delta(\mathbf{R}^n)$ and

$$\lim_{|\rho| \to \infty} \|\omega(\cdot, \rho)\|_{H^\varepsilon_\delta} = 0.$$

Using this result we can give a shorter proof of the following uniqueness result from [34].

Theorem 1.3. Let $\Omega \subseteq \mathbf{R}^n$ be a bounded domain with Lipschitz boundary, and assume $n \geq 3$. Then if $\sigma_j \in C^{3/2}(\Omega)$ are such that $0 < c \leq \sigma_j \leq C$ in Ω $(j = 1, 2)$, then $\Lambda_{\sigma_1} = \Lambda_{\sigma_2}$ implies $\sigma_1 = \sigma_2$ in Ω.

We remark that Theorems 1.1 and 1.3 and most of Theorem 1.2 are contained in [34], and Theorem 1.3 has been improved so that it holds for $W^{3/2,2n+\varepsilon}$ conductivities in Brown-Torres [10]. The results are included here because of the method of proof. The lack of regularity of the coefficient is handled by approximation similarly as in [34], but the proof of the norm estimates is more straightforward and combines two basic ideas: the reduction of a smooth elliptic equation into a Schrödinger equation, and a perturbation argument. The main subject of this thesis is the extension of this procedure to more general inverse problems which are considered below.

1.2 Norm estimates for general operators

Complex geometrical optics solutions have shown their usefulness in questions related to the inverse conductivity problem. In this section we want to consider constructing these solutions for more general equations. More precisely, we will consider equations of the form

$$(\Delta + W \cdot \nabla + q)u = 0 \quad \text{in } \Omega \tag{1.2}$$

where W is a nonsmooth vector field and q is a bounded measurable function in a domain Ω. We assume that W and q are complex valued in this section.

A complex geometrical optics solution to (1.2) is a solution $u = u(x, \rho)$ of the form

$$u = e^{\rho \cdot x}(\omega_0 + \omega) \tag{1.3}$$

where $\rho \in \mathbf{C}^n$ is a complex parameter with $\rho \cdot \rho = 0$, ω_0 depends on the equation, and ω is an error term which is small in suitable norms when ρ is large. Inserting (1.3) into (1.2) gives the equation

$$(\Delta_\rho + W \cdot \nabla_\rho + q)\omega = f \quad \text{in } \Omega \tag{1.4}$$

where $f = -(\Delta_\rho + W \cdot \nabla_\rho + q)\omega_0$. Thus, with a suitable choice of ω_0, we see that constructing complex geometrical optics solutions only needs norm estimates like the ones in Theorem 1.1 for the equation (1.4).

The required norm estimates are provided in a quite general setting by the following theorem.

Theorem 1.4. Let $\Omega \subseteq \mathbf{R}^n$ be a bounded open set, let $W \in C(\overline{\Omega}; \mathbf{C}^n)$ and let $q \in L^\infty(\Omega; \mathbf{C})$. If $\rho \in \mathbf{C}^n$ with $\rho \cdot \rho = 0$ and $|\rho|$ is large enough, then for any $f \in L^2(\Omega)$ the equation

$$(\Delta_\rho + W \cdot \nabla_\rho + q)u = f \quad \text{in } \Omega \tag{1.5}$$

has a solution $u \in H^1(\Omega)$ which satisfies

$$\|u\|_{L^2(\Omega)} \leq \frac{C}{|\rho|}\|f\|_{L^2(\Omega)},$$
$$\|u\|_{H^1(\Omega)} \leq C\|f\|_{L^2(\Omega)}$$

where C is independent of ρ and f.

This result was proved for C^∞ vector fields W in the fundamental paper of Nakamura and Uhlmann [31], where they introduced an intertwining method which used pseudodifferential operators depending on a complex parameter to remove the first order term in (1.5). This method was extended to $C^{2/3+\varepsilon}$ vector fields in Tolmasky [46] using symbol smoothing and paradifferential calculus. We obtain the result above for just continuous vector fields by combining the ideas in the proof of Theorem 1.1 with the Nakamura-Uhlmann pseudodifferential intertwining method.

1.3 Applications to inverse problems

Our aim is to use the norm estimates given above to prove uniqueness results for inverse problems. The inverse problems which we will consider are the Schrödinger equation in a magnetic field and the steady state heat equation with a convection term. The first problem is selfadjoint while the other is not, but their analysis may be carried out using similar arguments. Thus we first consider an auxiliary inverse problem and collect the required arguments there. The results are then used to study the other problems.

An auxiliary inverse problem

Let $\Omega \subseteq \mathbf{R}^n$ be a bounded open set. If $W \in L^\infty(\Omega; \mathbf{C}^n)$ and $q \in L^\infty(\Omega; \mathbf{C})$ consider the operator

$$L_{W,q} = \sum_{j=1}^{n} \Big(\frac{1}{i}\frac{\partial}{\partial x_j} + W_j\Big)^2 + q.$$

Assume for the moment that 0 is not a Dirichlet eigenvalue of $L_{W,q}$ and that $\partial\Omega$ is Lipschitz. Then the Dirichlet problem

$$\begin{cases} L_{W,q} u = 0 & \text{in } \Omega, \\ \quad\;\; u = f & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

has a unique solution $u \in H^1(\Omega)$ for any $f \in H^{1/2}(\partial\Omega)$. We may then define a Dirichlet-to-Neumann map formally by

$$\Lambda_{W,q} : f \mapsto \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} + i(W \cdot \nu) f. \tag{1.7}$$

This map has a natural weak formulation which gives that $\Lambda_{W,q}$ is a bounded map from $H^{1/2}(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$.

If Ω does not have Lipschitz boundary we may do as in Astala-Päivärinta [4] and define the trace space of $H^1(\Omega)$ abstractly as $H^1(\Omega)/H^1_0(\Omega)$. If $u \in H^1(\Omega)$ solves $L_{W,q} u = 0$ in Ω then we may use the equation and define $(\frac{\partial u}{\partial \nu} + i(W \cdot \nu)u)|_{\partial\Omega}$ in a natural way as an element of the dual $(H^1(\Omega)/H^1_0(\Omega))'$. This defines the Dirichlet-to-Neumann map also when no regularity is assumed of $\partial\Omega$, but it is still required that 0 is not a Dirichlet eigenvalue of $L_{W,q}$. To remove this extraneous assumption we introduce the Cauchy data set

$$C_{W,q} = \{(u|_{\partial\Omega}, (\frac{\partial u}{\partial \nu} + i(W \cdot \nu)u)|_{\partial\Omega})\,;\, u \in H^1(\Omega) \text{ and } L_{W,q} u = 0 \text{ in } \Omega\}.$$

With natural interpretations $C_{W,q} \subseteq H^1(\Omega)/H^1_0(\Omega) \times (H^1(\Omega)/H^1_0(\Omega))'$. If Ω has Lipschitz boundary and 0 is not a Dirichlet eigenvalue of $L_{W,q}$, then $C_{W,q}$ is just the graph of $\Lambda_{W,q}$ on $H^{1/2}(\partial\Omega)$.

Now given W and q, the set $C_{W,q}$ represents our boundary measurements, and the inverse problem is to determine W and q from $C_{W,q}$. Similarly as in Sun [41] there is an obstruction to uniqueness given by gauge equivalence: if $p \in W^{1,\infty}(\Omega)$ satisfies $p|_{\partial\Omega} = 0$, then $C_{W+\nabla p,q} = C_{W,q}$. Thus one can only hope to recover the curl of W, which is defined distributionally in Ω by

$$\operatorname{curl} W = \sum_{1 \leq j < k \leq n} \Big(\frac{\partial W_k}{\partial x_j} - \frac{\partial W_j}{\partial x_k}\Big) dx_j \wedge dx_k.$$

The curl may indeed be recovered under certain assumptions on W, q and $\partial\Omega$, as has been shown for $n \geq 3$ in [41], [28], [46], [35]. The following uniqueness theorem improves earlier results in several directions, the most important being that one has uniqueness in the class C^d of Dini continuous vector fields (see Section 4.2) instead of C^1 as in Tolmasky [46].

Theorem 1.5. Let $\Omega \subseteq \mathbf{R}^n$ be a bounded open set where $n \geq 3$, and assume that $W_1, W_2 \in C^d(\Omega; \mathbf{C}^n)$ and $q_1, q_2 \in L^\infty(\Omega; \mathbf{C})$. If $C_{W_1,q_1} = C_{W_2,q_2}$ and $W_1|_{\partial\Omega} = W_2|_{\partial\Omega}$, then $\operatorname{curl} W_1 = \operatorname{curl} W_2$ and $q_1 = q_2$ in Ω.

The proof uses an idea from Panchenko [35]: by gauge equivalence we can reduce questions concerning general vector fields to questions for divergence free fields. More precisely, if $W \in C^d(\Omega; \mathbf{C}^n)$ we use a Helmholtz decomposition $W = E + \nabla p$ where $\operatorname{div} E = 0$ in the sense of distributions. The Dini continuity of W ensures that E is continuous.

Now $L_{E,q} = -\Delta - 2iE \cdot \nabla + G$ is a nondivergence form operator with continuous coefficients in the first order part, so we may use the norm estimates of Theorem 1.4 to construct complex geometrical optics solutions to $L_{E,q}u = 0$. Gauge equivalence gives similar solutions to $L_{W,q}u = 0$, and these solutions yield the uniqueness result by the arguments in [41].

Schrödinger equation in a magnetic field

The Schrödinger operator with magnetic and electric potentials is given by

$$H_{W,q} = \sum_{j=1}^{n} \Big(\frac{1}{i}\frac{\partial}{\partial x_j} + W_j\Big)^2 + q, \tag{1.8}$$

where $W \in L^\infty(\Omega; \mathbf{R}^n)$ and $q \in L^\infty(\Omega; \mathbf{R})$ are the magnetic and electric potentials, respectively, and $\Omega \subseteq \mathbf{R}^n$ is a bounded Lipschitz domain. Note that this is exactly the operator $L_{W,q}$ considered above, but W and q are now assumed to be real. With this assumption $H_{W,q}$ is selfadjoint.

Assuming that 0 is not a Dirichlet eigenvalue of $H_{W,q}$, for any $f \in H^{1/2}(\partial\Omega)$ there is a unique solution $u \in H^1(\Omega)$ to the Dirichlet problem

$$\begin{cases} H_{W,q}u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

The boundary measurements are given by the Dirichlet-to-Neumann map $\Lambda_{W,q}$ defined formally by (1.7), and a weak formulation gives that $\Lambda_{W,q}$ is a bounded map from $H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$.

The inverse problem considered in [41], [28], [46], [35] is to determine W and q from the knowledge of $\Lambda_{W,q}$. One has here the same obstruction to uniqueness as in the auxiliary problem, so that if $p \in W^{1,\infty}(\Omega)$ with $p|_{\partial\Omega} = 0$, then $\Lambda_{W+\nabla p,q} = \Lambda_{W,q}$. The map

$$W \mapsto W + \nabla p$$

transforms the magnetic potential into a gauge equivalent potential but preserves the induced magnetic field, which is given by the rotation $\operatorname{curl} W$. The magnetic field is the physically observable quantity, so it is natural from this point of view to expect to recover $\operatorname{curl} W$ and q from $\Lambda_{W,q}$.

We improve known results for this problem to less regular coefficients and less regular domains. The first theorem is a boundary determination result which states that $\Lambda_{W,q}$ uniquely determines the tangential components of W on $\partial\Omega$. This is the best one can hope for since gauge transformations may alter the normal component.

Theorem 1.6. Let $\Omega \subseteq \mathbf{R}^n$ be a bounded open set with $C^{1,1}$ boundary, and let $W \in C(\overline{\Omega}; \mathbf{R}^n)$ and $q \in L^\infty(\Omega; \mathbf{R})$. Suppose that 0 is not a Dirichlet eigenvalue of $H_{W,q}$. Then $\Lambda_{W,q}$ uniquely determines the tangential components of W on $\partial\Omega$.

The assumption on Ω means that Ω is locally the region above the graph of a $C^{1,1}$ function. Our result is in fact more precise: if $W \in L^\infty(\Omega; \mathbf{R}^n)$ is continuous at $z \in \partial\Omega$ in a certain sense, then the local Dirichlet-to-Neumann map near z uniquely determines the tangential components of $W(z)$. There is also a formula which gives the tangential components. The method we use is due to Brown [9] in the case of the conductivity equation, and it employs oscillating solutions which concentrate near a boundary point.

The following global uniqueness theorem for Dini continuous vector fields now follows from Theorem 1.5.

Theorem 1.7. Let $\Omega \subseteq \mathbf{R}^n$ be a bounded open set with $C^{1,1}$ boundary, $n \geq 3$, let $W_1, W_2 \in C^d(\Omega; \mathbf{R}^n)$, and let $q_1, q_2 \in L^\infty(\Omega; \mathbf{R})$. Suppose that 0 is not a Dirichlet eigenvalue of H_{W_1,q_1} or H_{W_2,q_2}. Then $\Lambda_{W_1,q_1} = \Lambda_{W_2,q_2}$ implies $\operatorname{curl} W_1 = \operatorname{curl} W_2$ and $q_1 = q_2$ in Ω.

The boundary result, Theorem 1.6, was proved for $C^\infty(\overline{\Omega})$ coefficients and C^∞ domains in Nakamura-Sun-Uhlmann [28]. There is an error in the corresponding theorem in this article, but this is not difficult to fix. The global uniqueness result, Theorem 1.7, was known for C^1 vector fields vanishing near the boundary and is found in Tolmasky [46].

Finally, we mention that this inverse problem has applications to the inverse scattering problem for $H_{W,q}$ at a fixed energy. It is known that for compactly supported potentials the two problems are equivalent. For the inverse scattering problem for noncompactly supported potentials, see Novikov-Khenkin [33] and Eskin-Ralston [17].

Steady state heat equation with a convection term

Consider the problem of heat conduction in a body $\Omega \subseteq \mathbf{R}^n$, which is a bounded open set with Lipschitz boundary. Assume that the heat diffusion coefficient in Ω is constant and equal to one, and that there is a Lipschitz continuous velocity field $-W$ in Ω with represents convection of heat and is not affected by the warming of the body. Let f be a stationary temperature distribution at the boundary $\partial\Omega$, and suppose $\partial\Omega$ is kept at temperature f. Then the temperature distribution $u(\cdot, t)$ in Ω at time t satisfies the heat equation

$$\left\{ \begin{array}{ll} u_t = \Delta u + W \cdot \nabla u & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{array} \right. \tag{1.9}$$

After the system has stabilized, the steady state temperature u solves the Dirichlet problem

$$\left\{ \begin{array}{rll} (\Delta + W \cdot \nabla)u &= 0 & \text{in } \Omega, \\ u &= f & \text{on } \partial\Omega. \end{array} \right. \tag{1.10}$$

The problem (1.10) has a unique solution $u \in H^1(\Omega)$ for any $f \in H^{1/2}(\partial\Omega)$.

The quantity which is measured at the boundary is the steady state heat flow on $\partial\Omega$. Thus the measurements are described by the Dirichlet-to-Neumann map

$$\Lambda_W : f \mapsto \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega}.$$

A weak formulation gives that Λ_W is bounded from $H^{1/2}(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$. The inverse problem is to determine the convection term W from the boundary measurements Λ_W.

This inverse problem in the case $n \geq 3$ was studied in Cheng-Nakamura-Somersalo [13], where it was shown that Λ_W uniquely determines a C^∞ vector field W in a domain with C^∞ boundary. They used ideas from [41] and [28] where the related problem of the Schrödinger equation in a magnetic field was considered. The main point was again the construction of complex geometrical optics solutions to (1.10).

We improve the results of [13] to the case where W is Lipschitz continuous and Ω has Lipschitz boundary. The first step is a boundary determination result, and for this we use the method of singular solutions due to Alessandrini [3]. The idea is to construct solutions with a high order singularity near a boundary point, and such solutions are provided by the following theorem. To make the notation simpler we will use the summation convention whenever convenient.

Theorem 1.8. Let L be an operator in $B_{4R} = B(0, 4R) \subseteq \mathbf{R}^n$, $n \geq 3$, with

$$Lu = -\partial_{x_j}(a_{jk}\partial_{x_k}u + b_j u) + c_j \partial_{x_j} u + du$$

where $a_{jk}, b_j \in C^\alpha(B_{4R})$, $c_j, d \in L^\infty(B_{4R})$, $(a_{jk}) \geq \lambda I$, $a_{jk} = a_{kj}$, and one of the conditions $d - \partial_{x_j} b_j \geq 0$, $d - \partial_{x_j} c_j \geq 0$, holds. Assume also that $a_{jk}(0) = \delta_{jk}$. Then for every spherical harmonic S_m of degree $m = 0, 1, 2, \ldots$, there exists $u \in C^{1,\beta}_{\mathrm{loc}}(B_R \smallsetminus \{0\})$ such that

$$Lu = 0 \quad \text{in } B_R \smallsetminus \{0\},$$

and furthermore

$$u(x) = |x|^{2-n-m} S_m\Big(\frac{x}{|x|}\Big) + w(x),$$

where w satisfies

$$|w(x)| + |x||\nabla w(x)| \leq C|x|^{2-n-m+\beta} \quad \text{in } B_R \smallsetminus \{0\},$$

$$r^{1+\beta} \sup_{r<|x|,|y|<2r} \frac{|\nabla w(x) - \nabla w(y)|}{|x-y|^\beta} \leq Cr^{2-n-m+\beta} \quad \text{for } 0 < r < R/2.$$

Here β is any number with $0 < \beta < \alpha$.

This extends the results of [3] to operators with lower order terms and less regular coefficients. The boundary determination result is obtained by using suitable solutions of this type in an integral identity.

Theorem 1.9. Let $\Omega \subseteq \mathbf{R}^n$ be a bounded open set with Lipschitz boundary, and $n \geq 3$. If $W_1, W_2 \in C^\alpha(\Omega; \mathbf{R}^n)$ for some $\alpha > 0$, then $\Lambda_{W_1} = \Lambda_{W_2}$ implies $W_1 = W_2$ on $\partial\Omega$.

The global uniqueness theorem follows from the boundary result combined with Theorem 1.5. Here we have to assume that the vector fields are Lipschitz continuous.

Theorem 1.10. Let $\Omega \subseteq \mathbf{R}^n$ be a bounded open set with Lipschitz boundary, and suppose $n \geq 3$. If W_1 and W_2 are two Lipschitz continuous vector fields in Ω, then $\Lambda_{W_1} = \Lambda_{W_2}$ implies $W_1 = W_2$ in Ω.

1.4 Bibliographical notes

In this section we discuss in greater detail some earlier results on the problems considered in this thesis. We will mostly cover uniqueness results in dimensions $n \geq 3$.

Inverse conductivity problem

We start by discussing different aspects of this problem. A fundamental question is that of uniqueness, where one wishes to know whether Λ_σ uniquely determines σ in a given class of conductivities. For practical purposes it is important to have algorithms for reconstructing σ from Λ_σ. Next one could ask for stability: even though the problem is ill-posed so that σ does not depend continuously on Λ_σ, there are estimates which show, given some a priori information on σ, that two conductivities are close if the corresponding Dirichlet-to-Neumann maps are.

The pioneer contribution to the inverse conductivity problem was the article of Calderón [12] where a uniqueness result was obtained for a linearization of the problem at constant conductivities. This paper also contained an approximate reconstruction procedure for conductivities close to constant. Kohn and Vogelius [23], [24] proved that Λ_σ determines the Taylor series of σ at the boundary, which gives global uniqueness for real analytic σ.

The major breakthrough in the uniqueness question is due to Sylvester and Uhlmann [43], who showed global uniqueness for C^∞ conductivities when $n \geq 3$. The first step was to convert the conductivity equation into a zero order perturbation of Δ by an intertwining formula, and then construct complex geometrical optics solutions, which as described above are solutions depending on a complex parameter ρ and look like harmonic exponentials when ρ is large. The uniqueness result follows by inserting these solutions in an integral identity and letting $|\rho| \to \infty$. The method breaks down for $n = 2$, which may be explained by the fact that the problem is formally overdetermined for $n \geq 3$ and formally determined for $n = 2$.

The global uniqueness result of Sylvester-Uhlmann has been improved to less regular conductivities. Nachman-Sylvester-Uhlmann [27] proved the result for $\sigma \in W^{2,\infty}$, Brown [8] for $\sigma \in C^{3/2+\varepsilon}$ using singular zero order perturbations of Δ, Päivärinta-Panchenko-Uhlmann [34] for $\sigma \in C^{3/2}$ by convolution approximation, and Brown-Torres [10] for $\sigma \in W^{3/2,2n+\varepsilon}$. Uniqueness for $C^{1+\varepsilon}$ conormal conductivities was shown in Greenleaf-Lassas-Uhlmann [19]. A reconstruction algorithm for $n \geq 3$ was given by Nachman [25], and stability estimates were proved by Alessandrini [2]. All these developments use complex geometrical optics solutions.

For $n = 2$, the global uniqueness result was proved by Nachman [26] for $W^{2,p}$ $(p > 1)$, conductivities, and improved to $W^{1,p}$ $(p > 2)$ conductivities by Brown-Uhlmann [11]. Recently, the question was solved completely by Astala-Päivärinta [4], who showed using quasiconformal maps that the Dirichlet-to-Neumann map uniquely determines a L^∞ conductivity, thus proving the original conjecture of Calderón in two dimensions. The sharp results for $n = 2$ rely on complex analytic methods, and attempts to extend the methods to higher dimensions have not been successful so far.

Besides global uniqueness, also the question of uniqueness at the boundary has been studied. A typical result shows that Λ_σ determines the values of σ and its derivatives at the boundary. This question is usually easier than that of global uniqueness, the methods work in any dimension and allow for more general conductivities, and in fact most global uniqueness results for $n \geq 3$ use a boundary determination result at some stage. Also, a boundary determination result immediately implies global uniqueness in the class of piecewise analytic conductivities, as shown in [24].

The first boundary uniqueness results were the ones of Kohn and Vogelius [23], who considered a C^∞ conductivity and domain. Sylvester and Uhlmann [44] gave a different proof of this result, based on the fact that Λ_σ is a pseudodifferential operator, and the Taylor series of σ may be read off from the symbol of Λ_σ. Their method is very flexible and has been adapted to a various number of other situations. They also proved boundary stability results, and showed how these may be used to obtain boundary uniqueness for nonsmooth conductivities in C^∞ domains.

For nonsmooth domains, Alessandrini [3] used solutions with singularities of arbitrary order at a given point to obtain boundary uniqueness of σ and its derivatives in a Lipschitz domain. Nachman [26] and Brown [9] also have results for Lipschitz domains, now using solutions with highly oscillatory boundary data. More recent results are Nakamura-Tanuma [29], [30] and Kang-Yun [22], which extend the method of Brown to work for higher derivatives of σ and also for the anisotropic problem, where σ is a matrix.

Schrödinger equation in a magnetic field

The inverse problem of determining the magnetic field $\operatorname{curl} W$ and electric potential q from $\Lambda_{W,q}$ was first considered by Sun [41] in the case $n \geq 3$. As noted above, one may not recover the full vector field W because of gauge equivalence. He showed that $\Lambda_{W,q}$ uniquely determines $\operatorname{curl} W$ and q when $W \in W^{2,\infty}$, $q \in L^\infty$, and $\operatorname{curl} W$ is small in the L^∞ norm.

The proof in [41] is based on the Sylvester-Uhlmann result [43] for the conductivity equation, with a few notable exceptions. First of all, in this case there is no simple identity to intertwine the equation into a zero order perturbation of Δ. Therefore, the construction of complex geometrical optics solutions is more difficult, and the smallness assumption for $\operatorname{curl} W$ was required to achieve this.

Once the complex geometrical optics have been constructed, they are inserted in an integral identity, and one lets $|\rho| \to \infty$. For the conductivity equation this is enough for uniqueness, but for the magnetic Schrödinger equation one gets an identity which involves the coefficients in a nonlinear way. Sun gave a nontrivial argument which showed that this identity implies uniqueness.

The Schrödinger operator with a magnetic potential is in fact a general selfadjoint first order perturbation of Δ. That such operators can indeed be intertwined to zero order perturbations of Δ was shown by Nakamura and Uhlmann [31] (see also [32]). The method involves pseudodifferential operators depending on the parameter ρ. Using the Nakamura-Uhlmann result combined with the argument in [41], it was shown in Nakamura-Sun-Uhlmann [28] that $\Lambda_{W,q}$ uniquely determines $\mathrm{curl}\, W$ and q if $W \in C^\infty$, $q \in L^\infty$, and $W = 0$ near the boundary. The main point is the absence of smallness assumptions.

Tolmasky [46], using symbol smoothing and paradifferential type estimates for pseudodifferential operators depending on ρ, extended the result of [28] to C^1 vector fields. Recently, Panchenko [35] gave results for less regular W but had to assume a smallness condition. The result for lower regularity was made possible by an effective use of the gauge equivalence of the equation. All these results rely on Sylvester-Uhlmann type arguments, and only work for $n \geq 3$.

For $n = 2$, the problem has been considered in Sun [42]. The related problem for the Pauli Hamiltonian is studied in Kang-Uhlmann [21].

Boundary uniqueness results for this problem were given in [28]. The method there was to show that when everything is C^∞, $\Lambda_{W,q}$ is a pseudodifferential operator, and its symbol determines the Taylor series of W and q at the boundary. There is a small mistake in Theorem D in this paper: as described in Section 1.3, one can only determine the tangential components of W, and the proof when corrected gives this result.

Steady state heat equation with a convection term

This problem differs from the earlier ones in that the operator considered is not selfadjoint. The only reference for $n \geq 3$ that we are aware of is Cheng-Nakamura-Somersalo [13]. In this article, it is proved that Λ_W uniquely determines W if the vector field and domain are C^∞. It should be noted that there is no gauge equivalence in this problem, and the full vector field W may indeed be recovered.

In the uniqueness proof in [13], one first passes from Λ_W to an operator of the form $\Lambda_{W,q}$, proves a uniqueness result for $\Lambda_{W,q}$ as in [41] and [28], and uses this to get uniqueness for W. The method is based on complex geometrical optics solutions, constructed as in [28]. A boundary uniqueness result is also given, using the fact that Λ_W is a pseudodifferential operator.

For $n = 2$ the uniqueness question has been studied in Cheng-Yamamoto [14], [15]. In this case the problem is handled by similar methods as in [11] and falls within the framework of pseudoanalytic functions, and one has uniqueness for L^p coefficients, $p > 2$. Reconstruction algorithms are given in Tamasan [45] and Tong-Cheng-Yamamoto [47].

Chapter 2

Inverse conductivity problem

In this chapter we prove Theorems 1.1 to 1.3. The setup is the following. Let $\Omega \subseteq \mathbf{R}^n$ be a bounded open set with Lipschitz boundary, and assume $\sigma \in C^1(\overline{\Omega})$ is a conductivity with $\sigma \geq c > 0$ in $\overline{\Omega}$. Extend σ to a function in $C^1(\mathbf{R}^n)$ so that $\sigma = 1$ outside a large ball. Then the conductivity equation $\mathrm{div}(\sigma \nabla u) = 0$ in $\mathbf{R}^n$ may be written equivalently as

$$(\Delta + \nabla a \cdot \nabla) u = 0 \tag{2.1}$$

where $a = \log \sigma \in C^1_c(\mathbf{R}^n)$. Thus, if the conductivity has one derivative, the conductivity equation may be written in terms of a first order perturbation of the Laplace operator.

We look for complex geometrical optics solutions to (2.1). These are solutions of the form

$$u = e^{\rho \cdot x}(e^{-\frac{1}{2}a} + \omega) \tag{2.2}$$

where $\rho \in \mathbf{C}^n$ satisfies $\rho \cdot \rho = 0$, and $\omega \to 0$ in a suitable norm as $|\rho| \to \infty$. Note that $e^{-\frac{1}{2}a} = \sigma^{-1/2}$, so these solutions are the same as the ones introduced in Section 1.1. Substituting (2.2) to (2.1) gives that ω must satisfy

$$(\Delta_\rho + \nabla a \cdot \nabla_\rho)\omega = f \tag{2.3}$$

where $\Delta_\rho = e^{-\rho \cdot x} \Delta (e^{\rho \cdot x}\,\cdot\,) = \Delta + 2\rho \cdot \nabla$ and $\nabla_\rho = e^{-\rho \cdot x} \nabla (e^{\rho \cdot x}\,\cdot\,) = \nabla + \rho$ are operators depending on the parameter ρ, and $f = -(\Delta_\rho + \nabla a \cdot \nabla_\rho) e^{-\frac{1}{2}a}$. Since f may be singular if a has only one derivative, in practice one has to use a smooth approximation of a in the construction.

We have reduced the problem of finding complex geometrical optics solutions to having estimates for the inhomogeneous problem (2.3). These estimates are proved in the next section. The construction of complex geometrical optics solutions is given in Section 2.2, and the final section shows how to give a short proof of a lemma in [34], from which global uniqueness follows as in [34].

2.1 Estimates for the inhomogeneous problem

Motivated by the discussion above, we want to show existence, uniqueness, and norm estimates for solutions of

$$(\Delta_\rho + \nabla a \cdot \nabla_\rho)\omega = f \quad \text{in } \mathbf{R}^n, \tag{2.4}$$

where $a \in C_c^1(\mathbf{R}^n)$ and f is in the weighted space $L^2_{\delta+1}(\mathbf{R}^n)$ with $-1 < \delta < 0$. Recall that Δ_ρ^{-1} is well defined on $L^2_{\delta+1}(\mathbf{R}^n)$ and satisfies the estimates of Proposition 1.1. We look for a solution to (2.4) which has the form $\omega = \Delta_\rho^{-1} v$ where $v \in L^2_{\delta+1}(\mathbf{R}^n)$, so that (2.4) reads $T_\rho(a)v = f$ where

$$T_\rho(a) = I + \nabla a \cdot \nabla_\rho \Delta_\rho^{-1}.$$

Proposition 1.1 implies that $T_\rho(a)$ is bounded on $L^2_{\delta+1}(\mathbf{R}^n)$, and we need to show that it is invertible. There are two methods for inverting $T_\rho(a)$ in certain cases.

1. If $a \in C_c^2(\mathbf{R}^n)$ then one may use the intertwining identity

   $$(\Delta_\rho + \nabla a \cdot \nabla_\rho)e^{-\frac{1}{2}a} = e^{-\frac{1}{2}a}(\Delta_\rho - q)$$

   where $q = \frac{\Delta e^{\frac{1}{2}a}}{e^{\frac{1}{2}a}}$. This is just the usual method of converting the conductivity equation to a Schrödinger equation, which involves a zero order perturbation of Δ_ρ. The point is that $I - q\Delta_\rho^{-1}$ is invertible on $L^2_{\delta+1}$ for large ρ by Proposition 1.1, so one obtains that also $T_\rho(a)$ is invertible, regardless of the size of a.

2. If $a \in W_c^{1,\infty}$ has small Lipschitz norm (i.e. $\|\nabla a\|_{L^\infty}$ is small) then $T_\rho(a)$ is just a small perturbation of the identity on $L^2_{\delta+1}$, hence invertible.

Now if $a \in C_c^1$ and $\|\nabla a\|_{L^\infty}$ is large, we may combine the two methods and write $a = a^\sharp + a^\flat$ where $a^\sharp$ is a smooth approximation and $\|\nabla a^\flat\|_{L^\infty}$ is small. A similar argument was used in [34], [35]. Then

$$T_\rho(a) = \tilde{I} + \nabla a^\sharp \cdot \nabla_\rho \Delta_\rho^{-1}$$

where $\tilde{I} = I + \nabla a^\flat \cdot \nabla_\rho$ is close to the identity. One may now apply the intertwining idea to this operator, using the fact that $a^\sharp$ is smooth. If $a^\sharp$ is chosen in a suitable way so that the approximation improves as ρ grows, this method will show that $T_\rho(a)$ is invertible on $L^2_{\delta+1}$. Theorem 1.1 follows immediately from this argument.

We first set up the approximation procedure. Let $\phi \in C_c^\infty(\mathbf{R}^n)$ with $0 \le \phi \le 1$, ϕ radial, $\int \phi(x)\,dx = 1$, $\phi = 1$ for $|x| \le 1/2$, and $\phi = 0$ for $|x| \ge 1$. For $f \in L^1_{\text{loc}}(\mathbf{R}^n)$ define $f^\sharp = \hat{\phi}(D/r)f = r^n\phi(r\cdot) * f$. We have the following basic estimates.

Lemma 2.1. (a) If $f \in C_c^k(\mathbf{R}^n)$ then

$$\begin{aligned} \|\partial^\alpha f^\sharp\|_{L^\infty} &\le \|f\|_{W^{k,\infty}} && \text{for } |\alpha| \le k, \\ \|\partial^\alpha f^\sharp\|_{L^\infty} &= o(r^{|\alpha|-k}) && \text{for } |\alpha| > k, \\ \|\partial^\alpha (f - f^\sharp)\|_{L^\infty} &= o(1) && \text{for } |\alpha| = k, \end{aligned}$$

as $r \to \infty$.

(b) If $f \in C_c^{k+\varepsilon}(\mathbf{R}^n)$ where $0 \le \varepsilon \le 1$, then

$$\begin{aligned} \|\partial^\alpha f^\sharp\|_{L^2} &= o(r^{|\alpha|-k-\varepsilon}) && \text{for } |\alpha| > k, \\ \|\partial^\alpha (f - f^\sharp)\|_{L^2} &= o(r^{-\varepsilon}) && \text{for } |\alpha| = k \end{aligned}$$

as $r \to \infty$.

Proof. (a) It is enough to give the proof for $k = 0$. The first estimate is immediate. If $\alpha \neq 0$ then

$$\begin{aligned} \partial^\alpha f^\sharp(x) &= r^{|\alpha|} \int r^n \partial^\alpha \phi(r(x-y)) f(y)\,dy = r^{|\alpha|} \int \partial^\alpha \phi(y) f(x - r^{-1}y)\,dy \\ &= r^{|\alpha|} \int \partial^\alpha \phi(y)(f(x - r^{-1}y) - f(x))\,dy \end{aligned}$$

since $\int \partial^\alpha \phi(y)\,dy = 0$. We obtain the second estimate by using uniform continuity. Also,

$$(f - f^\sharp)(x) = \int \phi(y)(f(x) - f(x - r^{-1}y))\,dy$$

and uniform continuity gives the last estimate.

(b) Assume again that $k = 0$. Let $\alpha \neq 0$ and write

$$r^{-|\alpha|+\varepsilon} \|\partial^\alpha f^\sharp\|_{L^2} = r^{-|\alpha|+\varepsilon} \|\xi^\alpha \hat{\phi}(\xi/r) \hat{f}\|_{L^2} = \|g(\xi/r)|\xi|^\varepsilon \hat{f}\|_{L^2}$$

where $g(z) = |z|^{-\varepsilon} z^\alpha \hat{\phi}(z)$ is continuous and bounded. Lemma 2.2 implies $\||\xi|^\varepsilon \hat{f}\|_{L^2} \le C\|f\|_{C^\varepsilon}$. Since $g(0) = 0$, we may apply dominated convergence to obtain that $r^{-|\alpha|+\varepsilon}\|\partial^\alpha f^\sharp\|_{L^2} \to 0$ as $r \to \infty$.

Further, we have

$$r^\varepsilon \|f - f^\sharp\|_{L^2} = r^\varepsilon \|(1 - \hat{\phi}(\xi/r))\hat{f}\|_{L^2} = \|g(\xi/r)|\xi|^\varepsilon \hat{f}\|_{L^2}$$

where $g(z) = |z|^{-\varepsilon}(1 - \hat{\phi}(z))$ is continuous and bounded with $g(0) = 0$. Use Lemma 2.2 and dominated convergence to end the proof. □

Lemma 2.2. If $f \in C_c^{k+\varepsilon}(\mathbf{R}^n)$ and $0 \le \varepsilon \le 1$, then $\|f\|_{H^{k+\varepsilon}} \le C\|f\|_{C^{k+\varepsilon}}$, where C depends on n, k and the support of f.

Proof. Let $\chi \in C_c^\infty(\mathbf{R}^n)$ satisfy $\chi = 1$ on the support of f, and define $T : f \mapsto \chi f$. Then clearly $\|Tf\|_{H^l} \leq C(\chi, n, l)\|f\|_{W^{l,\infty}}$ for $l \in \mathbf{N}$. If $0 < c < 1$ then $\|f\|_{W^{l,\infty}} \leq \|f\|_{C^{l+c\varepsilon}}$, and interpolating these estimates for $l = k$ and $l = k+1$ results in

$$\|Tf\|_{H^{k+\varepsilon-c\varepsilon}} \leq C(\chi, n, k)\|f\|_{C^{k+\varepsilon}}.$$

Taking the limit as $c \to 0$ gives the claim. □

Next we prove the invertibility of $T_\rho(a)$ if $a \in C_c^1(\mathbf{R}^n)$ and $|\rho|$ is sufficiently large.

Proposition 2.1. Suppose $a \in C_c^1(\mathbf{R}^n)$ where $a = 0$ for $|x| \geq R$, and let $-1 < \delta < 0$. Then there exist constants $C_1 = C_1(\delta, n, a, R)$ and $C_2 = C_2(\delta, n, a, R)$ so that whenever $\rho \in \mathbf{C}^n$ with $\rho \cdot \rho = 0$ and

$$|\rho| \geq C_1, \tag{2.5}$$

then the operator

$$T_\rho = I + \nabla a \cdot \nabla_\rho \Delta_\rho^{-1} \tag{2.6}$$

is invertible on $L^2_{\delta+1}(\mathbf{R}^n)$, and the inverse satisfies

$$\|T_\rho^{-1}\|_{L^2_{\delta+1} \to L^2_{\delta+1}} \leq C_2. \tag{2.7}$$

Proof. Let $a^\sharp = r^n \phi(r\cdot) * a$, where $r = r(\rho)$, and let $a^\flat = a - a^\sharp$. Notice that

$$(\Delta_\rho + \nabla a^\sharp \cdot \nabla_\rho) e^{-\frac{1}{2}a^\sharp} = e^{-\frac{1}{2}a^\sharp}(\Delta_\rho - q^\sharp) \tag{2.8}$$

where $q^\sharp = \frac{\Delta e^{\frac{1}{2}a^\sharp}}{e^{\frac{1}{2}a^\sharp}} = \frac{1}{2}\Delta a^\sharp + \frac{1}{4}|\nabla a^\sharp|^2$. This implies that

$$\begin{aligned} T_\rho &= I + \nabla a^\sharp \cdot \nabla_\rho \Delta_\rho^{-1} + \nabla a^\flat \cdot \nabla_\rho \Delta_\rho^{-1} &(2.9)\\ &= e^{-\frac{1}{2}a^\sharp}(\Delta_\rho - q^\sharp) e^{\frac{1}{2}a^\sharp} \Delta_\rho^{-1} + \nabla a^\flat \cdot \nabla_\rho \Delta_\rho^{-1}. &(2.10)\end{aligned}$$

We write $T_\rho = A - B$ where $A = e^{-\frac{1}{2}a^\sharp}\Delta_\rho e^{\frac{1}{2}a^\sharp}\Delta_\rho^{-1}$ and $B = q^\sharp \Delta_\rho^{-1} - \nabla a^\flat \cdot \nabla_\rho \Delta_\rho^{-1}$. Now T_ρ, A and B are bounded operators on $L^2_{\delta+1}$ and A is invertible with inverse

$$\begin{aligned} A^{-1} &= \Delta_\rho e^{-\frac{1}{2}a^\sharp} \Delta_\rho^{-1} e^{\frac{1}{2}a^\sharp} \\ &= (\Delta_\rho(e^{-\frac{1}{2}a^\sharp}))\Delta_\rho^{-1} e^{\frac{1}{2}a^\sharp} + 2\nabla(e^{-\frac{1}{2}a^\sharp}) \cdot \nabla \Delta_\rho^{-1} e^{\frac{1}{2}a^\sharp} + e^{-\frac{1}{2}a^\sharp}\Delta_\rho \Delta_\rho^{-1} e^{\frac{1}{2}a^\sharp} \\ &= I + (-\frac{1}{2}\Delta a^\sharp + \frac{1}{4}|\nabla a^\sharp|^2) e^{-\frac{1}{2}a^\sharp}\Delta_\rho^{-1} e^{\frac{1}{2}a^\sharp} - (\rho \cdot \nabla a^\sharp) e^{-\frac{1}{2}a^\sharp} \Delta_\rho^{-1} e^{\frac{1}{2}a^\sharp} \\ &\quad - e^{-\frac{1}{2}a^\sharp} \nabla a^\sharp \cdot \nabla \Delta_\rho^{-1} e^{\frac{1}{2}a^\sharp}. \end{aligned}$$

Write $\langle x\rangle = (1+|x|^2)^{1/2}$. The norm of A^{-1} on $L^2_{\delta+1}$ satisfies

$$\begin{aligned}
\|A^{-1}\| &\le 1 + \|e^{-\frac{1}{2}a^\sharp}\|_{L^\infty}\|e^{\frac{1}{2}a^\sharp}\|_{L^\infty}(\|\langle x\rangle(-\frac{1}{2}\Delta a^\sharp + \frac{1}{4}|\nabla a^\sharp|^2)\|_{L^\infty}\frac{C_0}{|\rho|} \\
&\quad + |\rho|\|\langle x\rangle\nabla a^\sharp\|_{L^\infty}\frac{C_0}{|\rho|} + \|\langle x\rangle\nabla a^\sharp\|_{L^\infty}C_0) \\
&\le C_2(1 + e^{\|a\|_{L^\infty}}(\frac{\|\langle x\rangle\Delta a^\sharp\|_{L^\infty} + \|\langle x\rangle|\nabla a^\sharp|^2\|_{L^\infty}}{|\rho|} + \|\langle x\rangle\nabla a^\sharp\|_{L^\infty})) \\
&\le C_2(1 + \frac{\|\Delta a^\sharp\|_{L^\infty} + \||\nabla a^\sharp|^2\|_{L^\infty}}{|\rho|} + \|\nabla a^\sharp\|_{L^\infty})) \qquad (2.11)
\end{aligned}$$

where $C_2 = C_2(\delta, n, R, a)$ and C_0 is as in Proposition 1.1. We have used that $a^\sharp = 0$ when $|x| \ge R+1$. Here $\|\nabla a^\sharp\|_{L^\infty} \le \|\nabla a\|_{L^\infty}$ and $\|\Delta a^\sharp\|_{L^\infty} = o(r)$ by Lemma 2.1. The choice $r = |\rho|^\alpha$ for any α with $0 < \alpha \le 1$ then ensures that $\|A^{-1}\| \le C_2(\delta, n, a, R)$ when $|\rho| \ge C_1(\delta, n, \phi, \alpha, R, a)$.

To invert T_ρ we write $T_\rho = A(I - A^{-1}B)$ and note that

$$\|q^\sharp\Delta_\rho^{-1}\| \le \|\langle x\rangle q^\sharp\|_{L^\infty}\frac{C_0}{|\rho|} \le (1+R^2)^{1/2}(\|\Delta a^\sharp\|_{L^\infty} + \|\nabla a^\sharp\|^2_{L^\infty})\frac{C_0}{|\rho|} = \frac{o(r)}{|\rho|}$$

and

$$\|\nabla a^\flat \cdot \nabla_\rho\Delta_\rho^{-1}\| \le (1+R^2)^{1/2}\|\nabla(a - a^\sharp)\|_{L^\infty}2C_0 = o(1)$$

by Proposition 1.1 and Lemma 2.1. Again, the choice $r = |\rho|^\alpha$ for $0 < \alpha \le 1$ ensures that $\|B\| \le \frac{1}{2C_2}$ for $|\rho| \ge C_1$. Then $I - A^{-1}B$ is invertible with $\|(I - A^{-1}B)^{-1}\| \le 2$, so also T_ρ is invertible with $\|T_\rho^{-1}\| \le C_2$, for a new C_2. □

It is now easy to prove Theorem 1.1. We give a slightly more precise result.

Proposition 2.2. Suppose $a \in C^1_c(\mathbf{R}^n)$ with $a = 0$ for $|x| \ge R$, and let $-1 < \delta < 0$. Then there exist constants $C_1 = C_1(\delta, n, a, R)$ and $C_2 = C_2(\delta, n, a, R)$ so that whenever $\rho \in \mathbf{C}^n$ with $\rho \cdot \rho = 0$ and

$$|\rho| \ge C_1, \qquad (2.12)$$

then for any $f \in L^2_{\delta+1}(\mathbf{R}^n)$ the equation

$$(\Delta_\rho + \nabla a \cdot \nabla_\rho)u = f \qquad (2.13)$$

has a unique solution $u \in \Delta_\rho^{-1}L^2_{\delta+1}(\mathbf{R}^n)$. The solution u has the form $u = \Delta_\rho^{-1}v$ where $v \in L^2_{\delta+1}(\mathbf{R}^n)$ satisfies

$$\|v\|_{L^2_{\delta+1}} \le C_2\|f\|_{L^2_{\delta+1}}. \qquad (2.14)$$

Proof. We obtain a solution by setting $u = \Delta_\rho^{-1}T_\rho^{-1}f$, and then $v = T_\rho^{-1}f$ satisfies the desired estimate by Proposition 2.1. If u_1, u_2 are two solutions in $\Delta_\rho^{-1}L^2_{\delta+1}$ then $u_1 - u_2 = \Delta_\rho^{-1}w$ for some $w \in L^2_{\delta+1}$. Then w satisfies $T_\rho w = 0$, and the invertibility of T_ρ shows that $w = 0$, or $u_1 = u_2$. □

2.2 Complex geometrical optics solutions

We will now construct complex geometrical optics solutions to the conductivity equation $\mathrm{div}(\sigma \nabla u) = 0$, or the equivalent equation (2.1). Since this will be done for conductivities having only one derivative, the first result shows the existence of solutions of the form (2.2) but where a is replaced by a smooth approximation.

Proposition 2.3. Let $a \in C_c^{1+\varepsilon}(\mathbf{R}^n)$ where $0 \leq \varepsilon \leq 1$, and let $-1 < \delta < 0$. Let $a^\sharp = \hat{\phi}(D/r)a \in C_c^\infty(\mathbf{R}^n)$ be an approximation to a, where $r = r(\rho)$. Finally, suppose $\rho \in \mathbf{C}^n$ satisfies $\rho \cdot \rho = 0$ and assume that $|\rho|$ is sufficiently large. Then the equation

$$(\Delta + \nabla a \cdot \nabla)u = 0 \tag{2.15}$$

has a unique solution

$$u = e^{\rho \cdot x}(\omega_0 + \omega_1) \tag{2.16}$$

where $\omega_0 = e^{-\frac{1}{2}a^\sharp}$ and $\omega_1 \in \Delta_\rho^{-1} L^2_{\delta+1}$. Further, if $r(\rho) = |\rho|$ then ω_1 satisfies

$$\lim_{|\rho| \to \infty} \|\omega_1\|_{H^\varepsilon_\delta} = 0. \tag{2.17}$$

Proof. We use Proposition 2.2 and let $\omega_1 \in \Delta_\rho^{-1} L^2_{\delta+1}$ solve

$$(\Delta_\rho + \nabla a \cdot \nabla_\rho)\omega_1 = f_0 \tag{2.18}$$

where

$$\begin{aligned} f_0 &= -(\Delta_\rho + \nabla a \cdot \nabla_\rho)\omega_0 \\ &= -(\Delta_\rho + \nabla a^\sharp \cdot \nabla_\rho)e^{-\frac{1}{2}a^\sharp} - \nabla a^\flat \cdot \nabla_\rho e^{-\frac{1}{2}a^\sharp} \\ &= q^\sharp e^{-\frac{1}{2}a^\sharp} - \nabla a^\flat \cdot (-\frac{1}{2}\nabla a^\sharp + \rho)e^{-\frac{1}{2}a^\sharp}. \end{aligned}$$

We have written $q^\sharp = \frac{1}{2}\Delta a^\sharp + \frac{1}{4}|\nabla a^\sharp|^2$ and $a^\flat = a - a^\sharp$. Since a and $a^\sharp$ are supported in some ball $B(0, R)$, one has

$$\|f_0\|_{L^2_{\delta+1}} \leq C(R, \delta)e^{\frac{1}{2}\|a^\sharp\|_{L^\infty}}(\|\Delta a^\sharp\|_{L^2} + \|\nabla a^\sharp\|^2_{L^\infty} + \|\nabla a^\flat\|_{L^2}(\|\nabla a^\sharp\|_{L^\infty} + |\rho|)).$$

From Lemma 2.1 we obtain

$$\|f_0\|_{L^2_{\delta+1}} = o(r^{1-\varepsilon}) + |\rho| o(r^{-\varepsilon}).$$

The choice $r = |\rho|$ gives the smallest growth in $|\rho|$ for this expression. We obtain from (2.18), Proposition 2.2 and Proposition 1.1 (by interpolation) that

$$\|\omega_1\|_{H^\varepsilon_\delta} \leq \frac{C}{|\rho|^{1-\varepsilon}} \|f_0\|_{L^2_{\delta+1}} = o(1)$$

as $|\rho| \to \infty$. This shows (2.17). The function u given by (2.16) is a solution to (2.15) by the choice of ω_1, and uniqueness follows immediately from the uniqueness part of Proposition 2.2. □

The solutions (2.16) are in fact complex geometrical optics solutions of the form (2.2). To see this we need the following simple lemma.

Lemma 2.3. If $a \in C_c^{1+\varepsilon}(\mathbf{R}^n)$ where $0 \leq \varepsilon \leq 1$ and if $a^\sharp = \hat{\phi}(D/r)a$ is as above, then

$$\|e^{-\frac{1}{2}a} - e^{-\frac{1}{2}a^\sharp}\|_{H^1} = o(r^{-\varepsilon}) \tag{2.19}$$

as $r \to \infty$.

Proof. Write $F(t) = e^{-\frac{1}{2}t}$ and $g = F(a) - F(a^\sharp)$. Using the fact that $\|a^\sharp\|_{L^\infty} \leq \|a\|_{L^\infty}$, the mean value theorem gives

$$\|g\|_{L^2} \leq \sup_{|t| \leq \|a\|_{L^\infty}} |F'(t)| \cdot \|a - a^\sharp\|_{L^2} \leq Ce^{\frac{1}{2}\|a\|_{L^\infty}} \|a - a^\sharp\|_{L^2}.$$

For the derivatives one has

$$\begin{aligned} g_{x_k} &= F'(a)a_{x_k} - F'(a^\sharp)a^\sharp_{x_k} \\ &= F'(a)(a_{x_k} - a^\sharp_{x_k}) + a^\sharp_{x_k}(F'(a) - F'(a^\sharp)) \end{aligned}$$

so again by the mean value theorem

$$\begin{aligned} \|g_{x_k}\|_{L^2} \leq &\sup_{|t| \leq \|a\|_{L^\infty}} |F'(t)| \cdot \|\partial_k(a - a^\sharp)\|_{L^2} + \|\nabla a\|_{L^\infty} \cdot \\ &\sup_{|t| \leq \|a\|_{L^\infty}} |F''(t)| \cdot \|a - a^\sharp\|_{L^2} \leq Ce^{\frac{1}{2}\|a\|_{L^\infty}}(1 + \|\nabla a\|_{L^\infty})\|a - a^\sharp\|_{H^1}. \end{aligned}$$

By Lemma 2.1 we have $\|a - a^\sharp\|_{H^1} = o(r^{-\varepsilon})$, which proves the lemma. □

We may now prove our main theorem about complex geometrical optics solutions to the conductivity equation.

Proof. (of Theorem 1.2) Noting that $\sigma^{-\frac{1}{2}} = e^{-\frac{1}{2}a}$ where $a = \log \sigma$, the solution u in Proposition 2.3 may be written as

$$u = e^{\rho \cdot x}(\sigma^{-\frac{1}{2}} + \omega),$$

where $\omega = e^{-\frac{1}{2}a^\sharp} - e^{-\frac{1}{2}a} + \omega_1$ belongs to H^1_δ and satisfies

$$\lim_{|\rho| \to \infty} \|\omega\|_{H^\varepsilon_\delta} = 0$$

by Proposition 2.3 and Lemma 2.3. □

2.3 Uniqueness in the inverse conductivity problem

The global uniqueness result, Theorem 1.3, is proved by inserting the complex geometrical optics solutions of Theorem 1.2 in an appropriate integral identity involving the Dirichlet-to-Neumann maps and the unknown conductivities. In [34], a new such integral identity was used to obtain the uniqueness result. We will not repeat here the arguments of [34], but will only give a short proof of the following key lemma, Lemma 3.4 in [34], using Theorem 1.2.

Lemma 2.4. Let $\sigma \in C^{3/2}(\mathbf{R}^n)$ be strictly positive and equal to 1 outside a large ball. If ω_1 is as in Proposition 2.3 and $\xi \in \mathbf{R}^n$, then

$$\lim_{|\rho|\to\infty} \int_{\mathbf{R}^n} e^{ix\cdot\xi} \nabla\sigma^{1/2} \cdot \nabla\omega_1 \, dx = 0.$$

Proof. Since $\nabla\sigma^{1/2} = 0$ outside a large ball and $\sigma \in C^{3/2}$ we have

$$\Big| \int_{\mathbf{R}^n} e^{ix\cdot\xi} \nabla\sigma^{1/2} \cdot \nabla\omega_1 \, dx \Big| \le \|e^{ix\cdot\xi}\nabla\sigma^{1/2}\|_{H^{1/2}_{-\delta}} \|\nabla\omega_1\|_{(H^{1/2}_{-\delta})'} \le C\|\omega_1\|_{H^{1/2}_{\delta}}$$

by an easy duality argument. The claim follows from (2.17). □

Theorem 1.3 is now proved as in [34].

Chapter 3

Norm estimates for general operators

This section is devoted to the proof of Theorem 1.4. The main tool is the Nakamura-Uhlmann intertwining method, which transforms a first order perturbation of the Laplacian to a lower order perturbation. This will be achieved using pseudodifferential operators depending on a parameter, so we will first discuss these. The proof of the theorem is outlined in Section 3.2, and the two remaining sections contain the details for the construction of the intertwining operators and the solutions.

3.1 Pseudodifferential operators depending on a parameter

The operators Δ_ρ and ∇_ρ in Chapter 2 are examples of differential operators depending on a parameter. Taking Fourier transforms we have

$$\Delta_\rho f(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} p_\rho(\xi) \hat{f}(\xi)\, d\xi$$

where $p_\rho(\xi) = -|\xi|^2 + 2i\rho \cdot \xi$ is the symbol of Δ_ρ. This is a second order polynomial in ξ and ρ. We will define a class of pseudodifferential symbols modelled after degree m polynomials of the variables ξ and ρ, so that ξ and ρ are equally important in the growth estimates but no smoothness in ρ is assumed (hence ρ is the parameter).

Pseudodifferential operators depending on a parameter were considered in Shubin [37], where they were used to study the spectral theory of elliptic operators. In inverse problems such operators were introduced by Nakamura and Uhlmann in [31] as a tool to construct complex geometrical optics solutions to first order perturbations of the Laplacian.

We proceed to give the basic definitions related to pseudodifferential operators depending on a parameter. For details we refer to [37] (the parameter space in [37] is a subset of $\mathbf{C}$ instead of $\mathbf{C}^n$, but the proofs are identical).

Definition. (a) Let $Z = \{\rho \in \mathbf{C}^n\,;\, \rho \cdot \rho = 0, |\rho| \geq 1\}$ be the space of complex parameters that we will use.

(b) Let $m \in \mathbf{R}$ and $0 \leq r, \delta \leq 1$. The class $S^m_{r,\delta} = S^m_{r,\delta}(\mathbf{R}^n, Z)$ of pseudodifferential symbols depending on a parameter, of order m and type (r, δ), is defined as follows: $a = a(x, \xi, \rho)$ is in $S^m_{r,\delta}$ if $a(\cdot,\cdot,\rho) \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ for any $\rho \in Z$, and if for any compact set $K \subseteq \mathbf{R}^n$ and for all $\alpha, \beta \in \mathbf{N}^n$ there exists $C_{K,\alpha,\beta} > 0$ so that

$$\sup_{x \in K} |\partial_x^\alpha \partial_\xi^\beta a(x, \xi, \rho)| \leq C_{K,\alpha,\beta}(1 + |\xi| + |\rho|)^{m - r|\beta| + \delta|\alpha|}.$$

We will slightly abuse notation and write a_ρ both for the function $a(\cdot,\cdot,\rho) : \mathbf{R}^{2n} \to \mathbf{C}$ where ρ is fixed and for $a : \mathbf{R}^{2n} \times Z \to \mathbf{C}$.

(c) Let $S^{-\infty} = \bigcap_{m \in \mathbf{R}} S^m_{r,\delta}$ (this is independent of r, δ).

(d) If $a_\rho \in S^m_{r,\delta}$ define an operator $A_\rho = \mathrm{Op}(a_\rho)$ for $f \in C_c^\infty(\mathbf{R}^n)$ by

$$A_\rho f(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{ix\cdot\xi} a_\rho(x, \xi) \hat{f}(\xi)\, d\xi.$$

We write $\mathrm{Op}\, S^m_{r,\delta}$ for the set of operators corresponding to $S^m_{r,\delta}$.

(e) The class of smoothing operators depending on a parameter is the set $\Psi^{-\infty} = \Psi^{-\infty}(\mathbf{R}^n, Z)$ of all operators with an integral kernel $K_\rho(x, y)$ in $C^\infty(\mathbf{R}^{2n})$ where $\rho \in Z$, so that for any $N \in \mathbf{N}$, any compact set $K \subseteq \mathbf{R}^{2n}$ and any multi-indices α, β, there is $C_{N,K,\alpha,\beta} > 0$ so that

$$\sup_{(x,y) \in K} |\partial_x^\alpha \partial_y^\beta K_\rho(x, y)| \leq C_{N,K,\alpha,\beta} |\rho|^{-N}$$

for $\rho \in Z$.

(f) Define the full class $\Psi^m_{r,\delta} = \Psi^m_{r,\delta}(\mathbf{R}^n, Z)$ of pseudodifferential operators depending on a parameter as the set $\mathrm{Op}\, S^m_{r,\delta} + \Psi^{-\infty}$.

(g) Let $A_\rho \in \Psi^m_{r,\delta}$ have integral kernel $K_\rho(x, y)$. We say that A_ρ is (uniformly) properly supported if there is a closed set $L \subseteq \mathbf{R}^n \times \mathbf{R}^n$ so that $\mathrm{supp}(K_\rho) \subseteq L$ for any ρ, and the projections of L to the first and second components are proper (the preimages of compact sets are compact).

(h) An operator $B_\rho \in \operatorname{Op} S^m_{r,\delta}$ is called elliptic if the symbol satisfies the following: for any compact set $K \subseteq \mathbf{R}^n$ there is $\varepsilon = \varepsilon(K) > 0$ so that

$$|b_\rho(x, \xi)| \geq \varepsilon(1 + |\xi| + |\rho|)^m$$

whenever $x \in K$ and $|\xi| + |\rho| \geq \varepsilon^{-1}$.

Some elementary properties are collected in the following remarks.

Remarks. (i) The symbols we have introduced are indeed pseudodifferential symbols depending on a parameter: if $a_\rho \in S^m_{r,\delta}$ and ρ is fixed, then a_ρ is a symbol in the usual (local) Hörmander class $S^m_{r,\delta}(\mathbf{R}^n)$. The additional requirement is that when ρ varies, the growth of the symbol must also be controlled by ρ.

(ii) $S^m_{r,\delta}$ is a vector space which decreases if m increases, r increases, or δ decreases. If $a_\rho \in S^m_{r,\delta}$ and $b_\rho \in S^{m'}_{r,\delta}$ then $a_\rho b_\rho \in S^{m+m'}_{r,\delta}$. If $a_\rho \in S^0_{r,\delta}$ and $F \in C^\infty(\mathbf{C})$ then $F(a_\rho) \in S^0_{r,\delta}$.

(iii) For ρ fixed, $A_\rho \in \operatorname{Op} S^m_{r,\delta}$ is the usual pseudodifferential operator corresponding to the symbol a_ρ. Hence A_ρ is a map $C^\infty_c(\mathbf{R}^n) \to C^\infty(\mathbf{R}^n)$.

(iv) As with classical pseudodifferential operators, most computations can only be done modulo smoothing terms. In this situation the smoothing terms are given by the set $\Psi^{-\infty}$. If ρ is fixed, then $R_\rho \in \Psi^{-\infty}$ is a smoothing operator in the classical sense and $R_\rho : \mathscr{E}'(\mathbf{R}^n) \to C^\infty(\mathbf{R}^n)$.

(v) There is a one-to-one correspondence between operators in $\operatorname{Op} S^m_{r,\delta}$ and symbols in $S^m_{r,\delta}$, hence in this class we have "exact symbols". In the class $\Psi^m_{r,\delta}$ we have given up this requirement. Two operators in $\Psi^m_{r,\delta}$ which differ by a smoothing operator should be considered equal, and in this class we only work modulo smoothing.

(vi) If $A_\rho \in \operatorname{Op} S^m_{r,\delta}$ then A_ρ has an integral kernel $K_\rho(x, y)$ which is a distribution in $\mathbf{R}^{2n}$, so that formally $A_\rho f(x) = \int K_\rho(x, y) f(y)\, dy$ for $f \in C^\infty_c$. Here K_ρ is C^∞ outside the diagonal of $\mathbf{R}^n \times \mathbf{R}^n$, and for any N, α, β and compact K there is $C_{N,K,\alpha,\beta}$ so that $|\partial^\alpha_x \partial^\beta_\xi K_\rho(x, y)| \leq C_{N,K,\alpha,\beta} |\rho|^{-N} |x - y|^{-N}$ for $x \in K$ and $|x - y| \geq 1$. If $A_\rho \in \operatorname{Op} S^{-\infty}$ then $K_\rho \in C^\infty(\mathbf{R}^{2n})$.

(vii) If $A_\rho \in \operatorname{Op} S^m_{r,\delta}$ and $\varphi_1, \varphi_2 \in C^\infty_c$ with $\varphi_2 = 1$ near $\operatorname{supp}(\varphi_1)$, then $\varphi_1 A_\rho (1 - \varphi_2) \in \operatorname{Op} S^{-\infty}$. If $A_\rho \in \Psi^m_{r,\delta}$ then $\varphi_1 A_\rho (1 - \varphi_2) \in \Psi^{-\infty}$.

(viii) If $A_\rho \in \Psi^m_{r,\delta}$ is properly supported, then for any compact $K \subseteq \mathbf{R}^n$ there is a compact set $K_1 \subseteq \mathbf{R}^n$ so that $\operatorname{supp}(f) \subseteq K$ implies $\operatorname{supp}(A_\rho f) \subseteq K_1$. The same holds for the adjoint operator A^t, and consequently properly supported operators map C^∞_c to C^∞_c and C^∞ to C^∞.

(ix) One may compose pseudodifferential operators depending on a parameter, provided that all but one are properly supported. The composition is initially a map $C_c^\infty \to C^\infty$.

As mentioned above the basic examples of pseudodifferential operators depending on a parameter are the differential operators, in particular we have $\Delta_\rho \in \operatorname{Op} S^2_{1,0}$ and $\nabla_\rho \in \operatorname{Op} S^1_{1,0}$.

Proposition 3.1. Let $r > \delta$.

(a) Let $a_\rho^{(j)} \in S^{m_j}_{r,\delta}$ for $j \geq 0$ where $m_j \searrow -\infty$ as $j \to \infty$. Then there exists a_ρ with

$$a_\rho \sim \sum_{j=0}^{\infty} a_\rho^{(j)},$$

which means that $a_\rho \in S^{m_0}_{r,\delta}$ and $a_\rho - \sum_{j=0}^{k-1} a_\rho^{(j)} \in S^{m_k}_{r,\delta}$ for any $k \geq 1$. Such a symbol a_ρ is unique modulo $S^{-\infty}$.

(b) If $A_\rho \in \Psi^m_{r,\delta}$ and $B_\rho \in \Psi^{m'}_{r,\delta}$ and at least one operator is properly supported, then $A_\rho B_\rho \in \Psi^{m+m'}_{r,\delta}$. One has $A_\rho B_\rho = C_\rho + \Psi^{-\infty}$, where $C_\rho \in \operatorname{Op} S^{m+m'}_{r,\delta}$ and its symbol satisfies

$$c_\rho \sim \sum_\alpha \frac{\partial_\xi^\alpha a_\rho D_x^\alpha b_\rho}{\alpha!}.$$

(c) If $A_\rho \in \operatorname{Op} S^m_{r,\delta}$ then $\partial_{x_j} A_\rho \in \operatorname{Op} S^{m+1}_{r,\delta}$ and $\partial_{x_j} A_\rho = A_\rho \partial_{x_j} + \operatorname{Op}\big(\frac{\partial a_\rho}{\partial x_j}\big)$.

(d) If $B_\rho \in \operatorname{Op} S^m_{r,\delta}$ is elliptic, then there exists $C_\rho \in \operatorname{Op} S^{-m}_{r,\delta}$, elliptic and properly supported, so that

$$B_\rho C_\rho = I + R_\rho$$

where R_ρ is in $\Psi^{-\infty}$.

(e) Suppose $A_\rho \in \Psi^m_{r,\delta}$ where $m \leq 0$. Then for any $\varphi_1, \varphi_2 \in C_c^\infty(\mathbf{R}^n)$ one has $\varphi_1 A_\rho \varphi_2 : L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$ with

$$\|\varphi_1 A_\rho \varphi_2 f\|_{L^2} \leq C|\rho|^m \|f\|_{L^2}$$

where C does not depend on ρ or f.

(f) Let $A_\rho \in \operatorname{Op} S^{-\infty}$ and $\varphi \in C_c^\infty(\mathbf{R}^n)$. If $\alpha \in \mathbf{R}$ then $\varphi A_\rho : L^2_\alpha(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$, and for any $N > 0$ there is C_N with

$$\|\varphi A_\rho f\|_{L^2} \leq C_N |\rho|^{-N} \|f\|_{L^2_\alpha}$$

where C_N does not depend on ρ or f.

Proof. Parts (a) to (e) are contained in [37], and (f) follows easily by writing the operator in terms of its integral kernel. □

3.2 The main theorem

We repeat the statement of the main theorem.

Theorem 1.4. Let $\Omega \subseteq \mathbf{R}^n$ be a bounded open set, let $W \in C(\overline{\Omega}; \mathbf{C}^n)$ and let $q \in L^\infty(\Omega; \mathbf{C})$. If $\rho \in \mathbf{C}^n$ with $\rho \cdot \rho = 0$ and $|\rho|$ is large enough, then for any $f \in L^2(\Omega)$ the equation

$$(\Delta_\rho + W \cdot \nabla_\rho + q)u = f \quad \text{in } \Omega$$

has a solution $u \in H^1(\Omega)$ which satisfies

$$\|u\|_{L^2(\Omega)} \leq \frac{C}{|\rho|}\|f\|_{L^2(\Omega)},$$
$$\|u\|_{H^1(\Omega)} \leq C\|f\|_{L^2(\Omega)}$$

where C is independent of ρ and f.

Proof. First extend W to a vector field in $C_c(\mathbf{R}^n; \mathbf{C}^n)$ and q and f by zero to $\mathbf{R}^n$, and consider the equation in $\mathbf{R}^n$. The proof is given in three steps.

Step 1: Decomposition of W

The lack of smoothness in W is handled by approximation. We make the ρ-dependent decomposition

$$W = W_\rho^\sharp + W_\rho^\flat$$

where $W_\rho^\sharp = W * \phi_r$ with $\phi_r(x) = r^n\phi(rx)$ the usual mollifier, and where we make the specific choice

$$r = r(\rho) = |\rho|^\delta$$

for some fixed δ with $0 < \delta < 1/2$. Then $W_\rho^\sharp$ is a C^∞ vector field and Lemma 2.1 gives the estimates

$$\|\partial^\alpha W_\rho^\sharp\|_{L^\infty} \leq C_\alpha|\rho|^{\delta|\alpha|}, \tag{3.1}$$
$$\|W_\rho^\flat\|_{L^\infty} = o(1) \quad \text{as } |\rho| \to \infty. \tag{3.2}$$

Then the operator becomes

$$\Delta_\rho + W_\rho^\sharp \cdot \nabla_\rho + W_\rho^\flat \cdot \nabla_\rho + q.$$

By the norm estimates, the third term $W_\rho^\flat \cdot \nabla_\rho$ may be considered to be a small perturbation of Δ_ρ (in the sense that $W_\rho^\flat \cdot \nabla_\rho \Delta_\rho^{-1}$ has small norm on $L^2(\Omega)$ for ρ large), and the same holds for the term q. The real problem is the smooth first order term $W_\rho^\sharp \cdot \nabla_\rho$. We handle this by converting the term into a lower order term by pseudodifferential intertwining.

Step 2: Intertwining for the smooth part

Let δ be as in Step 1. We will construct elliptic operators $A_\rho, B_\rho \in \mathrm{Op}\, S^0_{1-\delta,\delta}$ and an operator $Q_\rho \in \mathrm{Op}\, S^{2\delta}_{1-\delta,\delta}$ so that

$$(\Delta_\rho + W^\sharp_\rho \cdot \nabla_\rho)A_\rho = B_\rho \Delta_\rho + Q_\rho.$$

This is the Nakamura-Uhlmann intertwining method, adapted to the present situation. Note that since $\delta < 1/2$ one has $2\delta < 1$ and so Q_ρ has order less than one. The construction of the intertwining operators A_ρ and B_ρ is given in Proposition 3.2 below.

Step 3: Construction of the solutions

The details of how to use the result in Step 2 to construct the solutions are given in Section 3.4. □

3.3 Construction of intertwining operators

We begin with some remarks on the operator Δ_ρ. If $\rho \in Z$ we will write $\rho = \eta + ik$ where $\eta, k \in \mathbf{R}^n$. Then $\rho \cdot \rho = 0$ means that $|\eta| = |k|$ and $\eta \cdot k = 0$, and we write $s = |\eta| = |k| = \frac{|\rho|}{\sqrt{2}}$.

Let $p_\rho(\xi) = -|\xi|^2 + 2i\rho \cdot \xi$ be the symbol of Δ_ρ. With the above notation $p_\rho(\xi) = -|\xi|^2 - 2k \cdot \xi + 2i\eta \cdot \xi$, so the characteristic set of Δ_ρ is the $(n-2)$-dimensional sphere

$$p_\rho^{-1}(0) = \{\xi \in \mathbf{R}^n\,;\, \eta \cdot \xi = 0, |\xi + k| = |k|\}.$$

There are zeros of p_ρ for arbitrarily large ξ and ρ, so Δ_ρ is not elliptic as an operator depending on a parameter.

In the construction of the intertwining operators we will need to deal separately with the cases where one is near or away from the characteristic set, so we introduce a neighborhood of $p_\rho^{-1}(0)$ by

$$U_\rho(\varepsilon) = \{\xi \in \mathbf{R}^n\,;\, (1-\varepsilon)|k| < |\xi + k| < (1+\varepsilon)|k|, |\langle \xi + k, \eta/|\eta|\rangle| < \varepsilon|\xi + k|\}. \tag{3.3}$$

With $\varepsilon > 0$ given take $\psi_1, \psi_2 \in C^\infty_c(\mathbf{R})$ with $\mathrm{supp}(\psi_1) \subseteq \{1-\varepsilon < |t| < 1+\varepsilon\}$, $\mathrm{supp}(\psi_2) \subseteq (-\varepsilon, \varepsilon)$, and $\psi_1 = 1$ near ± 1, $\psi_2 = 1$ near 0. Define

$$\psi_\rho(\xi) = \psi_1(s^{-1}|\xi + k|)\psi_2(\langle \frac{\xi + k}{|\xi + k|}, \frac{\eta}{s}\rangle). \tag{3.4}$$

Then $\psi_\rho(\xi) \in C^\infty_c(\mathbf{R}^n)$ and $\mathrm{supp}(\psi_\rho) \subseteq U_\rho(\varepsilon)$ with $\psi_\rho = 1$ near $p_\rho^{-1}(0)$. Also one has $\psi_\rho \in S^0_{1,0}$.

Proposition 3.2. There exist elliptic operators $A_\rho, B_\rho \in \operatorname{Op} S^0_{1-\delta,\delta}$ and an operator $Q_\rho \in \operatorname{Op} S^{2\delta}_{1-\delta,\delta}$ so that

$$(\Delta_\rho + W^\sharp_\rho \cdot \nabla_\rho)A_\rho = B_\rho \Delta_\rho + Q_\rho. \tag{3.5}$$

One may choose the symbols of A_ρ, B_ρ, and Q_ρ to be

$$a_\rho = e^{-\frac{1}{2}w_\rho}, \tag{3.6}$$

$$b_\rho = a_\rho + \frac{1-\psi_\rho(\xi)}{p_\rho(\xi)}[(i\xi+\rho)\cdot W^\sharp_\rho]a_\rho, \tag{3.7}$$

$$q_\rho = \Delta_x a_\rho + W^\sharp_\rho \cdot \nabla_x a_\rho \tag{3.8}$$

where ψ_ρ is as in (3.4), with ε chosen small enough, and $w_\rho \in S^0_{1-\delta,\delta}$ is given by

$$w_\rho = \frac{1}{2\pi s}\int_{\mathbf{R}^2} \frac{1}{y_1+iy_2}\psi_\rho(\xi)[(i\xi+\rho)\cdot W^\sharp_\rho(x - y_1(\frac{\eta}{s}) - y_2(\frac{\xi+k}{s}))]\,dy_1\,dy_2. \tag{3.9}$$

Proof. Suppose a_ρ is any symbol of order 0 and $A_\rho = \operatorname{Op}(a_\rho)$. If we commute A_ρ to the left of Δ_ρ in the left hand side of (3.5), we obtain

$$\begin{aligned}(\Delta_\rho + W^\sharp_\rho \cdot \nabla_\rho)A_\rho &= A_\rho\Delta_\rho \\ &+ \operatorname{Op}(2(i\xi+\rho)\cdot\nabla_x a_\rho + [(i\xi+\rho)\cdot W^\sharp_\rho]a_\rho) + \operatorname{Op}(\Delta_x a_\rho + W^\sharp_\rho\cdot\nabla_x a_\rho).\end{aligned} \tag{3.10}$$

The first and third terms are of the same form as in the right hand side of (3.5), but the second term is of order 1 and we would like its symbol to vanish. Setting $a_\rho = e^{-\frac{1}{2}w_\rho}$, this gives the equation

$$(i\xi+\rho)\cdot\nabla_x w_\rho = (i\xi+\rho)\cdot W^\sharp_\rho. \tag{3.11}$$

Here $i\xi+\rho = \eta + i(\xi+k)$. On $p_\rho^{-1}(0)$, $\eta \perp \xi+k$ and $|\eta| = |\xi+k|$, so $(i\xi+\rho)\cdot\nabla_x$ looks like $s(\partial_{x_1} + i\partial_{x_2})$ near the characteristic variety. In fact, when η and $\xi+k$ are not collinear we may change variables in (3.11) and reduce to a $\overline{\partial}$ equation. Using the fundamental solution of $\overline{\partial}$ and changing back to the original coordinates, we obtain a solution of (3.11) of the form

$$w_\rho(x,\xi) = \frac{1}{2\pi s}\int_{\mathbf{R}^2}\frac{1}{y_1+iy_2}[(i\xi+\rho)\cdot W^\sharp_\rho(x - y_1(\frac{\eta}{s}) - y_2(\frac{\xi+k}{s}))]\,dy_1\,dy_2. \tag{3.12}$$

The problem is that w_ρ defined by (3.12) may not behave like a pseudodifferential symbol away from the characteristic variety $p_\rho^{-1}(0)$. Thus we will only work in a sufficiently small neighborhood $U_\rho(\varepsilon)$ of $p_\rho^{-1}(0)$ and introduce the cutoff $\psi_\rho(\xi)$ as in (3.4). Precisely, we will define w_ρ by (3.9). Lemma 3.1 below will show that one may choose ε small enough so that w_ρ will then be a symbol in $S^0_{1-\delta,\delta}$.

We may now define $a_\rho = e^{-\frac{1}{2}w_\rho}$, so that $a_\rho \in S^0_{1-\delta,\delta}$ and a_ρ is elliptic. By direct differentiation we verify that a_ρ satisfies

$$2(i\xi + \rho) \cdot \nabla_x a_\rho + \psi_\rho(\xi)[(i\xi + \rho) \cdot W^\sharp_\rho(x)]a_\rho = 0.$$

We also define b_ρ by (3.7) and q_ρ by (3.8). One sees that $\frac{1-\psi_\rho}{p_\rho}$ is in $S^{-2}_{1,0}$ so we have $b_\rho = a_\rho + S^{-1}_{1-\delta,\delta}$, which implies that b_ρ is in $S^0_{1-\delta,\delta}$ and is elliptic. Clearly $q_\rho \in S^{2\delta}_{1-\delta,\delta}$. Taking (3.10) into account we obtain

$$(\Delta_\rho + W^\sharp_\rho(x) \cdot \nabla_\rho)A_\rho = B_\rho\Delta_\rho + Q_\rho$$

and the proposition is proved modulo Lemma 3.1. □

Lemma 3.1. One may choose $\varepsilon > 0$ small enough so that $w_\rho \in S^0_{1-\delta,\delta}$, where w_ρ is defined by (3.9), i.e.

$$w_\rho = \frac{1}{2\pi}\int_{\mathbf{R}^2} \frac{1}{y_1 + iy_2}\psi_\rho(\xi)[\frac{i\xi+\rho}{s} \cdot W^\sharp_\rho(x - y_1(\frac{\eta}{s}) - y_2(\frac{\xi+k}{s}))]\, dy_1\, dy_2. \tag{3.13}$$

Proof. Take $R > 0$ so that $W^\sharp_\rho$ has support contained in the ball $B(0, R)$, for any $\rho \in Z$. Then the integration in (3.13) is over the compact set

$$K(x, \xi, \rho) = \{(y_1, y_2) \in \mathbf{R}^2\,;\, x - y_1(\frac{\eta}{s}) - y_2(\frac{\xi+k}{s}) \in B(0, R)\}.$$

Note that the cutoff $\psi_\rho(\xi)$ in (3.13) forces $\xi \in U_\rho(\varepsilon)$ by (3.4). We make the following claim.

Lemma 3.2. If ε is small enough then there is $R' > 0$, independent of x, ξ and ρ, so that whenever $\xi \in U_\rho(\varepsilon)$ then $K(x, \xi, \rho) \subseteq B(z, R')$, where $z = z(x, \xi, \rho)$ is continuous in x.

Assuming this we complete the proof. Clearly w_ρ given by (3.13) is a smooth function of x and ξ. If $\xi \in U_\rho(\varepsilon)$ then $|\xi| \le |\xi + k| + |k| < (2 + \varepsilon)s$, so we have

$$s^{-1} \le C_\varepsilon(1 + |\xi| + |\rho|)^{-1}. \tag{3.14}$$

On the other hand we have $s \le 1 + |\xi| + |\rho|$, so we only need to obtain estimates in s to obtain the $S^0_{1-\delta,\delta}$ estimates for w_ρ. Now taking x-derivatives of w_ρ just corresponds to taking x-derivatives of $W^\sharp_\rho$ in (3.13). In the presence of ξ-derivatives one has to differentiate $\psi_\rho(\xi)$, $\frac{i\xi+\rho}{s}$, and $W^\sharp_\rho(x - y_1\frac{\eta}{s} - y_2\frac{\xi+k}{s})$, where the first two are symbols in $S^0_{1,0}$ when $\xi \in U_\rho(\varepsilon)$, so the worst behaviour in ξ and ρ occurs when all the ξ-derivatives fall on the $W^\sharp_\rho$ part. In $\partial^\alpha_x\partial^\beta_\xi w_\rho$ this worst term is

$$\frac{1}{2\pi}\int_{\mathbf{R}^2} \frac{1}{y_1 + iy_2}\psi_\rho(\xi)[\frac{i\xi+\rho}{s} \cdot \Big(-\frac{y_2}{s}\Big)^{|\beta|}\partial^{\alpha+\beta}_x W^\sharp_\rho(x - y_1(\frac{\eta}{s}) - y_2(\frac{\xi+k}{s}))]\, dy_1\, dy_2.$$

Taking absolute values gives

$$|\partial_x^\alpha \partial_\xi^\beta w_\rho(x,\xi)| \leq C s^{-|\beta|} \|\partial_x^{\alpha+\beta} W_\rho^\sharp\|_{L^\infty} \int_{K(x,\xi,\rho)} |y|^{|\beta|-1} \, dy_1 \, dy_2.$$

Suppose $x \in K$ with K compact. Using Lemma 3.2, $K(x,\xi,\rho) \subseteq B$ where B is a large ball depending on K, and the integral is $\leq \int_B |y|^{|\beta|-1} \, dy_1 \, dy_2 = C_K$. The estimates (3.1) imply $\|\partial_x^{\alpha+\beta} W_\rho^\sharp\|_{L^\infty} \leq C s^{\delta|\alpha+\beta|}$. All in all we obtain

$$|\partial_x^\alpha \partial_\xi^\beta w_\rho(x,\xi)| \leq C s^{-(1-\delta)|\beta|+\delta|\alpha|}$$

when $x \in K$, and (3.14) shows that w_ρ satisfies the correct estimates. □

Proof. (of Lemma 3.2) For any $\xi \in \mathbf{R}^n$, $\rho \in Z$ so that η and $\xi + k$ are not collinear, define $v_1(\xi,\rho) = \hat{\eta} = \frac{\eta}{s}$, $v_2(\xi,\rho) = \frac{\mathrm{proj}_{\eta^\perp}(\xi+k)}{|\mathrm{proj}_{\eta^\perp}(\xi+k)|}$, and take $v_3(\xi,\rho), \ldots, v_n(\xi,\rho)$ to be any vectors so that $\{v_1, \ldots, v_n\}$ forms an orthonormal basis of $\mathbf{R}^n$. Let

$$C_0(\xi,\rho) = \begin{pmatrix} v_1 & \ldots & v_n \end{pmatrix}$$

so that C_0 is an orthogonal matrix for any ξ and ρ, and let

$$C(\xi,\rho) = \begin{pmatrix} v_1 & \frac{\xi+k}{s} & v_3 & \ldots & v_n \end{pmatrix}.$$

Then $C = C_0 + E$ where $E = \begin{pmatrix} 0 & \frac{\xi+k}{s} - \frac{\mathrm{proj}_{\eta^\perp}(\xi+k)}{|\mathrm{proj}_{\eta^\perp}(\xi+k)|} & 0 & \ldots & 0 \end{pmatrix}$. On the other hand, one has

$$\begin{aligned} (y_1,y_2) \in K(x,\xi,\rho) &\Leftrightarrow C(y_1,y_2,0)^t \in x + B(0,R) \\ &\Leftrightarrow (y_1,y_2,0)^t \in C^{-1}x + C^{-1}B(0,R). \end{aligned} \tag{3.15}$$

We want that C^{-1} has bounded norm when $\xi \in U_\rho(\varepsilon)$, which will follow if E is small. This is achieved by (3.3) and some elementary estimates. Write $p = \mathrm{proj}_{\eta^\perp}(\xi+k)$. Then

$$\frac{\xi+k}{s} - \frac{p}{|p|} = \frac{1}{s}(\xi+k-p) + \frac{|p|-s}{s}\frac{p}{|p|}. \tag{3.16}$$

Here

$$|\xi+k-p| = |\langle \xi+k, \hat{\eta}\rangle \hat{\eta}| = |\langle \xi+k, \hat{\eta}\rangle| < \varepsilon|\xi+k| \tag{3.17}$$

by (3.3). Using the triangle inequality in (3.17) gives $(1-\varepsilon)|\xi+k| < |p| < (1+\varepsilon)|\xi+k|$, and using (3.3) again gives

$$(1-\varepsilon)^2 s < |p| < (1+\varepsilon)^2 s. \tag{3.18}$$

Also, (3.17) and (3.3) give

$$|\xi+k-p| < \varepsilon(1+\varepsilon)s. \tag{3.19}$$

Now combining (3.16), (3.18) and (3.19) yields

$$\Big|\frac{\xi+k}{s}-\frac{p}{|p|}\Big| < \varepsilon(1+\varepsilon)+(1+\varepsilon)^2-1=\varepsilon(3+2\varepsilon). \tag{3.20}$$

Finally, consider $M_n(\mathbf{R})$ with the norm $\|A\| = \sup_{|x|=1}|Ax|$. Then

$$\|C_0^{-1}E\| \leq \|C_0^{-1}\|\|E\| \leq \Big|\frac{\xi+k}{s}-\frac{p}{|p|}\Big| \leq \frac{1}{2}$$

if $\varepsilon < \frac{1}{10}$, by (3.20). Then $C = C_0(I + C_0^{-1}E)$ is invertible and $\|C^{-1}\| \leq 2$. Considering (3.15) let m_1^t, m_2^t be the first two row vectors of C^{-1}, so that $(y_1, y_2) \in K(x, \xi, \rho)$ implies that $y_j = m_j^t x + m_j^t w$ for some $w \in B(0, R)$. Here $|m_j^t w| \leq |C^{-1}w| \leq 2R$, and setting $z_j(x, \xi, \rho) = m_j(\xi, \rho)^t x$ we obtain $y \in B(z, 2\sqrt{2}R)$ whenever $y \in K(x, \xi, \rho)$. This concludes the proof. □

3.4 Construction of solutions

We now solve

$$(\Delta_\rho + W \cdot \nabla_\rho + q)u = f$$

near Ω. Using the decomposition for W and the intertwining operators of Proposition 3.2, we look for u of the form $u = A_\rho v$ where v satisfies

$$(B_\rho\Delta_\rho + Q_\rho + W_\rho^\flat \cdot \nabla_\rho A_\rho + qA_\rho)v = f$$

near Ω. Since B_ρ was elliptic, we can find $C_\rho \in \mathrm{Op}\, S^0_{1-\delta,\delta}$, elliptic and properly supported, so that

$$B_\rho C_\rho = I + R_\rho$$

where $R_\rho \in \Psi^{-\infty}$. We now try a solution v of the form

$$v = \varphi_2 \Delta_\rho^{-1} \varphi_3 C_\rho \varphi_4 w$$

for some $w \in L^2(\mathbf{R}^n)$. Here $\varphi_j \in C_c^\infty(\mathbf{R}^n)$ $(j = 1, 2, 3, 4)$ are cutoff functions so that $\varphi_1 = 1$ near Ω and $\varphi_{j+1} = 1$ near $\mathrm{supp}(\varphi_j)$. Inserting this to the equation gives

$$\begin{aligned}
&(B_\rho\Delta_\rho + Q_\rho + W_\rho^\flat \cdot \nabla_\rho A_\rho + qA_\rho)v = \\
&\quad (I\varphi_4 + R_\rho\varphi_4 - B_\rho(1-\varphi_3)C_\rho\varphi_4 - B_\rho\Delta_\rho(1-\varphi_2)\Delta_\rho^{-1}\varphi_3 C_\rho\varphi_4 \\
&\qquad + Q_\rho\varphi_2\Delta_\rho^{-1}\varphi_3C_\rho\varphi_4 + \sum_j W_{\rho,j}^\flat A_\rho \partial_{x_j}\varphi_2\Delta_\rho^{-1}\varphi_3C_\rho\varphi_4 \\
&\quad + \sum_j W_{\rho,j}^\flat \mathrm{Op}(\frac{\partial a_\rho}{\partial x_j})\varphi_2\Delta_\rho^{-1}\varphi_3C_\rho\varphi_4 + \sum_j W_{\rho,j}^\flat \rho_j A_\rho\varphi_2\Delta_\rho^{-1}\varphi_3C_\rho\varphi_4 \\
&\qquad\qquad + qA_\rho\varphi_2\Delta_\rho^{-1}\varphi_3C_\rho\varphi_4)w.
\end{aligned}$$

We call the last expression $D_\rho w$, and so we want to solve $D_\rho w = f$ near Ω, or $\varphi_1 D_\rho w = f$ in $\mathbf{R}^n$.

To get something invertible on $L^2(\mathbf{R}^n)$ we look at a related operator

$$T_\rho = I + \sum_{k=1}^{8} E_k$$

where

$$\begin{aligned}
&E_1 = \varphi_1 R_\rho \varphi_4,\ E_2 = -\varphi_1 B_\rho (1-\varphi_3) C_\rho \varphi_4,\\
&E_3 = -\varphi_1 B_\rho \Delta_\rho (1-\varphi_2)\Delta_\rho^{-1}\varphi_3 C_\rho \varphi_4,\ E_4 = \varphi_1 Q_\rho \varphi_2 \Delta_\rho^{-1}\varphi_3 C_\rho \varphi_4,\\
&E_5 = \sum_j \varphi_1 W^\flat_{\rho,j} A_\rho \partial_{x_j}\varphi_2 \Delta_\rho^{-1}\varphi_3 C_\rho \varphi_4,\\
&E_6 = \sum_j \varphi_1 W^\flat_{\rho,j} \mathrm{Op}(\frac{\partial a_\rho}{\partial x_j})\varphi_2 \Delta_\rho^{-1}\varphi_3 C_\rho \varphi_4,\\
&E_7 = \sum_j \varphi_1 W^\flat_{\rho,j} \rho_j A_\rho \varphi_2 \Delta_\rho^{-1}\varphi_3 C_\rho \varphi_4,\\
&E_8 = \varphi_1 q A_\rho \varphi_2 \Delta_\rho^{-1}\varphi_3 C_\rho \varphi_4.
\end{aligned}$$

We wish to show that each E_j is an operator $L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$ with $\|E_j\| = \|E_j\|_{L^2\to L^2}$ small when $|\rho|$ is large.

First, E_1 and E_2 contain a $\Psi^{-\infty}$ operator with cutoffs on either side, so $\|E_1\|$ and $\|E_2\|$ are small for large $|\rho|$ by Proposition 3.1 (e). Also, E_3 contains the operator $\varphi_1 B_\rho \Delta_\rho (1-\varphi_2) \in \mathrm{Op}\, S^{-\infty}$ which has norm $\leq C_{N,\alpha}|\rho|^{-N}$ from L^2_α to L^2 by Proposition 3.1 (f), for any α and N. Using the fact that $\Delta_\rho^{-1}\varphi_3$ has norm $\leq C_\alpha |\rho|^{-1}$ from L^2 to L^2_α with $-1 < \alpha < 0$, we obtain that $\|E_3\|$ is small for $|\rho|$ large.

For E_4 we insert an additional cutoff using $\varphi_j = \varphi_j \varphi_{j+1}$ so

$$\|E_4\| \leq \|\varphi_1 Q_\rho \varphi_2\|\, \|\varphi_3 \Delta_\rho^{-1}\varphi_3\|\, \|C_\rho \varphi_4\|.$$

We need to estimate $\|\varphi_1 Q_\rho \varphi_2\|$. Using the explicit formula (3.9) for w_ρ, the proof of Lemma 3.1 gives that $|\rho|^{-\delta}\partial_{x_j} w_\rho, |\rho|^{-2\delta}\partial^2_{x_j} w_\rho \in S^0_{1-\delta,\delta}$. Consequently

$$|\rho|^{-\delta}\partial_{x_j} a_\rho, |\rho|^{-2\delta}\partial^2_{x_j} a_\rho, |\rho|^{-2\delta} q_\rho \in S^0_{1-\delta,\delta}. \tag{3.21}$$

Then Proposition 3.1 (e) gives $\|\varphi_1 Q_\rho \varphi_2\| \leq C|\rho|^{2\delta}$. Since $\|\varphi_3 \Delta_\rho^{-1}\varphi_3\| \leq C|\rho|^{-1}$ and $\delta < 1/2$, we see that $\|E_4\|$ is small for large $|\rho|$.

For E_5 note that $\|\partial_{x_j}\varphi_2 \Delta_\rho^{-1}\varphi_3\| \leq C$ independently of ρ, and $\|W^\flat_{\rho,j}\|_{L^\infty}$ is small as $|\rho| \to \infty$. For E_6 we have $\|\varphi_2 \mathrm{Op}(\frac{\partial a_\rho}{\partial x_j})\varphi_3\|\, \|\varphi_2 \Delta_\rho^{-1}\varphi_3\| \leq C|\rho|^{\delta-1}$ by (3.21). Finally, E_7 has small norm for large ρ since $\|\rho_j \varphi_2 \Delta_\rho^{-1}\varphi_3\| \leq C$ and $\|W^\flat_{\rho,j}\|_{L^\infty} \to 0$ as $|\rho| \to \infty$, and $\|E_8\| \leq C|\rho|^{-1}$.

All this gives that T_ρ is an invertible operator on $L^2(\mathbf{R}^n)$ for $|\rho|$ large, and we may assume $\|T_\rho^{-1}\| \le 2$. Set

$$w = T_\rho^{-1} f.$$

Since $T_\rho = D_\rho + I(1-\varphi_4) - (1-\varphi_1)(D_\rho - I\varphi_4)$ we have $T_\rho w = D_\rho w$ in Ω, so that $D_\rho w = f$ in Ω. Chasing back the steps we see that $v = \varphi_2 \Delta_\rho^{-1} \varphi_3 C_\rho \varphi_4 w$ satisfies

$$(B_\rho \Delta_\rho + Q_\rho + W_\rho^\flat \cdot \nabla_\rho A_\rho + q A_\rho) v = f \quad \text{in } \Omega,$$

so $u = A_\rho v$ satisfies $(\Delta_\rho + W \cdot \nabla_\rho + q) u = f$ in Ω. The solution has therefore the form

$$u = A_\rho \varphi_2 \Delta_\rho^{-1} \varphi_3 C_\rho \varphi_4 T_\rho^{-1} f.$$

One obtains $\|u\|_{L^2(\Omega)} \le \frac{C}{|\rho|} \|f\|_{L^2(\Omega)}$ immediately. We have

$$\partial_{x_j} u = A_\rho \partial_{x_j} \varphi_2 \Delta_\rho^{-1} \varphi_3 C_\rho \varphi_4 T_\rho^{-1} f + \mathrm{Op}(\frac{\partial a_\rho}{\partial_{x_j}}) \varphi_2 \Delta_\rho^{-1} \varphi_3 C_\rho \varphi_4 T_\rho^{-1} f,$$

and since $\|\partial_{x_j} \varphi_2 \Delta_\rho^{-1} \varphi_3\| \le C$, $\|\varphi_1 \mathrm{Op}(\frac{\partial a_\rho}{\partial_{x_j}}) \varphi_2 \Delta_\rho^{-1} \varphi_3\| \le C|\rho|^{\delta-1}$, we have $\|\partial_{x_j} u\|_{L^2(\Omega)} \le C \|f\|_{L^2(\Omega)}$. This ends the proof.

Chapter 4

An auxiliary inverse problem

In this chapter we discuss the auxiliary inverse problem considered in Section 1.3. The main objective is to prove the uniqueness result, Theorem 1.5, following the argument in Sun [41]. In the first section we set up the inverse problem and discuss some of its basic properties. The next section contains a proof of the Helmholtz decomposition for Dini continuous vector fields, which will be used to decompose a vector field into divergence free and curl free parts.

In Section 4.3 we construct complex geometrical optics solutions for this problem, and use these to obtain an identity for two vector fields assuming that their Cauchy data sets coincide. Finally in Section 4.4 we give the rest of the details of the proof of Theorem 1.5.

4.1 Preliminaries

Let $\Omega \subseteq \mathbf{R}^n$ be a bounded open set, and assume $W \in L^\infty(\Omega; \mathbf{C}^n)$ and $q \in L^\infty(\Omega; \mathbf{C})$. Consider the operator

$$L_{W,q} = \sum_{j=1}^{n} \Big(\frac{1}{i}\frac{\partial}{\partial x_j} + W_j\Big)^2 + q.$$

If W and q are real then this is the selfadjoint Schrödinger operator in (1.8). Complex coefficients are needed later when we study the inverse problem for the steady state heat equation. In nondivergence form one has

$$L_{W,q} = -\Delta - 2iW \cdot \nabla + (W \cdot W - i(\nabla \cdot W) + q). \tag{4.1}$$

The bilinear form associated with $L_{W,q}$ is

$$(L_{W,q}u, v) = \int_\Omega (\nabla u \cdot \nabla \bar{v} + iW \cdot (u\nabla \bar{v} - \bar{v}\nabla u) + (W \cdot W + q)u\bar{v})\, dx \tag{4.2}$$

which makes sense if $u, v \in H^1(\Omega)$. One sees easily that $(L_{W,q}u, v) = (u, L_{\bar{W},\bar{q}}v)$. We define the set of solutions

$$M_{W,q} = \{u \in H^1(\Omega)\,;\, L_{W,q}u = 0 \text{ in } \Omega\}.$$

This set is always nontrivial by the Fredholm alternative.

We next want to define the Cauchy data set. First one has the abstract trace space $H^1(\Omega)/H^1_0(\Omega)$ where the trace map $T : H^1(\Omega) \to H^1(\Omega)/H^1_0(\Omega)$ is just the quotient map. If $u \in M_{W,q}$ is a solution one may define $N_{W,q}u = \frac{\partial u}{\partial \nu}\big|_{\partial\Omega} + i(W \cdot \nu)u|_{\partial\Omega}$ weakly as an element of the dual $(H^1(\Omega)/H^1_0(\Omega))'$ by

$$(N_{W,q}u, v) = (L_{W,q}u, v).$$

It follows that $N_{W,q}$ is a bounded linear map $M_{W,q} \to (H^1(\Omega)/H^1_0(\Omega))'$. The Cauchy data set is the set

$$C_{W,q} = \{(Tu, N_{W,q}u)\,;\, u \in M_{W,q}\}.$$

If Ω is a Lipschitz domain and 0 is not a Dirichlet eigenvalue of $L_{W,q}$, then the Cauchy data set is the graph of the Dirichlet-to-Neumann map $\Lambda_{W,q}$, defined by a natural weak formulation of (1.7) as in Section 5.1.

As noted in Chapter 1 there is gauge equivalence in this problem.

Lemma 4.1. If Ω, W, and q are as above and $p \in W^{1,\infty}(\Omega)$, then

$$L_{W+\nabla p,q} = e^{-ip} L_{W,q} e^{ip}, \tag{4.3}$$
$$M_{W+\nabla p,q} = e^{-ip} M_{W,q}. \tag{4.4}$$

If additionally $p|_{\partial\Omega} = 0$ then

$$C_{W+\nabla p,q} = C_{W,q}. \tag{4.5}$$

Proof. If $p \in W^{1,\infty}(\Omega)$ then $u \mapsto e^{-ip}u$ is a bounded map $H^1(\Omega) \to H^1(\Omega)$ and $H^1_0(\Omega) \to H^1_0(\Omega)$. A direct computation shows (4.3), and (4.4) follows immediately. If $p|_{\partial\Omega} = 0$ then $e^{-ip}u = u$ as elements of $H^1(\Omega)/H^1_0(\Omega)$. We have $(N_{W+\nabla p,q}(e^{-ip}u), v) = (N_{W+\nabla p,q}(e^{-ip}u), e^{-ip}v) = (N_{W,q}u, v)$ and

$$\begin{aligned} C_{W+\nabla p,q} &= \{(Tv, N_{W+\nabla p,q}v)\,;\, v \in M_{W+\nabla p,q}\} \\ &= \{(T(e^{-ip}u), N_{W+\nabla p,q}(e^{-ip}u)\,;\, u \in M_{W,q}\} = C_{W,q}. \end{aligned} \qquad \square$$

Next we discuss a reduction which allows us to replace the domain by a larger one if the coefficients coincide outside the smaller domain.

Lemma 4.2. Let $\Omega, \Omega' \subseteq \mathbf{R}^n$ be bounded open sets with $\overline{\Omega} \subseteq \Omega'$. If $W_1, W_2 \in L^\infty(\Omega'; \mathbf{C}^n)$ and $q_1, q_2 \in L^\infty(\Omega'; \mathbf{C})$, let C_{W_j,q_j} and C'_{W_j,q_j} be the Cauchy data sets for $L_{W_j|_\Omega, q_j|_\Omega}$ in Ω and L_{W_j,q_j} in Ω', respectively. If

$$W_1 = W_2 \text{ and } q_1 = q_2 \text{ in } \Omega' \smallsetminus \Omega, \tag{4.6}$$

then $C_{W_1,q_1} = C_{W_2,q_2}$ implies $C'_{W_1,q_1} = C'_{W_2,q_2}$.

Proof. Let $L_{W_1,q_1}u' = 0$ in Ω' and let $u = u'|_\Omega$. Then $u \in H^1(\Omega)$ satisfies $L_{W_1,q_1}u = 0$ in Ω, so from $C_{W_1,q_1} \subseteq C_{W_2,q_2}$ we know that there is $v_0 \in H^1(\Omega)$ with $L_{W_2,q_2}v_0 = 0$ in Ω and $Tv_0 = Tu$, $N_{W_2,q_2}v_0 = N_{W_1,q_1}u$ in Ω. This implies that $v_0 = u + \varphi$ where $\varphi \in H_0^1(\Omega)$, and we define

$$v' = u' + \varphi$$

so that $v' \in H^1(\Omega')$ and $v' = v_0$ in Ω, $v' = u'$ in $\Omega' \smallsetminus \Omega$. Now for $w' \in H^1(\Omega')$ we have

$$\begin{aligned}(L_{W_2,q_2}v', w')_{\Omega'} &= (L_{W_2,q_2}v_0, w')_\Omega + (L_{W_2,q_2}u', w')_{\Omega' \smallsetminus \Omega} \\ &= (L_{W_1,q_1}u, w')_\Omega + (L_{W_1,q_1}u', w')_{\Omega' \smallsetminus \Omega} = (L_{W_1,q_1}u', w')_{\Omega'}\end{aligned}$$

where we have used $N_{W_2,q_2}v_0 = N_{W_1,q_1}u$ in Ω and (4.6), and the subscript indicates the integration set. This shows that $L_{W_2,q_2}v' = 0$ in Ω' and $N_{W_2,q_2}v' = N_{W_1,q_1}u'$. Since also $Tv' = Tu'$ in Ω' we obtain $C'_{W_1,q_1} \subseteq C'_{W_2,q_2}$. The other direction is analogous and we get $C'_{W_1,q_1} = C'_{W_2,q_2}$. □

As the last fact which will be done without any further regularity assumptions on W, we derive an integral identity which will be used in the uniqueness proof.

Lemma 4.3. Let $\Omega \subseteq \mathbf{R}^n$ be a bounded open set, and suppose $W_1, W_2 \in L^\infty(\Omega; \mathbf{C}^n)$ and $q_1, q_2 \in L^\infty(\Omega; \mathbf{C})$. If $C_{W_1,q_1} = C_{W_2,q_2}$ then one has

$$\int_\Omega (i(W_1 - W_2) \cdot (u\nabla\bar{v} - \bar{v}\nabla u) + (W_1 \cdot W_1 - W_2 \cdot W_2 + q_1 - q_2)u\bar{v})\, dx = 0$$

for any $u \in M_{W_1,q_1}$ and $v \in M_{\bar{W}_2,\bar{q}_2}$.

Proof. If u and v are as stated then $C_{W_1,q_1} = C_{W_2,q_2}$ implies that there is $v_0 \in M_{W_2,q_2}$ with $Tv_0 = Tu$, $N_{W_2,q_2}v_0 = N_{W_1,q_1}u$. Then

$$(N_{W_1,q_1}u, v) = (N_{W_2,q_2}v_0, v) = (v_0, N_{\bar{W}_2,\bar{q}_2}v) = (u, N_{\bar{W}_2,\bar{q}_2}v) = (N_{W_2,q_2}u, v)$$

and the identity follows from the definition of $N_{W,q}$. □

4.2 Helmholtz decomposition

In this section we discuss the Helmholtz decomposition of a vector field W as $W = E + \nabla p$, where E is a divergence free vector field and ∇p is curl free. The motivation comes from the construction of complex geometrical optics solutions for $L_{W,q}$. The tool for doing this, Theorem 1.4, requires a nondivergence form operator $\Delta + W_1 \cdot \nabla + q_1$ where W_1 is continuous and q_1 is L^∞. From (4.1) we see that $L_{W,q}$ is of this form if $W \in C(\overline{\Omega}; \mathbf{C}^n)$ and $\nabla \cdot W \in L^\infty(\Omega; \mathbf{C})$.

For more general W with $\nabla \cdot W \notin L^\infty$ we may do as in [35] and use the gauge equivalence of $L_{W,q}$. Lemma 4.1 shows that if $W = E + \nabla p$ with $p \in W^{1,\infty}$, then solutions to $L_{W,q}u = 0$ are easily obtained from solutions to $L_{E,q}u = 0$. Here we want that E is in $C(\overline{\Omega}; \mathbf{C}^n)$ and is divergence free, so that $L_{E,q}$ is of the desired form.

If $W \in L^p(\Omega; \mathbf{C}^n)$ with $1 < p < \infty$ and Ω has smooth boundary, then one has Helmholtz decompositions $W = E + \nabla p$ where $E \in L^p(\Omega; \mathbf{C}^n)$ is divergence free and $p \in W^{1,p}(\Omega)$ (Schwarz [36]). This fails for $p = \infty$. In our situation we need a condition for W which ensures that $E \in C(\overline{\Omega}; \mathbf{C}^n)$, and the right condition turns out to be Dini continuity. It is interesting that this is also the right condition for the L^∞ decomposition to exist: we give an example of a uniformly continuous vector field W which is not Dini continuous, for which there is no decomposition $W = E + \nabla p$ where E would be in L^∞_{loc} and divergence free.

We begin with some elementary remarks. Let Ω be a bounded open subset of $\mathbf{R}^n$. Then every function in $C(\overline{\Omega})$ is uniformly continuous in Ω, and conversely any uniformly continuous function in Ω has a unique extension into a function in $C(\overline{\Omega})$.

We call a function $\omega : [0, \infty) \to [0, \infty)$ a modulus of continuity if ω is continuous, nondecreasing, and $\omega(0) = 0$. A function $f : \Omega \to \mathbf{C}$ is continuous with modulus ω if $|f(x) - f(y)| \leq \omega(|x - y|)$ for $x, y \in \Omega$. The same condition is valid for $x, y \in \overline{\Omega}$ if f is replaced with the unique extension in $C(\overline{\Omega})$. For any $f \in C(\overline{\Omega})$, the function $\omega(t) = \sup\{|f(x) - f(y)|\,;\, x, y \in \overline{\Omega}, |x - y| \leq t\}$ is a modulus of continuity for f and is the smallest such modulus. Since Ω is bounded also ω is bounded.

We will consider moduli of continuity ω which satisfy the Dini condition

$$\int_0^\varepsilon \omega(t)\,\frac{dt}{t} < \infty \quad \text{for some } \varepsilon > 0, \tag{4.7}$$

and the minor technical condition

$$\frac{\omega(t_1)}{t_1} \geq \frac{\omega(t_2)}{t_2} \quad \text{when } t_1 < t_2. \tag{4.8}$$

If $f \in C(\overline{\Omega})$ is continuous with some modulus ω satisfying (4.7) and (4.8), we say that f is Dini continuous and write $f \in C^d(\Omega)$. Examples of admissible moduli are $\omega(t) = t^\alpha$ with $0 < \alpha < 1$ (so Hölder continuous functions are included) and $\omega(t) = |\log t|^{-1-\alpha}$ for $\alpha > 0$.

We will need an extension result, which is the only place where the condition (4.8) is used.

Lemma 4.4. Let $\Omega \subseteq \mathbf{R}^n$ be a bounded open set and let $f \in C^d(\Omega)$. Then there is an extension F of f so that $F \in C^d_c(\mathbf{R}^n)$.

Proof. The Whitney extension procedure, [39], gives the desired result. □

The Helmholtz decomposition will be a consequence of the following estimates for the generalized Newtonian potential of a vector field F. We write $\Gamma(x) = -c_n|x|^{2-n}$ for the fundamental solution of Δ in $\mathbf{R}^n$, $n \geq 3$. This lemma is similar to [18, Chapter 4] and [7].

Lemma 4.5. Let $\Omega \subseteq \mathbf{R}^n$ be a bounded open set, where $n \geq 3$, and let $F \in C^d(\Omega; \mathbf{C}^n)$. Fix some ball Ω_0 with $\overline{\Omega} \subseteq \Omega_0$, and extend F to a vector field in $C^d(\Omega_0; \mathbf{C}^n)$. When $x \in \overline{\Omega}$ define

$$w(x) = \int_{\Omega_0} \partial_k \Gamma(x-y) F_k(y)\, dy. \tag{4.9}$$

Then $w \in C^1(\overline{\Omega})$ and $\partial_{x_j} w = w_j$, where

$$w_j(x) = \int_{\Omega_0} \partial_j \partial_k \Gamma(x-y)[F_k(y) - F_k(x)]\, dy - F_k(x) \int_{\partial\Omega_0} \partial_k \Gamma(x-y) \nu_j\, dS(y). \tag{4.10}$$

Proof. First note that (4.9) and (4.10) are well defined for $x \in \overline{\Omega}$. For (4.10) this follows from

$$\begin{aligned} |w_j(x)| &\leq C\Big(\int_{\Omega_0} \frac{\omega(|x-y|)}{|x-y|^n}\, dy + \|F\|_{L^\infty} \int_{\partial\Omega_0} |x-y|^{1-n}\, dS(y) \Big) \\ &\leq C\Big(\int_0^R \omega(t)\, \frac{dt}{t} + r^{1-n} \int_{\partial\Omega_0} dS \Big) < \infty \end{aligned}$$

where $R = \operatorname{diam}(\Omega_0)$ and $r = \operatorname{dist}(\overline{\Omega}, \partial\Omega_0)$.

Let $\eta \in C^\infty(\mathbf{R}^n)$ with $0 \leq \eta \leq 1$, $\eta = 0$ for $|x| \leq 1/2$, and $\eta = 1$ for $|x| \geq 1$. Define $\eta_\varepsilon(x) = \eta(x/\varepsilon)$, so that $|\partial^\alpha \eta_\varepsilon| \leq C_\alpha \varepsilon^{-|\alpha|}$. Now ε and $|x-y|$ are comparable on $\operatorname{supp}(\partial^\alpha \eta_\varepsilon(x - \,\cdot\,))$ for $|\alpha| \geq 1$, and

$$|\partial^\alpha \eta_\varepsilon(x-y)| \leq C_\alpha |x-y|^{-|\alpha|}. \tag{4.11}$$

For $x \in \overline{\Omega}$ define

$$w^\varepsilon(x) = \int_{\Omega_0} \partial_k \Gamma(x-y) \eta_\varepsilon(x-y) F_k(y)\, dy.$$

Then $w^\varepsilon \to w$ uniformly in $\overline{\Omega}$ since

$$w^\varepsilon(x) - w(x) = \int_{\Omega_0} \partial_k \Gamma(x-y)(\eta_\varepsilon(x-y) - 1) F_k(y)\, dy,$$

and the integral is bounded by $C\|F\|_{L^\infty} \int_{|z| \leq \varepsilon} |z|^{1-n}\, dz$.

The function w^ε is C^∞. If $x \in \overline{\Omega}$ we obtain by differentiating and integrating by parts that

$$\begin{aligned} \partial_{x_j} w^\varepsilon(x) - w_j(x) &= \int_{\Omega_0} \partial_{x_j}(\partial_k \Gamma(x-y)(\eta_\varepsilon(x-y) - 1))[F_k(y) - F_k(x)]\, dy \\ &\quad - F_k(x) \int_{\partial\Omega_0} \partial_k \Gamma(x-y)(\eta_\varepsilon(x-y) - 1) \nu_j\, dS(y). \end{aligned}$$

Since $|x-y| \geq r$ for $y \in \partial\Omega_0$ the boundary integral vanishes for small ε. Using the Leibniz rule in the first integrand gives terms which are bounded by $C|x-y|^{-n}\omega(|x-y|)$ by (4.11), and the support of each term is contained in $|x-y| \leq \varepsilon$. This shows that $\partial_{x_j} w^\varepsilon \to w_j$ uniformly in $\overline{\Omega}$, which implies that $w \in C^1(\overline{\Omega})$ and $\partial_{x_j} w = w_j$. □

Proposition 4.1. Let $\Omega \subseteq \mathbf{R}^n$ be a bounded open set and $W \in C^d(\Omega; \mathbf{C}^n)$. Then there is a decomposition

$$W = E + \nabla p$$

where $E \in C(\overline{\Omega}; \mathbf{C}^n)$ is divergence free and $p \in C^1(\overline{\Omega})$.

Proof. Extend W to $C_c^d(\mathbf{R}^n; \mathbf{C}^n)$ and fix a ball Ω_0 with $\mathrm{supp}(W) \subseteq \Omega_0$. Let $p \in C^1(\overline{\Omega})$ be the generalized Newtonian potential given by (4.9). Then $p = \Gamma * \mathrm{div}\, W$ and $\Delta p = \mathrm{div}\, W$ since W is compactly supported, and $E = W - \nabla p$ has the desired properties. □

Remark. If Ω has C^2 boundary then a modification of Lemma 4.5, where $\Gamma(x-y)$ is replaced by the Green function $G(x,y)$ for Δ in Ω, gives a unique decomposition in Proposition 4.1 if one requires $p|_{\partial\Omega} = 0$. This uses estimates for the Green function as in [20].

We conclude the section with a counterexample from [18, Problem 4.9], which shows that Dini continuity is required for Proposition 4.1.

Let $P(x) = x_1^2 - x_2^2$ be a homogeneous harmonic polynomial of degree 2 in $\mathbf{R}^n$. Note that $\partial_{x_1}^2 P \neq 0$. Let $\eta \in C_c^\infty(\mathbf{R}^n)$, $0 \leq \eta \leq 1$, with $\eta = 1$ near $|x| \leq 1$ and $\mathrm{supp}(\eta) \subseteq \{|x| < 2\}$, let $t_k = 2^k$, and let (c_k) be a sequence of positive real numbers with $c_k \to 0$ as $k \to \infty$, and $\sum_{k=0}^\infty c_k$ divergent. Define

$$f(x) = \sum_{k=0}^\infty c_k \Delta(\eta P)(t_k x).$$

Now $\mathrm{supp}(\Delta(\eta P)(t_k x)) \subseteq \{2^{-k} < |x| < 2^{-k+1}\}$, so f is C^∞ in $\mathbf{R}^n \smallsetminus \{0\}$. Since $f(0) = 0$ and $|f(x)| \leq c_k \|\Delta(\eta P)\|_{L^\infty}$ for $2^{-k} \leq |x| \leq 2^{-k+1}$, we see that f is continuous at 0 and uniformly continuous in $\mathbf{R}^n$.

One has

$$f(x) = \sum_{k=0}^\infty c_k t_k^{-2} \Delta((\eta P)(t_k x))$$

with convergence in the sense of distributions, and we obtain

$$\Gamma * f(x) = \sum_{k=0}^\infty c_k t_k^{-2} \eta(t_k x) P(t_k x) = P(x) \sum_{k=0}^\infty c_k \eta(t_k x).$$

This is C^∞ in $\mathbf{R}^n \setminus \{0\}$. Writing $\Gamma * f = Pg$, we have for $x \neq 0$

$$\partial_{x_1}^2(\Gamma * f)(x) = 2g(x) + 4x_1\partial_{x_1}g(x) + (x_1^2 - x_2^2)\partial_{x_1}^2 g(x).$$

By a similar argument as above we see that the last two terms are continuous functions in $\mathbf{R}^n$ with value 0 at $x = 0$. But $g(x) \geq \sum_{k=0}^m c_k$ for $0 < |x| \leq 2^{-m}$, so $\partial_{x_1}^2(\Gamma * f)$ is not bounded near 0.

Let now $\Omega = B(0,2)$ and $W = (f, 0, \ldots, 0) \in C(\overline{\Omega}; \mathbf{R}^n)$. Then $p_0 = \partial_{x_1}(\Gamma * f)$ solves $\Delta p_0 = \operatorname{div} W$ in Ω, but $\partial_{x_1} p_0$ is not bounded near 0. Now if $W = E + \nabla p$ is a decomposition of W where E is divergence free, then $\Delta p = \operatorname{div} W$ in Ω, so that $p = p_0 + v$ where v is a harmonic function. This shows that $\partial_{x_1} p$ can not be bounded near 0, and the same is true for E.

4.3 Complex geometrical optics solutions

The next step is to construct complex geometrical optics solutions to the equation $L_{W,q}u = 0$, where W is a Dini continuous vector field. For this we first need a simple result concerning a first order equation. Let $\zeta = \gamma_1 + i\gamma_2$ be a vector with $\gamma_j \in \mathbf{R}^n$, $|\gamma_j| = 1$, and $\gamma_1 \perp \gamma_2$. Then $N_\zeta = \zeta \cdot \nabla$ is the $\overline{\partial}$ operator in different coordinates, so that there is an inverse operator defined by

$$N_\zeta^{-1} f = \frac{1}{2\pi} \int_{\mathbf{R}^2} \frac{1}{y_1 + iy_2} f(x - y_1\gamma_1 - y_2\gamma_2)\, dy_1\, dy_2.$$

The proof of the following lemma is immediate (see also [41]).

Lemma 4.6. Let $f \in W^{k,\infty}(\mathbf{R}^n)$, $k \geq 0$, with $\operatorname{supp}(f) \subseteq B(0,R)$. Then $\phi = N_\zeta^{-1} f \in W^{k,\infty}(\mathbf{R}^n)$ solves $N_\zeta \phi = f$ in $\mathbf{R}^n$, and satisfies

$$\|\phi\|_{W^{k,\infty}} \leq C\|f\|_{W^{k,\infty}} \tag{4.12}$$

where $C = C(R)$. If $f \in C_c(\mathbf{R}^n)$ then $\phi \in C(\mathbf{R}^n)$.

If $\rho \in \mathbf{C}^n$ satisfies $\rho \cdot \rho = 0$ we will write $\rho = s\zeta$, where ζ is of the above form and $s = \frac{|\rho|}{\sqrt{2}}$. With this notation we have the following proposition.

Proposition 4.2. Assume $\Omega \subseteq \mathbf{R}^n$ is a bounded open set, $W \in C^d(\Omega; \mathbf{C}^n)$, and $q \in L^\infty(\Omega; \mathbf{C})$. Let $\tilde{W}$ be any $C_c^d(\mathbf{R}^n; \mathbf{C}^n)$ extension of W. Then for $\rho \in \mathbf{C}^n$ satisfying $\rho \cdot \rho = 0$ and $|\rho|$ large enough, there exist complex geometrical optics solutions

$$u = e^{\rho \cdot x}(e^{\phi^\sharp} + \omega) \tag{4.13}$$

to the equation $L_{W,q}u = 0$ in Ω, where $\phi^\sharp \in C^1(\mathbf{R}^n)$ converges uniformly in $\mathbf{R}^n$ to $N_\zeta^{-1}(-i\zeta \cdot \tilde{W})$ as $s \to \infty$, and

$$\|\phi^\sharp\|_{W^{1,\infty}(\Omega)} = o(|\rho|^{1/2}), \tag{4.14}$$

$$\|\omega\|_{L^2(\Omega)} = o(1), \tag{4.15}$$

$$\|\omega\|_{H^1(\Omega)} = o(|\rho|) \tag{4.16}$$

as $|\rho| \to \infty$.

Proof. We first assume $W = E$ is any divergence free $C(\overline{\Omega}; \mathbf{C}^n)$ vector field, and look for a solution $v = e^{\rho\cdot x}(e^{\phi_E^\sharp} + \omega_E)$. Note that $L_{E,q}v = 0$ in Ω is equivalent with

$$(\Delta + 2iE\cdot\nabla + G)v = 0 \qquad \text{in } \Omega$$

where $G = -E\cdot E - q \in L^\infty(\Omega; \mathbf{C})$. Let $\tilde{E} \in C_c(\mathbf{R}^n; \mathbf{C}^n)$ be any extension of E, and decompose $\tilde{E}$ as

$$\tilde{E} = \tilde{E}_\rho^\sharp + \tilde{E}_\rho^\flat$$

where $\tilde{E}_\rho^\sharp = \tilde{E} * \phi_r$ is a smooth approximation to $\tilde{E}$ so that $r = r(\rho) = |\rho|^{1/2}$. Then by Lemma 2.1 we have the estimates

$$\begin{aligned} \|\tilde{E}_\rho^\sharp\|_{W^{1,\infty}} &= o(|\rho|^{1/2}), \\ \|\tilde{E}_\rho^\sharp\|_{W^{2,\infty}} &= o(|\rho|), \\ \|\tilde{E}_\rho^\flat\|_{L^\infty} &= o(1) \end{aligned} \tag{4.17}$$

as $|\rho| \to \infty$.

Writing $\rho = s\zeta$ we choose

$$\phi_E^\sharp = \phi_E^\sharp(x,\zeta,s) = N_\zeta^{-1}(-i\zeta\cdot\tilde{E}_{s\zeta}^\sharp), \tag{4.18}$$

so that $\rho\cdot\nabla\phi_E^\sharp = -i\rho\cdot\tilde{E}_\rho^\sharp$. Now ω_E must satisfy

$$(\Delta_\rho + 2iE\cdot\nabla_\rho + G)\omega_E = f \quad \text{in } \Omega$$

where $f = -(\Delta_\rho + 2iE\cdot\nabla_\rho + G)e^{\phi_E^\sharp}$. But one has $(2\rho\cdot\nabla + 2i\tilde{E}_\rho^\sharp\cdot\rho)e^{\phi_E^\sharp} = 2(\rho\cdot\nabla\phi_E^\sharp + i\rho\cdot\tilde{E}_\rho^\sharp)e^{\phi_E^\sharp} = 0$ by the choice of $\phi_E^\sharp$, and we have

$$\begin{aligned} f &= -(\Delta + 2iE\cdot\nabla + 2i\tilde{E}_\rho^\flat\cdot\rho + G)e^{\phi_E^\sharp} \\ &= -(\Delta\phi_E^\sharp + \nabla\phi_E^\sharp\cdot\nabla\phi_E^\sharp + 2iE\cdot\nabla\phi_E^\sharp + 2i\tilde{E}_\rho^\flat\cdot\rho + G)e^{\phi_E^\sharp}. \end{aligned}$$

Since Ω is bounded we get the estimate

$$\begin{aligned} \|f\|_{L^2(\Omega)} \leq C(\|\phi_E^\sharp\|_{W^{2,\infty}(\Omega)} + \|\phi_E^\sharp\|^2_{W^{1,\infty}(\Omega)} + \|\phi_E^\sharp\|_{W^{1,\infty}(\Omega)} \\ + |\rho|\|\tilde{E}_\rho^\flat\|_{L^\infty(\Omega)} + 1)e^{\|\phi_E^\sharp\|_{L^\infty(\Omega)}} \end{aligned}$$

where C depends on Ω and $\tilde{E}$, q. From (4.12) and (4.17) we have $\|f\|_{L^2(\Omega)} = o(|\rho|)$. Using Theorem 1.4 gives the desired estimates for ω_E.

Now assume $W \in C^d(\Omega; \mathbf{C}^n)$, and $\tilde{W}$ is a given $C_c^d(\mathbf{R}^n; \mathbf{C}^n)$ extension of W. Let Ω_0 be a ball with $\overline{\Omega} \subseteq \Omega_0$, and use the Helmholtz decomposition of Proposition 4.1 in Ω_0 to write $\tilde{W} = \tilde{E} + \nabla\tilde{p}$ where $\tilde{E} \in C(\overline{\Omega}_0; \mathbf{C}^n)$ is divergence free and $\tilde{p} \in C^1(\overline{\Omega}_0)$. Choose any $C_c^1(\mathbf{R}^n)$ extension of $\tilde{p}$, and define $\tilde{E} = \tilde{W} - \nabla\tilde{p}$ outside Ω_0. We then have a complex geometrical optics solution $v = e^{\rho\cdot x}(e^{\phi_E^\sharp} + \omega_E)$ of $L_{E,q}v = 0$ in Ω, with $\phi_E^\sharp$ given by (4.18).

Write $u = e^{-ip}v$. Lemma 4.1 implies that $L_{W,q}u = 0$ in Ω, and u is of the form (4.13) with

$$\phi^\sharp = \phi_E^\sharp - i\tilde{p} = N_\zeta^{-1}(-i\zeta\cdot(\tilde{E}_{s\zeta}^\sharp + \nabla\tilde{p})) \tag{4.19}$$

and $\omega = e^{-ip}\omega_E$. Obviously $\phi^\sharp \in C^1(\mathbf{R}^n)$, and (4.12), (4.17) show that $\phi^\sharp \to N_\zeta^{-1}(-i\zeta\cdot\tilde{W})$ uniformly in $\mathbf{R}^n$ as $s \to \infty$. We have (4.14) by (4.17), and (4.15), (4.16) follow from the corresponding estimates for ω_E. □

Remark. For further use we note that the result is also valid with the choice $\phi^\sharp = N_\zeta^{-1}(-i\zeta\cdot(\tilde{E}_{s\zeta}^\sharp + \nabla\tilde{p})) + t(\zeta\cdot x) \to N_\zeta^{-1}(-i\zeta\cdot\tilde{W}) + t(\zeta\cdot x)$, where $t \in \mathbf{R}$.

Heading toward the uniqueness result for the inverse problem, the following proposition shows what is obtained when the complex geometrical optics solutions are used in the identity of Lemma 4.3.

Proposition 4.3. Let $\Omega \subseteq \mathbf{R}^n$ be a bounded open set where $n \geq 3$, let $W_1, W_2 \in C^d(\Omega;\mathbf{C}^n)$, and let $q_1, q_2 \in L^\infty(\Omega;\mathbf{C})$. Then $C_{W_1,q_1} = C_{W_2,q_2}$ implies

$$\int_\Omega e^{ik\cdot x+\phi_1+\phi_2}(\zeta\cdot(W_1 - W_2))\,dx = 0$$

for any $k \in \mathbf{R}^n$ and $\zeta = \gamma_1 + i\gamma_2$ where $k, \gamma_1, \gamma_2 \in \mathbf{R}^n$ are mutually orthogonal with $|\gamma_1| = |\gamma_2| = 1$, and $\phi_j(\cdot,\zeta) \in C(\mathbf{R}^n)$ are defined by

$$\phi_1 = N_\zeta^{-1}(-i\zeta\cdot\tilde{W}_1), \tag{4.20}$$

$$\phi_2 = N_\zeta^{-1}(i\zeta\cdot\tilde{W}_2) \tag{4.21}$$

where $\tilde{W}_j$ are any $C_c^d(\mathbf{R}^n;\mathbf{C}^n)$ extensions of W_j.

Proof. Let $\tilde{W}_j = \tilde{E}_j + \nabla\tilde{p}_j$ be Helmholtz decompositions in a larger ball Ω_0 given by Proposition 4.1, choose a $C_c^1(\mathbf{R}^n)$ extension of $\tilde{p}_j$, and define $\tilde{E}_j = \tilde{W}_j - \nabla\tilde{p}_j$. From Proposition 4.2 we know that there are complex geometrical optics solutions to $L_{W_1,q_1}u = 0$ and $L_{\bar{W}_2,\bar{q}_2}v = 0$ in Ω, which have the form

$$u = e^{\rho_1\cdot x}(e^{\phi_1^\sharp} + \omega_1), \tag{4.22}$$

$$\bar{v} = e^{\rho_2\cdot x}(e^{\phi_2^\sharp} + \omega_2). \tag{4.23}$$

We have done some relabeling, so that $\rho_j = s\zeta_j \in \mathbf{C}^n$ are any large vectors with $\rho_j\cdot\rho_j = 0$, $\phi_1^\sharp = N_{\zeta_1}^{-1}(-i\zeta_1\cdot(\tilde{E}_{1,s\zeta_1}^\sharp + \nabla\tilde{p}_1))$ and $\phi_2^\sharp = N_{\zeta_2}^{-1}(i\zeta_2\cdot(\tilde{E}_{2,s\zeta_2}^\sharp + \nabla\tilde{p}_2))$, and ω_j satisfy (4.15), (4.16). Now

$$\nabla\bar{v} = \rho_2 e^{\rho_2\cdot x}(e^{\phi_2^\sharp} + \omega_2) + e^{\rho_2\cdot x}(e^{\phi_2^\sharp}\nabla\phi_2^\sharp + \nabla\omega_2),$$

$$\nabla u = \rho_1 e^{\rho_1\cdot x}(e^{\phi_1^\sharp} + \omega_1) + e^{\rho_1\cdot x}(e^{\phi_1^\sharp}\nabla\phi_1^\sharp + \nabla\omega_1)$$

and assuming $\rho_1 + \rho_2 = ik$ with $k \in \mathbf{R}^n$,

$$\begin{aligned} u\nabla\bar{v} - \bar{v}\nabla u = (\rho_2 - \rho_1)e^{ik\cdot x+\phi_1^\sharp+\phi_2^\sharp} \\ + (\rho_2 - \rho_1)e^{ik\cdot x}(e^{\phi_1^\sharp}\omega_2 + e^{\phi_2^\sharp}\omega_1 + \omega_1\omega_2) \\ + e^{ik\cdot x}(e^{\phi_1^\sharp+\phi_2^\sharp}(\nabla\phi_2^\sharp - \nabla\phi_1^\sharp) + e^{\phi_1^\sharp}\nabla\omega_2 - e^{\phi_2^\sharp}\nabla\omega_1 \\ + e^{\phi_2^\sharp}\nabla\phi_2^\sharp\omega_1 - e^{\phi_1^\sharp}\nabla\phi_1^\sharp\omega_2 + \omega_1\nabla\omega_2 - \omega_2\nabla\omega_1) \end{aligned}$$

Now inserting the solutions u and v in the identity of Lemma 4.3 gives

$$A + B + C + D = 0 \tag{4.24}$$

where

$$\begin{aligned} A &= i\int_\Omega e^{ik\cdot x+\phi_1^\sharp+\phi_2^\sharp}((\rho_2 - \rho_1)\cdot(W_1 - W_2))\,dx, \\ B &= i\int_\Omega e^{ik\cdot x}((\rho_2 - \rho_1)\cdot(W_1 - W_2))(e^{\phi_1^\sharp}\omega_2 + e^{\phi_2^\sharp}\omega_1 + \omega_1\omega_2)\,dx, \\ C &= i\int_\Omega e^{ik\cdot x}(W_1 - W_2)\cdot(e^{\phi_1^\sharp+\phi_2^\sharp}(\nabla\phi_2^\sharp - \nabla\phi_1^\sharp) + e^{\phi_1^\sharp}\nabla\omega_2 - e^{\phi_2^\sharp}\nabla\omega_1 \\ &\quad + e^{\phi_2^\sharp}\nabla\phi_2^\sharp\omega_1 - e^{\phi_1^\sharp}\nabla\phi_1^\sharp\omega_2 + \omega_1\nabla\omega_2 - \omega_2\nabla\omega_1)\,dx, \\ D &= \int_\Omega (W_1\cdot W_1 - W_2\cdot W_2 + q_1 - q_2)e^{ik\cdot x}(e^{\phi_1^\sharp+\phi_2^\sharp} + e^{\phi_1^\sharp}\omega_2 + e^{\phi_2^\sharp}\omega_1 \\ &\quad + \omega_1\omega_2)\,dx. \end{aligned}$$

Let k, γ_1 and γ_2 be three mutually orthogonal vectors in $\mathbf{R}^n$ with $|\gamma_1| = |\gamma_2| = 1$, and let $s > |k|/2$. We make the specific choice

$$\rho_1 = s\gamma_1 + i(\frac{k}{2} + s\sqrt{1 - \frac{|k|^2}{4s^2}}\gamma_2), \tag{4.25}$$

$$\rho_2 = -s\gamma_1 + i(\frac{k}{2} - s\sqrt{1 - \frac{|k|^2}{4s^2}}\gamma_2). \tag{4.26}$$

Then $\rho_1\cdot\rho_1 = \rho_2\cdot\rho_2 = 0$, $\rho_1 + \rho_2 = ik$ and $\rho_1 - \rho_2 = 2s(\gamma_1 + i\sqrt{1 - \frac{|k|^2}{4s^2}}\gamma_2)$.

We will multiply the equation (4.24) by $\frac{1}{s}$ and let $s \to \infty$. We first note that $\phi_1^\sharp$ is of the form

$$\begin{aligned} \phi_1^\sharp(x) = \frac{1}{2\pi}\int_{\mathbf{R}^2}\frac{1}{y_1 + iy_2}[-i(\gamma_1 + i(\frac{k}{2s} + \sqrt{1 - \frac{|k|^2}{4s^2}}\gamma_2))\cdot \\ (\tilde{E}_{1,\rho_1}^\sharp + \nabla\tilde{p}_1)(x - y_1\gamma_1 - y_2(\frac{k}{2s} + \sqrt{1 - \frac{|k|^2}{4s^2}}\gamma_2))]\,dy_1\,dy_2 \end{aligned}$$

where $\tilde{E}^{\sharp}_{1,\rho_1} \to \tilde{E}_1$ uniformly in $\mathbf{R}^n$ as $s \to \infty$. Dominated convergence shows that as $s \to \infty$ this converges pointwise in $\mathbf{R}^n$ to

$$\phi_1(x,\zeta) = \frac{1}{2\pi} \int_{\mathbf{R}^2} \frac{1}{y_1 + iy_2} [-i\zeta \cdot (\tilde{E}_1 + \nabla \tilde{p}_1)(x - y_1\gamma_1 - y_2\gamma_2)] \, dy_1 \, dy_2$$

where $\zeta = \gamma_1 + i\gamma_2$. Now $\phi_1 = N_\zeta^{-1}(-i\zeta \cdot \tilde{W}_1)$. Similarly $\phi_2^\sharp \to \phi_2$ in $\mathbf{R}^n$ as $s \to \infty$, where $\phi_2 = N_\zeta^{-1}(i\zeta \cdot \tilde{W}_2)$. Since $\|\phi_j^\sharp\|_{L^\infty(\Omega)}, \|W_j\|_{L^\infty(\Omega)}, \|q_j\|_{L^\infty(\Omega)} \leq C$ with C independent of ρ and since $\|\nabla \phi_j^\sharp\|_{L^\infty(\Omega)} = o(s^{1/2})$, the estimates (4.15), (4.16) and dominated convergence imply that

$$\lim_{s\to\infty} \frac{1}{s} A = i \int_\Omega e^{ik\cdot x + \phi_1 + \phi_2} (-2\zeta \cdot (W_1 - W_2)) \, dx,$$
$$\lim_{s\to\infty} \frac{1}{s} B = \lim_{s\to\infty} \frac{1}{s} C = \lim_{s\to\infty} \frac{1}{s} D = 0.$$

This gives the claim. □

The conclusion in Proposition 4.3 is not strong enough to give the uniqueness result. The following improvement is needed.

Proposition 4.4. In the situation of Proposition 4.3, one has for $|t| < 1$

$$\int_\Omega e^{ik\cdot x + \phi_1 + \phi_2 + t(\zeta\cdot x)} (\zeta \cdot (W_1 - W_2)) \, dx = 0 \tag{4.27}$$

for the appropriate k, ζ. Consequently

$$\int_\Omega e^{ik\cdot x + \phi_1 + \phi_2} (\zeta \cdot (W_1 - W_2))(\zeta \cdot x)^m \, dx = 0 \tag{4.28}$$

for such k, ζ and any integer $m \geq 0$.

Proof. In the proof of the Proposition 4.3, replace $\phi_1^\sharp$ by $\phi_1^\sharp + t(\zeta_1 \cdot x)$ and ϕ_1 by $\phi_1 + t(\zeta \cdot x)$. This is possible because of the remark after Proposition 4.2. The proof then yields (4.27), and (4.28) follows by differentiating (4.27) m times with respect to t and by evaluating at 0. □

Remark. The methods in this section, as well as in the following section, are mostly due to Sun [41] except for some modifications required because of the nonsmooth situation. The construction of complex geometrical optics solutions using a Helmholtz decomposition and convolution approximation is similar to Panchenko [35]. The existence of complex geometrical optics solutions for Dini continuous vector fields is a new result, and was made possible by the norm estimates of Theorem 1.4. Also the proof of (4.28) is new and avoids an additional argument in [41].

4.4 A uniqueness result

In this section we will prove Theorem 1.5. The main difficulty is to show that $C_{W_1,q_1} = C_{W_2,q_2}$ implies $\mathrm{curl}(W_1 - W_2) = 0$. This will follow from Proposition 4.4 and a sequence of lemmas. The first is an elementary result on integration by parts which is needed in the arguments below.

Lemma 4.7. Let $\Omega \subseteq \mathbf{R}^n$ be a bounded open set with smooth boundary, let $f \in C(\mathbf{R}^n)$ and $\zeta \in \mathbf{C}^n$, and suppose $\zeta \cdot \nabla f \in L^1(\Omega)$. Then one has

$$\int_\Omega \zeta \cdot \nabla f \, dx = \int_{\partial\Omega} f(\zeta \cdot \nu) \, dS.$$

Proof. This follows by approximation from the corresponding result for smooth functions. □

The next two lemmas consider characterizations for $\mathrm{curl}\, W = 0$.

Lemma 4.8. Let $\Omega = B(0,R) \subseteq \mathbf{R}^n$ be a ball, and let $W \in C(\overline{\Omega}; \mathbf{C}^n)$. Then $\mathrm{curl}\, W = 0$ if

$$\zeta \cdot \int_\Omega e^{ik\cdot x} W(x) \, dx = 0$$

whenever $\zeta = \gamma_1 + i\gamma_2$ and $k, \gamma_1, \gamma_2 \in \mathbf{R}^n$ where $|\gamma_j| = 1$, and $\{k, \gamma_1, \gamma_2\}$ is orthogonal.

Proof. The given condition implies that $\gamma \cdot (\chi_\Omega W)\hat{\ }(\xi) = 0$ whenever $\gamma \perp \xi$. Assume $\xi \neq 0$ and let $\gamma_{jk}(\xi) = \xi_j e_k - \xi_k e_j$ for $j \neq k$. Then $\gamma_{jk}(\xi) \perp \xi$ and so

$$\xi_j (\chi_\Omega W_k)\hat{\ }(\xi) - \xi_k (\chi_\Omega W_j)\hat{\ }(\xi) = 0.$$

Consequently $\partial_j W_k - \partial_k W_j = 0$ in Ω for $j \neq k$. □

Lemma 4.9. Let Ω and W be as in Lemma 4.8, let $\tilde{W}$ be any $C_c(\mathbf{R}^n; \mathbf{C}^n)$ extension of W, and define

$$\Psi = N_\zeta^{-1}(\zeta \cdot \tilde{W}).$$

Then $\mathrm{curl}\, W = 0$ in Ω if

$$\int_{\partial\Omega\cap T} (\zeta \cdot \nu_T) \Psi \, dS = 0 \tag{4.29}$$

whenever $\zeta = \gamma_1 + i\gamma_2$ with $|\gamma_j| = 1$ and $\gamma_1 \perp \gamma_2$, and whenever T is a two-dimensional plane parallel to γ_1 and γ_2. Here $\nu_T = (\nu \cdot \gamma_1)\gamma_1 + (\nu \cdot \gamma_2)\gamma_2$ and dS is the surface measure of $\partial\Omega \cap T$.

Proof. Fix γ_1, γ_2 with $|\gamma_j| = 1$ and $\gamma_1 \perp \gamma_2$. Extend these two vectors into a positive orthonormal basis $\{\gamma_1, \ldots, \gamma_n\}$ of $\mathbf{R}^n$. Then any k orthogonal to γ_1 and γ_2 has the form $k = \sum_{j=3}^n k_j \gamma_j$, and for such k one has

$$\begin{aligned}\zeta \cdot \int_\Omega e^{ik\cdot x} W(x)\,dx &= \int_\Omega e^{ik\cdot x}(\zeta \cdot \nabla \Psi)\,dx \\ &= \int_\Omega \zeta \cdot \nabla(e^{ik\cdot x}\Psi)\,dx = \int_{\partial\Omega} e^{ik\cdot x}(\zeta\cdot\nu)\Psi\,dS\end{aligned}$$

using $\zeta \cdot k = 0$ and Lemma 4.7. Writing $x = \lambda_1\gamma_1 + \ldots + \lambda_n\gamma_n$ and splitting the integral over $\lambda' = (\lambda_1, \lambda_2)$ and $\lambda'' = (\lambda_3, \ldots, \lambda_n)$ gives

$$\int_{\partial\Omega} e^{ik\cdot x}(\zeta\cdot\nu)\Psi\,dS = \int_{\mathbf{R}^{n-2}} e^{ik''\cdot\lambda''} \int_{\partial\Omega\cap T_{\lambda''}} (\zeta\cdot\nu)\Psi\,dS(\lambda')\,d\lambda''$$

where $k'' = (k_3, \ldots, k_n)$ and $T_{\lambda''} = \sum_{j=3}^n \lambda_j\gamma_j + T_0$ where T_0 is the two-dimensional plane spanned by γ_1 and γ_2. Here $\zeta \cdot \nu = \zeta \cdot \nu_{T_{\lambda''}}$, and using the inverse Fourier transform gives the claim. □

Now we assume $C_{W_1,q_1} = C_{W_2,q_2}$ and start working toward the condition of Lemma 4.9. The next lemma is a restatement of Proposition 4.4.

Lemma 4.10. Let $\Omega = B(0,R) \subseteq \mathbf{R}^n$ be a ball, and let $n \geq 3$. Assume $W_1, W_2 \in C^d(\Omega; \mathbf{C}^n)$ and $q_1, q_2 \in L^\infty(\Omega; \mathbf{C})$. Suppose that $C_{W_1,q_1} = C_{W_2,q_2}$. Then

$$\int_{\partial\Omega\cap T} (\zeta\cdot\nu_T)(\zeta\cdot x_T)^m e^\Psi\,dS = 0 \tag{4.30}$$

whenever $\zeta = \gamma_1 + i\gamma_2$ with $|\gamma_j| = 1$ and $\gamma_1 \perp \gamma_2$, and whenever T is a two-dimensional plane parallel to γ_1 and γ_2. Here $x_T = (x\cdot\gamma_1)\gamma_1 + (x\cdot\gamma_2)\gamma_2$, and

$$\Psi = N_\zeta^{-1}(-i\zeta\cdot(\tilde{W}_1 - \tilde{W}_2)) \tag{4.31}$$

where $\tilde{W}_j$ are any $C_c^d(\mathbf{R}^n; \mathbf{C}^n)$ extensions of W_j.

Proof. This follows from Proposition 4.4 similarly as Lemma 4.9. Note that $\Psi = \phi_1 + \phi_2$ where ϕ_j are defined by (4.20), (4.21). □

The next lemma is the main step in the proof of Theorem 1.5, and shows how the condition (4.30), which depends nonlinearly on Ψ, may be used to obtain the condition (4.29). The assumptions that Ω is a ball and W_1 and W_2 vanish near $\partial\Omega$ are removed later.

Lemma 4.11. Let $\Omega = B(0,R) \subseteq \mathbf{R}^n$ with $n \geq 3$, let $W_1, W_2 \in C_c^d(\Omega; \mathbf{C}^n)$, and let $q_1, q_2 \in L^\infty(\Omega; \mathbf{C})$. Suppose that $C_{W_1,q_1} = C_{W_2,q_2}$. Then $\operatorname{curl} W_1 = \operatorname{curl} W_2$ in Ω.

Proof. We let $\tilde{W}_j$ be the $C_c^d(\mathbf{R}^n;\mathbf{C}^n)$ extension of W_j which is zero outside Ω, and define Ψ by (4.31). Then we are in the situation of Lemma 4.10.

Fix γ_1, γ_2 with $|\gamma_j| = 1$ and $\gamma_1 \perp \gamma_2$, and let $T = T_0 + \gamma''$ be a two-dimensional plane parallel to $T_0 = \mathrm{span}(\gamma_1, \gamma_2)$ with $\gamma'' \perp T_0$ and $\partial\Omega \cap T \neq \emptyset$. Note that $\partial\Omega \cap T$ is a circle with center at the origin and some radius r. Let $f(x_1, x_2) = \Psi(rx_1\gamma_1 + rx_2\gamma_2 + \gamma'')$, so that f is continuous in $\mathbf{R}^2$ and f restricted to the unit disc $\mathbb{D}$ corresponds to $\Psi|_{\Omega\cap T}$. In the coordinates (x_1, x_2) on $\partial\mathbb{D}$ we have $x_T = rx_1\gamma_1 + rx_2\gamma_2$ and $\nu_T = x_T/R$, so $\zeta \cdot x_T = R(\zeta \cdot \nu_T) = r(x_1 + ix_2)$. Now (4.30) may be written as

$$\int_0^{2\pi} e^{i(m+1)\theta} e^{f(e^{i\theta})}\, d\theta = 0$$

for any integer $m \geq 0$. This shows that the negative Fourier coefficients of $e^f|_{\partial\mathbb{D}}$ are all zero.

On the other hand, one has in the sense of distributions

$$\begin{aligned} \overline{\partial} f(x_1, x_2) &= \frac{r}{2}\zeta \cdot \nabla\Psi(rx_1\gamma_1 + rx_2\gamma_2 + \gamma'') \\ &= -\frac{ir}{2}\zeta \cdot (\tilde{W}_1 - \tilde{W}_2)(rx_1\gamma_1 + rx_2\gamma_2 + \gamma'') \end{aligned}$$

which shows that $\overline{\partial} f = 0$ for $|x| > 1$. Thus f is holomorphic in $\{|x| > 1\}$ and bounded and continuous in $\{|x| \geq 1\}$, so the same holds for e^f and we obtain that the positive Fourier coefficients of $e^f|_{\partial\mathbb{D}}$ must be zero. This shows that e^f is constant on $\partial\mathbb{D}$ and then f is also constant there. Consequently we have

$$0 = \int_0^{2\pi} e^{i\theta} f(e^{i\theta})\, d\theta = \frac{R}{r}\int_{\partial\Omega\cap T} (\zeta \cdot \nu_T)\Psi\, dS$$

which is (4.29). It follows from Lemma 4.9 that $\mathrm{curl}(-i(W_1 - W_2)) = 0$ and $\mathrm{curl}\, W_1 = \mathrm{curl}\, W_2$ in Ω. ☐

The proof of Theorem 1.5 follows easily using Lemma 4.11.

Theorem 1.5. Let $\Omega \subseteq \mathbf{R}^n$ be a bounded open set where $n \geq 3$, and assume that $W_1, W_2 \in C^d(\Omega;\mathbf{C}^n)$ and $q_1, q_2 \in L^\infty(\Omega;\mathbf{C})$. If $C_{W_1,q_1} = C_{W_2,q_2}$ and $W_1|_{\partial\Omega} = W_2|_{\partial\Omega}$, then $\mathrm{curl}\, W_1 = \mathrm{curl}\, W_2$ and $q_1 = q_2$ in Ω.

Proof. First extend W_1 to a vector field in $C_c^d(\mathbf{R}^n;\mathbf{C}^n)$ using Lemma 4.4. The fact that $W_1 = W_2$ on $\partial\Omega$ ensures that W_2, defined for $x \notin \Omega$ by $W_2(x) = W_1(x)$, will also be in $C_c^d(\mathbf{R}^n;\mathbf{C}^n)$. Let $\Omega' = B(0, R)$ be a ball so that $\overline{\Omega}$ and the supports of W_1 and W_2 are contained in Ω', and extend q_1 and q_2 to Ω' so that they are zero outside Ω. Since $C_{W_1,q_1} = C_{W_2,q_2}$ in Ω, we obtain from Lemma 4.2 that $C_{W_1,q_1} = C_{W_2,q_2}$ in Ω'.

Now Lemma 4.11 gives that $\operatorname{curl} W_1 = \operatorname{curl} W_2$ in Ω'. Since Ω' has trivial cohomology we have $W_2 - W_1 = \nabla p$, where p is in fact given by

$$p(x) = \int_0^1 (W_2 - W_1)(tx) \cdot x \, dt.$$

This defines a function in $C^1(\mathbf{R}^n)$ with $\nabla p = 0$ near $\partial\Omega'$, so by substracting a constant we may assume that $p \in C^1(\overline{\Omega}')$ and $p|_{\partial\Omega'} = 0$. Thus p determines a gauge transformation which preserves Cauchy data sets, and we obtain

$$C_{W_1,q_1} = C_{W_2,q_2} = C_{W_1+\nabla p,q_2} = C_{W_1,q_2} \quad \text{in } \Omega'$$

by Lemma 4.1.

Since $C_{W_1,q_1} = C_{W_1,q_2}$ in Ω', Lemma 4.3 gives

$$\int_{\Omega'} (q_1 - q_2) u\bar{v} \, dx = 0 \tag{1.32}$$

for any $u, v \in H^1(\Omega')$ satisfying $L_{W_1,q_1} u = 0$ and $L_{\bar{W}_1,\bar{q}_2} v = 0$ in Ω'. Fix $k \in \mathbf{R}^n$ and let γ_1, γ_2 be any unit vectors with $\{k, \gamma_1, \gamma_2\}$ orthogonal. Choose u, v to be the complex geometrical optics solutions in Ω' given by (4.22), (4.23), where ρ_1 and ρ_2 are given by (4.25), (4.26) and $\phi_1^\sharp \to N_\zeta(-i\zeta \cdot W_1)$, $\phi_2^\sharp \to N_\zeta^{-1}(i\zeta \cdot W_1)$ in $\mathbf{R}^n$ as $s \to \infty$.

Plugging u and v in (1.32) gives

$$\int_{\Omega'} e^{ik\cdot x + \phi_1^\sharp + \phi_2^\sharp}(q_1 - q_2) \, dx = -\int_{\Omega'} e^{ik\cdot x}(q_1 - q_2)(e^{\phi_1^\sharp}\omega_2 + e^{\phi_2^\sharp}\omega_1 + \omega_1\omega_2) \, dx.$$

Letting $s \to \infty$ this becomes

$$\int_{\Omega'} e^{ik\cdot x}(q_1 - q_2) \, dx = 0$$

using that $\phi_1^\sharp + \phi_2^\sharp \to 0$ and $\|\omega_j\|_{L^2(\Omega)} \to 0$. Thus $(\chi_{\Omega'}(q_1 - q_2))\hat{} = 0$, which implies $q_1 = q_2$ in Ω'. $\square$

Chapter 5

Applications to inverse problems

We proceed to give uniqueness results for the two inverse problems considered in the introduction. In fact global uniqueness will follow almost immediately from Theorem 1.5, as soon as one knows that the Dirichlet to Neumann map determines the boundary values of the coefficients in some sense. Therefore, most of this chapter is devoted to boundary determination results.

For the magnetic Schrödinger equation, we adapt the method of Brown [9], originally used for the conductivity equation, to obtain that the Dirichlet to Neumann map uniquely determines the tangential components of the magnetic potential at the boundary. The argument requires a $C^{1,1}$ domain. In the case of the steady state heat equation with a convection term, the method of singular solutions due to Alessandrini [3] gives a sharper result in terms of boundary regularity, and we are able to handle Lipschitz domains.

5.1 Schrödinger equation in a magnetic field

Let $\Omega \subseteq \mathbf{R}^n$ be a bounded open set with Lipschitz boundary, and suppose $W \in L^\infty(\Omega; \mathbf{R}^n)$ and $q \in L^\infty(\Omega; \mathbf{R})$. Define the Schrödinger operator

$$H_{W,q} = \sum_{j=1}^{n} \Big(\frac{1}{i}\frac{\partial}{\partial x_j} + W_j\Big)^2 + q.$$

The operator $H_{W,q}$ is selfadjoint. We assume that 0 is not a Dirichlet eigenvalue of $H_{W,q}$, so that the problem

$$\begin{cases} H_{W,q} u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega \end{cases}$$

has a unique solution $u \in H^1(\Omega)$ for any $f \in H^{1/2}(\partial\Omega)$.

We may define a Dirichlet to Neumann map formally by

$$\Lambda_{W,q} : f \mapsto \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} + i(W \cdot \nu) f.$$

More precisely, $\Lambda_{W,q}$ is defined by the equivalent weak formulations

$$\begin{aligned}(\Lambda_{W,q} f, g) &= \int_\Omega (\nabla u_f \cdot \nabla \bar{e}_g + iW \cdot (u_f \nabla \bar{e}_g - \bar{e}_g \nabla u_f) + (|W|^2 + q) u_f \bar{e}_g)\, dx \\ &= \int_\Omega (\nabla e_f \cdot \nabla \bar{u}_g + iW \cdot (e_f \nabla \bar{u}_g - \bar{u}_g \nabla e_f) + (|W|^2 + q) e_f \bar{u}_g)\, dx\end{aligned}$$

where $u_h \in H^1(\Omega)$ is the solution to $H_{W,q} u_h = 0$ in Ω with $u_h = h$ on $\partial\Omega$, and e_h is any $H^1(\Omega)$ function with $e_h = h$ on $\partial\Omega$. Then $\Lambda_{W,q}$ is a bounded map $H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$.

In the notation of Section 4.1, one has $H_{W,q} = L_{W,q}$, and the Cauchy data set is $C_{W,q} = \{(f, \Lambda_{W,q} f)\,;\, f \in H^{1/2}(\partial\Omega)\}$. In particular one has gauge equivalence as in Lemma 4.1, so that $\Lambda_{W+\nabla p,q} = \Lambda_{W,q}$ whenever $p \in W^{1,\infty}(\Omega; \mathbf{R})$ with $p|_{\partial\Omega} = 0$.

We want to discuss the determination of boundary values of W from $\Lambda_{W,q}$. Because of gauge equivalence, we see that only the tangential components of W on the boundary may be determined from $\Lambda_{W,q}$. This follows since even if W and ∇p are continuous in $\overline{\Omega}$, the tangential components of ∇p are zero but the normal component may be nonzero.

We will prove boundary identifiability of tangential components in a $C^{1,1}$ domain. To be able to speak of boundary values of a L^∞ vector field, we introduce the following definition.

Definition. We say that $W \in L^\infty(\Omega; \mathbf{R}^n)$ is continuous at $z \in \partial\Omega$ if there exists $\eta \in \mathbf{R}^n$ so that

$$\operatorname{ess\,sup}_{x \in \Omega \cap B(z,r)} |W(x) - \eta| \to 0 \tag{5.1}$$

as $r \to 0$.

Note that if W is continuous at z, then the vector η is unique and given by $\lim_{r\to 0} \frac{1}{|\Omega \cap B(z,r)|} \int_{\Omega \cap B(z,r)} W(x)\, dx$, and we will define $W(z) = \eta$.

The boundary result we intend to prove is the following.

Proposition 5.1. Let $\Omega \subseteq \mathbf{R}^n$ be a bounded open set with $C^{1,1}$ boundary, and let $W \in L^\infty(\Omega; \mathbf{R}^n)$ and $q \in L^\infty(\Omega; \mathbf{R})$. Assume 0 is not a Dirichlet eigenvalue of $H_{W,q}$, and suppose $z \in \partial\Omega$ is a boundary point so that W is continuous at z. Then for any $\alpha \in T_z(\partial\Omega)$ with $|\alpha| = 1$, there exists a sequence $(f_N) \subseteq H^{1/2}(\partial\Omega)$ so that

$$\lim_{N\to\infty} ((\Lambda_{W,q} - \Lambda_{0,0}) f_N, f_N) = W(z) \cdot \alpha. \tag{5.2}$$

The sequence is independent of W and q. Furthermore, if U is any neighborhood of z in $\partial\Omega$, one may assume that $\operatorname{supp}(f_N) \subseteq U$ for all N.

Remarks. (a) If W, q, and the domain are C^∞, then $\Lambda_{W,q}$ is a pseudodifferential operator of order one on $\partial\Omega$ and its symbol may be explicitly computed ([28]). The principal symbol of $\Lambda_{W,q}$ is independent of W and q. Therefore, we consider the order 0 operator $\Lambda_{W,q} - \Lambda_{0,0}$, whose principal symbol contains the tangential components of W. Then (5.2) corresponds to the fact that the principal symbol of a pseudodifferential operator may be obtained by testing against highly oscillatory functions.

(b) The result also holds under a slightly weaker condition than (5.1), which is similar to the condition (H1) in [9].

(c) The result is completely local. If U is any neighborhood of z in $\partial\Omega$, then one may determine the tangential components of $W(z)$ from the knowledge of $(\Lambda_{W,q} f, g)$ for all $f, g \in H^{1/2}(\partial\Omega)$ which are supported in U.

The proof is based on the following identity, which is a direct consequence of the definition of $\Lambda_{W,q}$.

Lemma 5.1. Let $\Omega \subseteq \mathbf{R}^n$ be a bounded open set with Lipschitz boundary, let $W \in L^\infty(\Omega; \mathbf{R}^n)$ and $q \in L^\infty(\Omega; \mathbf{R}^n)$. Suppose 0 is not a Dirichlet eigenvalue of $H_{W,q}$. Then one has

$$((\Lambda_{W,q} - \Lambda_{0,0})f, f) = \int_\Omega (iW \cdot (u\nabla\bar{v} - \bar{v}\nabla u) + (|W|^2 + q)u\bar{v})\, dx$$

for any $f \in H^{1/2}(\partial\Omega)$, where $u, v \in H^1(\Omega)$ satisfy $H_{W,q}u = 0$ in Ω and $u|_{\partial\Omega} = f$, and $\Delta v = 0$ in Ω and $v|_{\partial\Omega} = f$.

We will use oscillatory solutions u and v which concentrate near a boundary point z. The construction is easier to do when Ω is flat near z, so we need to discuss a transformation which achieves this. The first step is to fix a convenient coordinate system at z.

From the definition of a $C^{1,1}$ domain, we know that there exist $r > 0$ and a coordinate system (x', x_n) in $\mathbf{R}^n$, isometric to the usual one, so that z is 0 in these coordinates, and one has $\Omega \cap B(0, r) = \{x_n > \phi(x')\} \cap B(0, r)$ where ϕ is a $C_c^{1,1}$ function $\mathbf{R}^{n-1} \to \mathbf{R}$. Furthermore, we may assume $\nabla\phi(0) = 0$, which follows since the inverse function theorem gives a $C^{1,1}$ local inverse when the original function is $C^{1,1}$. Then $T_z(\partial\Omega) = \mathbf{R}^{n-1} \times \{0\}$.

With the coordinate system (x', x_n) where z is the origin, define a bilipschitz homeomorphism F of $\mathbf{R}^n$ by $F(x', x_n) = (x', \phi(x') + x_n)$. Note that $DF(x', x_n) = (\frac{\partial F_j}{\partial x_k}) = \begin{pmatrix} I & 0 \\ \nabla\phi(x') & 1 \end{pmatrix}$, which shows that $\det DF = 1$ and $DF(0) = I$. We let $\tilde{\Omega} = F^{-1}(\Omega)$ be the domain corresponding to Ω in the (x', x_n) coordinates. Then $\tilde{\Omega}$ is a bilipschitz image of a bounded $C^{1,1}$ domain and is flat near the boundary point 0.

For $u \in L^1_{\rm loc}(\Omega)$ define $\tilde u \in L^1_{\rm loc}(\tilde\Omega)$ by $\tilde u = u \circ F$. One has $(H_{W,q}u, v)_\Omega = (\tilde H_{W,q}\tilde u, \tilde v)_{\tilde\Omega}$, where $\tilde H_{W,q}$ is the operator in $\tilde\Omega$ corresponding to $H_{W,q}$ in the transformation F, and is given by

$$\tilde H_{W,q}\tilde u = -\partial_{x_j}(a_{jk}\partial_{x_k}\tilde u + b_j \tilde u) - b_j\partial_{x_j}\tilde u + c\tilde u$$

where $A(x') = (a_{jk}) = (DF)^{-1}(DF)^{-t}$, $b(x', x_n) = (b_j) = i(DF)^{-1}\tilde W$, and $c = |\tilde W|^2 + \tilde q$. Then a_{jk} is $W^{1,\infty}$ and b and c are L^∞ in $\tilde\Omega$, and 0 is not a Dirichlet eigenvalue of $\tilde H_{W,q}$ since it was not one for $H_{W,q}$. Also, $H_{0,0} = -\Delta$ becomes $\tilde H_{0,0} = -\tilde\Delta = -\partial_{x_j}(a_{jk}\partial_{x_k})$ in these coordinates.

Let $\eta \in C_c^\infty(\mathbf{R})$ be a function with $0 \le \eta \le 1$, $\eta = 1$ for $|x| \le 1/2$, and $\eta = 0$ for $|x| \ge 1$. Let $\alpha = (\alpha', 0) \in \mathbf{R}^n$ be a unit vector tangent to $\partial\Omega$ at 0. For $N \in \mathbf{Z}_+$ we define an approximate solution

$$v_N(x) = \eta(N^{1/2}x_1)\cdots\eta(N^{1/2}x_n)e^{N(i\alpha - e_n)\cdot x}$$

so that v_N is C^∞ in $\mathbf{R}^n$ and localized near 0 when N is large. We write $v_N = \psi E$ where $\psi(x) = \eta(N^{1/2}x_1)\cdots\eta(N^{1/2}x_n)$ and $E(x) = e^{N(i\alpha - e_n)\cdot x}$. Note that if $L_0 = \mathrm{div}(A(0)\nabla) = \Delta$ is the operator $\tilde\Delta$ with coefficients frozen at 0, then $L_0 E = 0$. The scalings are chosen so that E dominates the cutoff ψ for large N, so v_N is indeed an approximate solution for the operator L_0 and then also for $\tilde H_{W,q}$ and $\tilde H_{0,0}$ when N is large.

Since v_N has an explicit form one obtains the following estimates. We write $\delta(x) = \mathrm{dist}(x, \partial\tilde\Omega)$ for $x \in \tilde\Omega$, so that $\delta(x) = x_n$ for x close to 0.

Lemma 5.2. One has in $\tilde\Omega$

$$\begin{aligned} \|v_N\|_{L^2} &= O(N^{\frac{-1-n}{4}}), \quad \|\nabla v_N\|_{L^2} = O(N^{\frac{3-n}{4}}), \\ \|\delta v_N\|_{L^2} &= O(N^{\frac{-5-n}{4}}), \quad \|\delta \nabla v_N\|_{L^2} = O(N^{\frac{-1-n}{4}}), \end{aligned}$$

as $N \to \infty$.

Proof. We begin by computing

$$\int_0^\infty \eta(N^{1/2}x_n)^2 e^{-2Nx_n}\,dx_n = \int_0^\infty e^{-2Nx_n}\,dx_n - \\ \int_0^\infty (1 - \eta(N^{1/2}x_n)^2)e^{-2Nx_n}\,dx_n = \frac{1}{2}N^{-1} + O(e^{-\frac{1}{2}N^{1/2}}N^{-1})$$

and

$$\begin{aligned}\int_0^\infty x_n^2\eta(N^{1/2}x_n)^2 e^{-2Nx_n}\,dx_n &= \int_0^\infty x_n^2 e^{-2Nx_n}\,dx_n + O(e^{-\frac{1}{2}N^{1/2}}N^{-3}) \\ &= c_0 N^{-3} + O(e^{-\frac{1}{2}N^{1/2}}N^{-3})\end{aligned}$$

where c_0 is an absolute constant. We obtain

$$\begin{aligned}\int_{\tilde{\Omega}} |v_N|^2\,dx &= \int_{\mathbf{R}^{n-1}} \eta(N^{1/2}x_1)^2 \cdots \eta(N^{1/2}x_{n-1})^2 \Big(\frac{1}{2}N^{-1} + o(N^{-1})\Big)\,dx' \\ &= \frac{1}{2}\Big(\int_{\mathbf{R}} \eta(t)^2\,dt\Big)^{n-1} N^{\frac{-1-n}{2}} + o(N^{\frac{-1-n}{2}}).\end{aligned}$$

This gives the estimate for $\|v_N\|_{L^2}$, and the case for $\|\delta v_N\|_{L^2}$ is similar. For the derivatives we note that $\partial_{x_j}(\psi E) = N^{1/2}\psi_j E + N(i\alpha_j - \delta_{jn})\psi E$ where ψ_j is of the same form as ψ, so the same computations as for v_N and δv_N give $\|\nabla v_N\|_{L^2} = O(N^{\frac{3-n}{4}})$ and $\|\delta \nabla v_N\|_{L^2} = O(N^{\frac{-1-n}{4}})$. □

We then want to move from the approximate solutions v_N to solutions $\tilde{u}$ and $\tilde{v}$ which solve $\tilde{H}_{W,q}\tilde{u} = 0$ and $\tilde{H}_{0,0}\tilde{v} = 0$ in $\tilde{\Omega}$, whose boundary values on $\tilde{\Omega}$ are v_N. These solutions are given by $\tilde{u} = v_N + w_N$, $\tilde{v} = v_N + w_N'$, where w_N, w_N' are the $H_0^1(\tilde{\Omega})$ solutions to $\tilde{H}_{W,q}w_N = -\tilde{H}_{W,q}v_N$ and $\tilde{H}_{0,0}w_N' = -\tilde{H}_{0,0}v_N$.

Several times below we will need Hardy's inequality. It is typically applied in the form $\big|\int_{\tilde{\Omega}} f\varphi\,dx\big| \leq \|\delta f\|_{L^2}\|\delta^{-1}\varphi\|_{L^2} \leq C\|\delta f\|_{L^2}\|\nabla\varphi\|_{L^2}$ when $\varphi \in H_0^1(\tilde{\Omega})$.

Lemma 5.3. Let $\tilde{\Omega} \subseteq \mathbf{R}^n$ be a bilipschitz image of a bounded open set with Lipschitz boundary. Then for any $\varphi \in H_0^1(\tilde{\Omega})$ one has

$$\int_{\tilde{\Omega}} \frac{|\varphi|^2}{\delta^2}\,dx \leq C \int_{\tilde{\Omega}} |\nabla\varphi|^2\,dx.$$

Proof. For sets with Lipschitz boundary see Davies [16]. The result follows for bilipschitz images of such sets by a change of coordinates. □

The next three lemmas are concerned with estimating the remainder terms w_N and w_N'. The objective is to show that they are in a suitable sense smaller than v_N, which will then be the dominating part in the solutions. The gradient L^2 estimates are obtained from standard estimates for weak solutions.

Lemma 5.4. One has $\|\nabla w_N\|_{L^2(\tilde{\Omega})}, \|\nabla w_N'\|_{L^2(\tilde{\Omega})} = o(N^{\frac{3-n}{4}})$ as $N \to \infty$.

Proof. Since 0 is not a Dirichlet eigenvalue of $H_{W,q}$ or $H_{0,0}$ in Ω, the equations for w_N and w_N' above have unique solutions in $H_0^1(\tilde{\Omega})$, and $\|\nabla w_N\|_{L^2(\tilde{\Omega})}$ and $\|\nabla w_N'\|_{L^2(\tilde{\Omega})}$ will be bounded by a constant times the $H^{-1}(\tilde{\Omega})$ norm of the right hand sides. Thus it will be enough to show that $\|\tilde{H}_{W,q}v_N\|_{H^{-1}(\tilde{\Omega})} = o(N^{\frac{3-n}{4}})$ as $N \to \infty$, and $W = q = 0$ will be a special case of this.

We have

$$\tilde{H}_{W,q}v_N = -L_0(\psi E) - \operatorname{div}(A - A(0))\nabla(\psi E) - \partial_{x_j}(b_j v_N) - b_j\partial_{x_j}v_N + cv_N$$

where $L_0 = \mathrm{div}\, A(0)\nabla = \Delta$. Note that $L_0(\psi E) = (L_0\psi)E + 2\nabla\psi\cdot\nabla E$ since $L_0 E = 0$. If $\varphi \in C_c^\infty(\tilde{\Omega})$ then in the distribution pairing we have

$$\begin{aligned}\langle \tilde{H}_{W,q} v_N, \varphi\rangle = \int_{\tilde{\Omega}} \Big(&- (L_0\psi)E\varphi - 2N(\nabla\psi\cdot(i\alpha - e_n))E\varphi \\ &+ (A - A(0))\nabla(\psi E)\cdot\nabla\varphi + v_N(b\cdot\nabla\varphi) - (b\cdot\nabla v_N)\varphi + c v_N\varphi\Big)\, dx\end{aligned}$$

We split this into a sum of six integrals as $\langle \tilde{H}_{W,q} v_N, \varphi\rangle = \sum_{j=1}^6 I_j$ and estimate each integral.

First, $(L_0\psi)E = N\psi_j E$ where ψ_j are of the same form as ψ. Consequently Hardy's inequality and the computation in Lemma 5.2 give $|I_1| \leq CN\|\delta\psi_j E\|_{L^2}\|\varphi\|_{H^1} = O(N^{\frac{-1-n}{4}})\|\varphi\|_{H^1}$. A similar argument shows that $|I_2| \leq CN^{3/2}O(N^{\frac{-5-n}{4}})\|\varphi\|_{H^1} = O(N^{\frac{1-n}{4}})\|\varphi\|_{H^1}$. Lemma 5.2 and Hardy's inequality also give $|I_j| = O(N^{\frac{-1-n}{4}})\|\varphi\|_{H^1}$ for $j = 4, 5, 6$.

It remains to estimate I_3. One has $\nabla(\psi E) = N^{1/2}(\psi_j)E + N(i\alpha - e_n)\psi E$ where ψ_j have the same form as ψ. Again the computation of Lemma 5.2 gives

$$\begin{aligned}\Big|\int_{\tilde{\Omega}} (A - A(0))N^{1/2}\psi_j E\cdot\nabla\varphi\, dx\Big| &\leq C\|A\|_{L^\infty} N^{1/2}\|\psi_j E\|_{L^2}\|\varphi\|_{H^1} \\ &= O(N^{\frac{1-n}{4}})\|\varphi\|_{H^1}.\end{aligned}$$

Finally,

$$\Big|\int_{\tilde{\Omega}} (A - A(0))N(i\alpha - e_n)\psi E\cdot\nabla\varphi\, dx\Big| \leq CN\|(A - A(0))\psi E\|_{L^2}\|\varphi\|_{H^1}.$$

This is $o(N^{\frac{3-n}{4}})\|\varphi\|_{H^1}$ by the continuity of A at 0 and by Lemma 5.2. We obtain $\|\tilde{H}_{W,q} v_N\|_{H^{-1}(\tilde{\Omega})} = o(N^{\frac{3-n}{4}})$ as desired. □

Next we need L^2 estimates. These are easier to prove for w_N' since on the Ω side everything reduces to the following properties of harmonic functions.

Lemma 5.5. Let $\Omega \subseteq \mathbf{R}^n$ be a bounded open set, and let u be a harmonic function in Ω.

(a) If $u \in L^2(\Omega)$ then $\|\delta\nabla u\|_{L^2(\Omega)} \leq C\|u\|_{L^2(\Omega)}$.

(b) If $u \in H^1(\Omega)$ and $\partial\Omega$ is $C^{1,1}$, then $\|u\|_{L^2(\Omega)} \leq C\|u|_{\partial\Omega}\|_{H^{-1/2}(\partial\Omega)}$.

Proof. (a) If $x \in \Omega$ and $B = B(x, \delta(x)/2)$ then the mean-value property implies

$$|\nabla u(x)| \leq \frac{C}{\delta(x)^{n+1}}\int_B |u(y) - u(x)|\, dy \leq \frac{C}{\delta(x)}\Big(\frac{1}{|B|}\int_B |u|\, dy + |u(x)|\Big)$$

with $C = C(n)$. Thus $\delta|\nabla u| \leq C(M(\chi_\Omega u) + |u|)$ where M is the Hardy-Littlewood maximal function in $\mathbf{R}^n$. By the mapping properties of this function, see Stein [39], we obtain (a).

(b) The proof is by duality. Let $\varphi \in L^2(\Omega)$ and let v be the $H_0^1(\Omega)$ solution to $\Delta v = \varphi$ in Ω. By [18, 9.6], $v \in H^2(\Omega)$ with $\|v\|_{H^2(\Omega)} \leq C\|\varphi\|_{L^2(\Omega)}$. Then

$$\int_\Omega u\varphi\, dx = \int_\Omega u\Delta v\, dx = \int_{\partial\Omega} u\frac{\partial v}{\partial \nu}\, dS - \int_\Omega \nabla u \cdot \nabla v\, dx.$$

Since u is harmonic the last integral is zero, and

$$\Big|\int_\Omega u\varphi\, dx\Big| \leq \|u|_{\partial\Omega}\|_{H^{-1/2}(\partial\Omega)}\|\frac{\partial v}{\partial \nu}\|_{H^{1/2}(\partial\Omega)}.$$

Here $\|\frac{\partial v}{\partial \nu}\|_{H^{1/2}(\partial\Omega)} \leq C\|v\|_{H^2(\Omega)} \leq C\|\varphi\|_{L^2(\Omega)}$, which shows (b). □

We remark that Lemma 5.5 (b) is the only place where extra regularity of $\partial\Omega$ is needed, in the sense that all other parts of the argument work for Lipschitz domains with small modifications. The estimate in part (b) is probably false for Lipschitz domains. The author would like to thank Carlos Kenig for clarifying this point.

The following estimates will be the last ones needed for the proof of Proposition 5.1.

Lemma 5.6. If $\partial\Omega$ is $C^{1,1}$ then $\|w_N'\|_{L^2(\tilde\Omega)}, \|\delta\nabla w_N'\|_{L^2(\tilde\Omega)} = O(N^{\frac{-1-n}{4}})$.

Proof. It is enough to prove this for $v = v_N + w_N'$, since v_N satisfies these estimates by Lemma 5.2. Furthermore, since F is bilipschitz, we may consider $v \circ F^{-1} \in H^1(\Omega)$ instead of v. Now $v \circ F^{-1}$ is a harmonic function in Ω with boundary values $v_N \circ F^{-1}$ on $\partial\Omega$. We have

$$\|v_N \circ F^{-1}\|_{H^{-1/2}(\partial\Omega)} \leq C\|v_N(x', 0)\|_{H^{-1/2}(\mathbf{R}^{n-1})}$$

Let $f(x') = v_N(x', 0) = \psi_0(N^{1/2}x')e^{iN\alpha'\cdot x'}$. Then

$$\|f\|_{L^2} = CN^{\frac{1-n}{4}}. \tag{5.3}$$

Choose $j, 1 \leq j \leq n-1$, with $\alpha_j \neq 0$. We have $\partial_{x_j} f = N^{1/2}f_1 + iN\alpha_j f$ and $\partial_{x_j} f_1 = N^{1/2}f_2 + iN\alpha_j f_1$, where f_1 and f_2 have the same form as f and satisfy (5.3). Then

$$\Big|\int f\varphi\, dx'\Big| \leq C(N^{-1}\|f\|_{L^2}\|\varphi\|_{H^1} + N^{-3/2}\|f_1\|_{L^2}\|\varphi\|_{H^1} + N^{-1}\|f_2\|_{L^2}\|\varphi\|_{L^2}).$$

We obtain from (5.3) that $\|f\|_{H^{-1}} = O(N^{\frac{-3-n}{4}})$, and interpolation gives $\|f\|_{H^{-1/2}} = O(N^{\frac{-1-n}{4}})$. The result now follows from Lemma 5.5. □

Proof. (of Proposition 5.1) We choose the coordinate system (x', x_n) as above and take $f_N = c_N v_N \circ F^{-1}|_{\partial\Omega}$, where $c_N > 0$ is a constant to be determined later. Using Lemma 5.1 we have

$$
\begin{aligned}
((\Lambda_{W,q} - \Lambda_{0,0})f_N, f_N) &= c_N^2 \int_\Omega (iW \cdot (u\nabla\bar{v} - \bar{v}\nabla u) + (|W|^2 + q)u\bar{v})\,dx \\
&= c_N^2 \int_{\tilde{\Omega}} (b \cdot (\tilde{u}\nabla\bar{\tilde{v}} - \bar{\tilde{v}}\nabla\tilde{u}) + c\tilde{u}\bar{\tilde{v}})\,dx
\end{aligned}
$$

where $\tilde{u}$ solves $\tilde{H}_{W,q}\tilde{u} = 0$ in $\tilde{\Omega}$, $\tilde{v}$ solves $\tilde{H}_{0,0}\tilde{v} = 0$ in $\tilde{\Omega}$, and $\tilde{u} = \tilde{v} = v_N$ on $\partial\tilde{\Omega}$. Thus $\tilde{u} = v_N + w_N$ and $\tilde{v} = v_N + w'_N$ according to our earlier notation, and we have

$$
\begin{aligned}
c_N^{-2}((\Lambda_{W,q} - \Lambda_{0,0})f_N, f_N) &= \int_{\tilde{\Omega}} b(0) \cdot (v_N\nabla\bar{v}_N - \bar{v}_N\nabla v_N)\,dx \\
&\quad + \int_{\tilde{\Omega}} (b - b(0)) \cdot (v_N\nabla\bar{v}_N - \bar{v}_N\nabla v_N)\,dx \\
+ \int_{\tilde{\Omega}} b \cdot (v_N\nabla\bar{w}'_N &+ w_N\nabla\bar{v}_N + w_N\nabla\bar{w}'_N - \bar{v}_N\nabla w'_N - \bar{w}_N\nabla v_N - \bar{w}_N\nabla w'_N)\,dx \\
&\quad + \int_{\tilde{\Omega}} c(v_N\bar{v}_N + v_N\bar{w}'_N + w_N\bar{v}_N + w_N\bar{w}'_N)\,dx. \quad (5.4)
\end{aligned}
$$

We write the right hand side as $I_1 + I_2 + I_3 + I_4$ and estimate each integral.

Note that $v_N\nabla\bar{v}_N - \bar{v}_N\nabla v_N = -2Ni\alpha|v_N|^2$ and $b(0) = iW(z)$, so

$$
I_1 = 2N(W(z) \cdot \alpha) \int_{\tilde{\Omega}} |v_N|^2\,dx = k_0(W(z) \cdot \alpha)N^{\frac{1-n}{2}} + o(N^{\frac{1-n}{2}})
$$

by Lemma 5.2, where $k_0 = (\int \eta(t)^2\,dt)^{n-1}$. The continuity of W at z implies $\operatorname{ess\,sup}_{x\in\tilde{\Omega}\cap B(0,N^{-1/2})} |b(x) - b(0)| \to 0$ as $N \to 0$, so that $I_2 = o(N^{\frac{1-n}{2}})$. We have

$$
\begin{aligned}
|I_3| \le C(&\|v_N\|\,\|\nabla w'_N\| + \|\delta^{-1}w_N\|\,\|\delta\nabla v_N\| + \|\delta^{-1}w_N\|\,\|\delta\nabla w'_N\| \\
&+ \|v_N\|\,\|\nabla w'_N\| + \|\delta^{-1}w_N\|\,\|\delta\nabla v_N\| + \|\delta^{-1}w_N\|\,\|\delta\nabla w'_N\|)
\end{aligned}
$$

where all the norms are in $L^2(\tilde{\Omega})$. We obtain from Lemmas 5.2 to 5.6 that $I_3 = o(N^{\frac{1-n}{2}})$. Finally,

$$
|I_4| \le C(\|v_N\|^2 + \|v_N\|\,\|w'_N\| + \|\delta^{-1}w_N\|\,\|\delta v_N\| + \|\delta^{-1}w_N\|\,\|\delta w'_N\|).
$$

Since $\|\delta w'_N\| \le C\|w'_N\|$, using the lemmas gives $I_4 = o(N^{\frac{1-n}{2}})$.

Setting $c_N = k_0^{-1/2}N^{\frac{n-1}{4}}$, using the estimates for the integrals, and letting $N \to \infty$ in (5.4), we obtain the desired result. □

We may now prove the theorems from the introduction.

Theorem 1.6. Let $\Omega \subseteq \mathbf{R}^n$ be a bounded open set with $C^{1,1}$ boundary, and let $W \in C(\overline{\Omega}; \mathbf{R}^n)$ and $q \in L^\infty(\Omega; \mathbf{R})$. Suppose that 0 is not a Dirichlet eigenvalue of $H_{W,q}$. Then $\Lambda_{W,q}$ uniquely determines the tangential components of W on $\partial\Omega$.

Proof. Follows directly from Proposition 5.1. □

For the proof of the global uniqueness theorem, we need to have a special gauge transformation which preserves the tangential components of a vector field but sets the normal component to zero. In the next lemma, $\phi_\varepsilon(x') = \varepsilon^{-(n-1)}\phi(x'/\varepsilon)$ with $\phi \in C_c^\infty(\mathbf{R}^{n-1})$, $0 \leq \phi \leq 1$, $\phi = 1$ for $|x| \leq 1/2$, and $\phi = 0$ for $|x| \geq 1$.

Lemma 5.7. Let $g \in C_c^d(\mathbf{R}^{n-1})$, and define $G(x', t) = (\phi_\varepsilon * g)(x')$ for $t > 0$, where $\varepsilon = \varepsilon(t) = t^{1/2}$. Let

$$p(x', x_n) = \int_0^{x_n} G(x', t)\, dt.$$

Then $p \in C^{1,d}(\mathbf{R}^{n-1} \times [0, 1])$ and $p|_{\mathbf{R}^{n-1}} = 0$, $\frac{\partial p}{\partial x_n}|_{\mathbf{R}^{n-1}} = g$.

Proof. Let ω be a Dini modulus for g. One has $|G(x', t)| \leq \|g\|_{L^\infty}$, $|G(x', t) - G(y', t)| \leq \omega(|x' - y'|)$ and $|G(x', s) - G(x', t)| \leq \omega(|s^{1/2} - t^{1/2}|) \leq \omega(|s - t|^{1/2})$, where $\omega(t^{1/2})$ is another Dini modulus. We easily see that p is continuous in $\mathbf{R}^{n-1} \times [0, 1]$.

We have $|\partial_{x_j} G(x', t)| \leq Ct^{-1/2}$, so $\partial_{x_j} p(x', x_n) = \int_0^{x_n} \partial_{x_j} G(x', t)$ and clearly $\partial_{x_n} p(x', x_n) = G(x', x_n)$. One also has the estimate $|\partial_{x_j} G(x', t) - \partial_{x_j} G(y', t)| \leq Ct^{-1/2}\omega(|x' - y'|)$. We obtain that

$$\begin{aligned}
|\partial_{x_j} p(x', x_n) - \partial_{x_j} p(y', x_n)| &\leq C\omega(|x' - y'|), \\
|\partial_{x_j} p(x', x_n) - \partial_{x_j} p(x', y_n)| &\leq C|x_n - y_n|^{1/2}, \\
|\partial_{x_n} p(x', x_n) - \partial_{x_n} p(y', x_n)| &\leq \omega(|x' - y'|), \\
|\partial_{x_n} p(x', x_n) - \partial_{x_n} p(x', y_n)| &\leq \omega(|x_n - y_n|^{1/2}).
\end{aligned}$$

This shows that $p \in C^{1,d}(\mathbf{R}^{n-1} \times [0, 1])$. □

Lemma 5.8. Let $\Omega \subseteq \mathbf{R}^n$ be a bounded open set with $C^{1,d}$ boundary, and let $W \in C^d(\Omega; \mathbf{R}^n)$. Then there is $p \in C^{1,d}(\Omega; \mathbf{R})$ which satisfies $p|_{\partial\Omega} = 0$ and $(W + \nabla p) \cdot \nu|_{\partial\Omega} = 0$.

Proof. Letting $g = W \cdot \nu \in C^d(\partial\Omega)$, we need a function $p \in C^{1,d}(\Omega)$ with $p|_{\partial\Omega} = 0$ and $-\frac{\partial p}{\partial \nu}|_{\partial\Omega} = g$. We may construct p locally near a boundary point and use a suitable partition of unity to get the desired function in Ω. Thus, assume 0 is a boundary point, and Ω is given near 0 by $\{y_n > \phi(y')\}$ where $\phi \in C_c^{1,d}(\mathbf{R}^{n-1})$, $\phi(0) = 0$. By the inverse function theorem we may assume $\nabla\phi(0) = 0$.

We first construct approximate boundary normal coordinates near 0. Let B be a small ball in $\mathbf{R}^n$ with center 0, and let $B_+ = B \cap \{x_n > 0\}$ and $B_- = B \cap \{x_n < 0\}$. Define for $x \in B_+$

$$F(x', x_n) = (x', \phi(x')) + \int_0^{x_n} N(x', t)\, dt$$

where $N(x', t) = (\phi_\varepsilon * n)(x')$ is as in Lemma 5.7, and $n(x') = -\nu(x', \phi(x'))$ where ν is the outer unit normal of $\partial\Omega$. By Lemma 5.7 we have $F \in C^{1,d}(\overline{B}_+)$. If $x \in B_-$ define

$$F(x', x_n) = (x', \phi(x')) - \int_0^{-x_n} N(x', t)\, dt$$

so that $F \in C^{1,d}(\overline{B}_-)$. Now the two definitions for F and DF coincide on $\mathbf{R}^{n-1}$, so $F \in C^{1,d}(\overline{B})$ and $DF(x', 0) = \begin{pmatrix} I & n'(x') \\ \nabla\phi(x') & n_n(x') \end{pmatrix}$. In particular $DF(0) = I$, so the inverse function theorem shows that F is a $C^{1,d}$ diffeomorphism from $U \ni 0$ onto $V \ni 0$.

Shrink B so that $\overline{B} \subseteq U$, let $\tilde{g} = g \circ F \in C^d(B \cap \mathbf{R}^{n-1})$, and let $\tilde{p}(x', x_n) = \int_0^{x_n} \tilde{G}(x', t)\, dt$ as in Lemma 5.7. Then $\tilde{p} \in C^{1,d}(\overline{B}_+)$ and $\tilde{p}|_{\mathbf{R}^{n-1}} = 0$, $\frac{\partial \tilde{p}}{\partial x_n}|_{\mathbf{R}^{n-1}} = \tilde{g}$. We define $p = \tilde{p} \circ F^{-1}$ near 0. This gives a $C^{1,d}$ function in $B(0, r) \cap \overline{\Omega}$ for some r, and p is zero on $\partial\Omega$. Finally, for $y \in B(0, r) \cap \partial\Omega$

$$\frac{\partial p}{\partial \nu}(y) = \nabla\tilde{p}(F^{-1}(y)) \cdot DF(F^{-1}(y))^{-1}\nu(y) = -\frac{\partial \tilde{p}}{\partial x_n}(F^{-1}(y)) = -g(y).$$

This ends the proof. □

Theorem 1.7. Let $\Omega \subseteq \mathbf{R}^n$ be a bounded open set with $C^{1,1}$ boundary, $n \geq 3$, let $W_1, W_2 \in C^d(\Omega; \mathbf{R}^n)$, and let $q_1, q_2 \in L^\infty(\Omega; \mathbf{R})$. Suppose that 0 is not a Dirichlet eigenvalue of H_{W_1,q_1} or H_{W_2,q_2}. Then $\Lambda_{W_1,q_1} = \Lambda_{W_2,q_2}$ implies $\operatorname{curl} W_1 = \operatorname{curl} W_2$ and $q_1 = q_2$ in Ω.

Proof. Theorem 1.6 implies that the tangential components of W_1 and W_2 on $\partial\Omega$ coincide. Applying the gauge transformation of Lemma 5.8 to W_1 and W_2 will preserve the tangential components and will make the normal components equal to zero. The new W_1 and W_2 will satisfy the hypotheses of the theorem, and one has $\Lambda_{W_1,q_1} = \Lambda_{W_2,q_2}$ and $W_1 = W_2$ on $\partial\Omega$. We are now in the situation of Theorem 1.5, and the result follows. □

5.2 Steady state heat equation with a convection term

Let $\Omega \subseteq \mathbf{R}^n$ be a bounded open set with Lipschitz boundary, and let $W \in L^\infty(\Omega; \mathbf{R}^n)$. Consider the Dirichlet problem

$$\begin{cases} (\Delta + W \cdot \nabla)u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

This problem has a unique solution $u \in H^1(\Omega)$ for any $f \in H^{1/2}(\partial\Omega)$ by [18, 8.2]. We may define a Dirichlet to Neumann map formally by

$$\Lambda_W : f \mapsto \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega}.$$

More precisely, we define Λ_W with the equivalent weak formulations

$$\langle \Lambda_W f, g \rangle = \int_\Omega (\nabla u_f \cdot \nabla e_g - W \cdot (\nabla u_f) e_g)\, dx \tag{5.5}$$

$$= \int_\Omega (\nabla e_f \cdot \nabla v_g - W \cdot (\nabla e_f) v_g)\, dx \tag{5.6}$$

where $u_f \in H^1(\Omega)$ solves $(\Delta + W \cdot \nabla)u_f = 0$ in Ω with $u_f|_{\partial\Omega} = f$, $v_g \in H^1(\Omega)$ solves the adjoint equation $\Delta v_g - \nabla \cdot (W v_g) = 0$ in Ω with $v_g|_{\partial\Omega} = g$, and e_f, e_g are any functions in $H^1(\Omega)$ with $e_f|_{\partial\Omega} = f$ and $e_g|_{\partial\Omega} = g$. We have that Λ_W is a bounded map from $H^{1/2}(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$.

We will start heading toward a proof of Theorem 1.9, which shows that Λ_W determines the boundary values of a Hölder continuous vector field W. The proof is based on the following integral identity. If $\Lambda_{W_1} = \Lambda_{W_2}$ then

$$\int_\Omega (W_1 - W_2) \cdot (\nabla u) v\, dx = 0 \tag{5.7}$$

where u, v are any $H^1(\Omega)$ functions satisfying $\Delta u + W_1 \cdot \nabla u = 0$ and $\Delta v - \nabla \cdot (W_2 v) = 0$ in Ω. The identity follows immediately when one uses (5.5) for W_1 and (5.6) for W_2 and chooses $e_f = u$, $e_g = v$.

For the determination of boundary values, we use the method of singular solutions due to Alessandrini [3]. The point is to find solutions u and v so that the integrand in (5.7) will blow up at a given boundary point $z \in \partial\Omega$ unless $W_1(z) = W_2(z)$. In [3] such solutions were constructed for second order divergence form elliptic operators with $W^{1,p}$ coefficients, $p > n$, which have no lower order terms. In our case lower order terms are present and the construction of [3] needs to be modified. Below we will repeat arguments from [3] and supply the necessary modifications, extending the results from $W^{1,p}$ to Hölder continuous coefficients in the process.

Consider the divergence form operator

$$Lu = -\partial_{x_j}(a_{jk}\partial_{x_k}u + b_j u) + c_j\partial_{x_j}u + du \tag{5.8}$$

where

$$\text{the domain is } \Omega = B_{4R} = B(0,4R) \subseteq \mathbf{R}^n, n \geq 3, \tag{5.9}$$

$$a_{jk}, b_j \in C^\alpha(\Omega) \text{ with } 0 < \alpha < 1,\ c_j, d \in L^\infty(\Omega), \tag{5.10}$$

$$a_{jk}\xi_j\xi_k \geq \lambda|\xi|^2 \text{ for } \xi \in \mathbf{R}^n, \text{ and} \tag{5.11}$$

$$a_{jk} = a_{kj}. \tag{5.12}$$

All functions in this section are real valued. We will also need that at least one of the positivity conditions

$$d - \partial_{x_j}b_j \geq 0, \tag{5.13}$$

$$d - \partial_{x_j}c_j \geq 0, \tag{5.14}$$

is valid in Ω. These conditions are understood in the sense of distributions.

If L is as above, then the equation $Lu = T$ in Ω has a unique solution $u \in H_0^1(\Omega)$ for any $T \in H^{-1}(\Omega)$ by [18, 8.2]. The Green function for L in Ω is the distribution kernel $G(x,y)$ of the solution operator $T \mapsto u$, and it satisfies $LG(x,\,\cdot\,) = \delta_x$ in Ω in a suitable sense. Unfortunately we could not find a reference for the following estimates for $G(x,y)$, and therefore we will very briefly indicate how to prove the estimates.

Lemma 5.9. Let L satisfy (5.8) - (5.12) and one of (5.13), (5.14). For any $x \in \Omega$, $G(x,\,\cdot\,) \in C^{1,\alpha}_{\mathrm{loc}}(\Omega \smallsetminus \{x\})$, and one has

$$|G(x,y)| \leq C|x-y|^{2-n} \qquad \text{for } x,y \in B_{4R}, \tag{5.15}$$

$$|\partial_{y_j}G(x,y)| \leq C|x-y|^{1-n} \qquad \text{for } x,y \in B_{2R}. \tag{5.16}$$

Proof. The estimate (5.15) is in fact valid for L^∞ coefficients and is found in Stampacchia [38], provided one assumes $d - \partial_{x_j}b_j \geq c_0 > 0$ instead of (5.13). When the methods of [38] are combined with the maximum principle and global boundedness and continuity results of [18, Chapter 8], which are stronger than the corresponding results in [38], one obtains (5.15) under the weaker assumption (5.13). Since the results of [18] are valid also when (5.13) is replaced by (5.14), small modifications of the argument give (5.15) also when (5.14) holds.

The estimates (5.16) follow from (5.15) and interior Hölder estimates as in Lemma 5.10 below, since one has $LG(x,\,\cdot\,) = 0$ in $B_{4R} \smallsetminus \{x\}$. □

We will use the notations

$$A_{r_1,r_2}(x_0) = \{x \in \mathbf{R}^n\,;\, r_1 < |x-x_0| < r_2\},\ A_{r_1,r_2} = A_{r_1,r_2}(0),$$

$$\|u\|'_{C^{k,\alpha}(\Omega)} = \sum_{|\beta|\leq k} d^{|\beta|}\|\partial^\beta u\|_{L^\infty(\Omega)} + \sum_{|\beta|=k} d^{k+\alpha}[\partial^\beta u]_{C^\alpha(\Omega)},$$

where $d = \mathrm{diam}(\Omega)$ and $[u]_{C^\alpha(\Omega)} = \sup_{x,y\in\Omega} \frac{|u(x)-u(y)|}{|x-y|^\alpha}$.

Lemma 5.10. Let L satisfy (5.8) - (5.12) and assume $f \in L^\infty(\Omega)$, $f_j \in C^\alpha(\Omega)$. Suppose $u \in H^1(\Omega)$ solves $Lu = f + \partial_{x_j} f_j$ in Ω. Then $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$, and if $B(x_0, r_0) \subseteq \Omega$ then one has

$$\|u\|'_{C^{1,\alpha}(A_1)} \leq C(\|u\|_{L^\infty(A_2)} + r^2 \|f\|_{L^\infty(A_2)} + r\|f_j\|'_{C^\alpha(A_2)}),$$

where $A_1 = A_{r,2r}(x_0)$, $A_2 = A_{r/2,4r}(x_0)$, $4r < r_0$, and C is independent of r, x_0, r_0.

Proof. This result is from [18, 8.11], except that we have paid closer attention to constants. □

We may now begin the construction of singular solutions. The following two lemmas correspond to Lemmas 2.2 and 2.3 in [3].

Lemma 5.11. Let L be as in (5.8) - (5.12), and suppose one of (5.13), (5.14) holds. Let $2 < s < n$, and let $f \in L^\infty_{\mathrm{loc}}(B_R \smallsetminus \{0\})$, $f_j \in C^\alpha_{\mathrm{loc}}(B_R \smallsetminus \{0\})$ satisfy

$$|f(x)| \leq A|x|^{-s} \qquad \text{in } B_R \smallsetminus \{0\}, \tag{5.17}$$
$$\|f_j\|'_{C^\alpha(A_{r,2r})} \leq Ar^{1-s} \qquad \text{for } 0 < r < R/2. \tag{5.18}$$

Then there exists a solution $u \in C^{1,\alpha}_{\mathrm{loc}}(B_R \smallsetminus \{0\})$ to

$$Lu = f + \partial_{x_j} f_j \qquad \text{in } B_R \smallsetminus \{0\},$$

which satisfies

$$|u(x)| + |x||\nabla u(x)| \leq C|x|^{2-s} \quad \text{in } B_R \smallsetminus \{0\}, \tag{5.19}$$
$$\|u\|'_{C^{1,\alpha}(A_{r,2r})} \leq Cr^{2-s} \quad \text{for } 0 < r < R/2. \tag{5.20}$$

Proof. We begin with some preparations. First extend f and f_j to $\mathbf{R}^n \smallsetminus \{0\}$ so that local boundedness and Hölder continuity are preserved, the supports are contained in B_{2R}, and (5.17), (5.18) are satisfied in B_{2R} with a new A only depending on the old value of A. Let $G(x, y)$ be the Green function of L in B_{4R}. The case where only f is present is handled exactly as in Lemma 2.2 of [3], using now the estimate (5.15) and the approximation argument in the end of this proof, so we may assume $f = 0$. Also, we make the temporary assumption $f_j \in L^\infty(B_{4R})$.

Define

$$u(x) = -\int_{B_{2R}} \partial_{y_j} G(x, y) f_j(y)\, dy. \tag{5.21}$$

Then u solves $Lu = \partial_{x_j} f_j$ in B_{4R}. From (5.16) and (5.18) we obtain

$$|u(x)| \leq C[I_1 + I_2 + I_3]$$

where

$$I_j = \int_{E_j} |x-y|^{1-n}|y|^{1-s}\,dy,$$

and $E_1 = \{|y| < |x|/2\}$, $E_2 = \{|x|/2 < |y| < 2|x|\}$, and $E_3 = \{|y| > 2|x|\}$. If $|y| < |x|/2$ then $|x-y| > |x|/2$, so that

$$I_1 \le C|x|^{1-n} \int_{|y|<|x|/2} |y|^{1-s}\,dy \le C|x|^{2-s}.$$

For I_2 we have

$$I_2 \le C|x|^{1-s} \int_{|z|<4|x|} |z|^{1-n}\,dy \le C|x|^{2-s}.$$

Finally, when $|y| > 2|x|$ then $|x-y| > |y|/2$ and

$$I_3 \le C \int_{|y|>2|x|} |y|^{2-n-s}\,dy \le C|x|^{2-s}.$$

Thus u satisfies $|u(x)| \le C|x|^{2-s}$, and (5.19), (5.20) follow from Lemma 5.10.

Next we remove the assumption $f_j \in L^\infty(B_{4R})$. We define

$$f_{j,N} = \begin{cases} N & \text{when } f_j > N, \\ f_j, & \text{when } |f_j| \le N, \\ -N, & \text{when } f_j < -N. \end{cases}$$

Then $f_{j,N} \in C^\alpha_{\mathrm{loc}}(B_{4R} \smallsetminus \{0\})$ with $\|f_{j,N}\|'_{C^\alpha(\Omega')} \le \|f_j\|'_{C^\alpha(\Omega')}$ when $\overline{\Omega}' \subseteq B_{4R} \smallsetminus \{0\}$, and $f_{j,N} \in L^\infty(B_{4R})$. Let u_N be the corresponding solution of $Lu_N = \partial_{x_j} f_{j,N}$ in $B_{4R} \smallsetminus \{0\}$ obtained from (5.21). Then u_N satisfies (5.19), (5.20) with C independent of N, so (u_N) is a bounded sequence in $C^{1,\alpha}_{\mathrm{loc}}(B_R \smallsetminus \{0\})$ and there is a subsequence which converges weakly in $C^{1,\alpha}_{\mathrm{loc}}(B_R \smallsetminus \{0\})$ and strongly in $C^1_{\mathrm{loc}}(B_R \smallsetminus \{0\})$. The limit u satisfies (5.19), (5.20) and $Lu = \partial_{x_j} f_j$ in $B_R \smallsetminus \{0\}$. □

Lemma 5.12. Let $s > n$ be a nonintegral real number, and suppose $f \in L^\infty_{\mathrm{loc}}(B_R \smallsetminus \{0\})$, $f_j \in C^\alpha_{\mathrm{loc}}(B_R \smallsetminus \{0\})$ satisfy (5.17), (5.18). Then there exists a solution $u \in C^{1,\alpha}_{\mathrm{loc}}(B_R \smallsetminus \{0\})$ to

$$\Delta u = f + \partial_{x_j} f_j \qquad \text{in } B_R \smallsetminus \{0\}, \tag{5.22}$$

which satisfies (5.19), (5.20).

Proof. We make similar preparations as in the proof of Lemma 5.11, assuming $f = 0$ by [3], Lemma 2.3, and $f_j \in L^\infty(B_{4R})$ by the approximation. We need some properties of Gegenbauer polynomials C_k^α from [1] and [40].

(a) $(1-2xz+z^2)^{-\alpha} = \sum_{k=0}^{\infty} C_k^{\alpha}(x)z^k$ for $\alpha > 0, |x| \leq 1, |z| < 1$,

(b) $|C_k^{\alpha}(x)| \leq \binom{k+2\alpha-1}{k}$ for $|x| \leq 1$,

(c) $(C_k^{\alpha})'(x) = 2\alpha C_{k-1}^{\alpha+1}(x)$ for $k \geq 1$.

By (a) we have $\Gamma(x-y) = -c_n|x-y|^{2-n} = \sum_{k=0}^{\infty} H_k(x,y)$ for $x \neq 0$ and $|y| < |x|$, where

$$H_k(x,y) = -c_n \frac{|y|^k}{|x|^{k+n-2}} C_k^{(n-2)/2}\Big(\frac{y}{|y|} \cdot \frac{x}{|x|}\Big).$$

From [40, Theorem 2.14] we have that for fixed $x \neq 0$, $H_k(x,\,\cdot\,)$ is a homogeneous harmonic polynomial of degree k. This also shows, upon changing the roles of x and y and after a computation, that $\Delta_x H_k(x,y) = 0$ for $x \neq 0$. By (b) and (c) we obtain $|C_k^{(n-2)/2}(x)| \leq Ck^{n-3}$ and $|(C_k^{(n-2)/2})'(x)| \leq Ck^{n-1}$ where C only depends on n, and this implies that

$$|\partial_{y_j} H_k(x,y)| \leq Ck^{n-1} \frac{|y|^{k-1}}{|x|^{k+n-2}}. \tag{5.23}$$

Let now $\nu = [s] - n$ and define $\Gamma_\nu(x,y) = \Gamma(x-y) - \sum_{k=0}^{\nu} H_k(x,y)$. Then the function

$$u(x) = -\int_{B_{2R}} \partial_{y_j} \Gamma_\nu(x,y) f_j(y)\, dy$$

solves $\Delta u = \partial_{x_j} f_j$ in $B_{4R} \smallsetminus \{0\}$. We estimate

$$|u(x)| \leq C[I_2 + I_3 + I_4 + I_5]$$

where I_2 and I_3 are as in Lemma 5.11 and are $\leq C|x|^{2-s}$, and

$$I_4 = \sum_{k=0}^{\nu} k^{n-1} \int_{|y|>|x|/2} \frac{|y|^{k-1}}{|x|^{k+n-2}} |y|^{1-s}\, dy,$$

$$I_5 = \sum_{k=\nu+1}^{\infty} k^{n-1} \int_{|y|<|x|/2} \frac{|y|^{k-1}}{|x|^{k+n-2}} |y|^{1-s}\, dy.$$

By the choice of ν we obtain

$$I_4 \leq C \sum_{k=0}^{\nu} k^{n-1} |x|^{2-k-n} |x|^{k+n-s} \leq C|x|^{2-s},$$

$$I_5 \leq C \sum_{k=\nu+1}^{\infty} k^{n-1} |x|^{2-k-n} \Big(\frac{|x|}{2}\Big)^{k+n-s} \leq C|x|^{2-s}.$$

Thus $|u(x)| \leq C|x|^{2-s}$, and again (5.19), (5.20) follow from Lemma 5.10. □

We may now give the result, corresponding to Theorem 1.1 of [3], which ensures the existence of solutions with a singularity of arbitrarily high order at a given point. The result extends [3] to operators with lower order terms and also a larger class of coefficients: recall that $W^{1,p} \subseteq C^{1-n/p}$ when $p > n$, but there are Hölder continuous functions which are nowhere differentiable. The assumption $a_{jk}(0) = \delta_{jk}$ is just a normalization which may be removed by introducing a constant matrix in the solution as in [3].

Theorem 1.8. Let L be as in (5.8) - (5.12), and suppose one of (5.13), (5.14) holds. Assume also that $a_{jk}(0) = \delta_{jk}$. Then for every spherical harmonic S_m of degree $m = 0, 1, 2, \ldots$, there exists $u \in C^{1,\beta}_{\mathrm{loc}}(B_R \smallsetminus \{0\})$ such that

$$Lu = 0 \quad \text{in } B_R \smallsetminus \{0\},$$

and furthermore

$$u(x) = |x|^{2-n-m} S_m\Big(\frac{x}{|x|}\Big) + w(x),$$

where w satisfies

$$|w(x)| + |x||\nabla w(x)| \leq C|x|^{2-n-m+\beta} \quad \text{in } B_R \smallsetminus \{0\}, \tag{5.24}$$

$$\|w\|'_{C^{1,\beta}(A_{r,2r})} \leq Cr^{2-n-m+\beta} \quad \text{for } 0 < r < R/2. \tag{5.25}$$

Here β is any number with $0 < \beta < \alpha$.

Proof. If α is rational, we decrease α so that it is larger than β and irrational. Choose $K = [m/\alpha]$ and let $H(x) = |x|^{2-n-m} S_m\big(\frac{x}{|x|}\big)$, so that $\Delta H = 0$ in $B_R \smallsetminus \{0\}$. We have

$$LH = (\Delta + L)H = \partial_{x_j}((a_{jk}(0) - a_{jk})\partial_{x_k} H - b_j H) + c_j \partial_{x_j} H + dH$$

so $LH = \partial_{x_j} f_j + f$, where $|f(x)| \leq C|x|^{1-n-m}$, $|f_j(x)| \leq C|x|^{1-n-m+\alpha}$ and

$$[f_j]_{C^\alpha(A_{r,2r})} \leq \|a_{jk}(0) - a_{jk}\|_{L^\infty(A_{r,2r})}[\partial_{x_k} H]_{C^\alpha(A_{r,2r})}$$
$$+ [a_{jk}]_{C^\alpha(A_{r,2r})}\|\partial_{x_k} H\|_{L^\infty(A_{r,2r})} + \|b_j\|_{C^\alpha(A_{r,2r})}\|H\|_{C^\alpha(A_{r,2r})} \leq Cr^{1-n-m}.$$

Thus f, f_j satisfy the conditions of Lemma 5.12 with $s = n+m-\alpha$. Let w_0 be the corresponding solution of $\Delta w_0 = LH$ which satisfies $\|w_0\|'_{C^{1,\alpha}(A_{r,2r})} \leq Cr^{2-n-m+\alpha}$. Inductively, we define w_j for $1 \leq j \leq K-1$ as the solution of $\Delta w_j = (\Delta + L)w_{j-1}$ given by Lemma 5.12. The solutions w_j satisfy $\|w_j\|'_{C^{1,\alpha}(A_{r,2r})} \leq Cr^{2-n-m+(j+1)\alpha}$, and then $(\Delta+L)w_{K-1} = f + \partial_{x_j} f_j$ where f, f_j satisfy the conditions of Lemma 5.11 with $s = n + m - (K+1)\alpha < n$. Finally, let W_K be the solution to $LW_K = -(\Delta + L)w_{K-1}$ obtained from Lemma 5.11.

Set

$$w = \sum_{j=0}^{K-1} w_j + W_K.$$

Then w satisfies (5.24), (5.25) and

$$Lw = \sum_{j=0}^{K-1} (\Delta + L) w_j - \sum_{j=0}^{K-1} \Delta w_j + L W_K = -LH.$$

This shows that $u = H + w$ is indeed a solution of the desired form. □

We now prove Theorem 1.9. The author gratefully acknowledges the help of Giovanni Alessandrini in the choice of the singular solutions. We remark that the restriction to $n \geq 3$ in Theorems 1.8 and 1.9 is for convenience only, and similar results hold also for $n = 2$.

Theorem 1.9. Let $\Omega \subseteq \mathbf{R}^n$ be a bounded open set with Lipschitz boundary, and $n \geq 3$. If $W_1, W_2 \in C^\alpha(\Omega; \mathbf{R}^n)$ for some $\alpha > 0$, then $\Lambda_{W_1} = \Lambda_{W_2}$ implies $W_1 = W_2$ on $\partial\Omega$.

Proof. We argue by contradiction and assume that $W_1(z_0) \neq W_2(z_0)$ for some $z_0 \in \partial\Omega$. We may choose coordinates so that $z_0 = 0$ and for some r_0 we have $\Omega \cap B(0, r_0) = \{x_n > \phi(x')\} \cap B(0, r_0)$ where $\phi : \mathbf{R}^{n-1} \to \mathbf{R}$ is a Lipschitz function. Let $\eta = W_1(0) - W_2(0)$; since $\partial\Omega$ is Lipschitz we may rotate the coordinates slightly so that $\eta_n \neq 0$ in the new coordinates.

We need some more geometric preliminaries. For $0 < \varepsilon < 1$ and $z \in \mathbf{R}^n$ define the cones

$$C_\varepsilon(z) = \{x \,;\, \Big(\frac{x - z}{|x - z|}\Big)_n > 1 - \varepsilon\}, \quad C_\varepsilon^-(z) = \{x \,;\, \Big(\frac{x - z}{|x - z|}\Big)_n < -(1 - \varepsilon)\}.$$

By the Lipschitz condition there is ε_0 with $0 < \varepsilon_0 < 1$ so that $C_{\varepsilon_0}(0) \cap B(0, r_0) \subseteq \Omega$ and $C_{\varepsilon_0}^-(0) \cap B(0, r_0) \subseteq \mathbf{R}^n \smallsetminus \overline{\Omega}$. Let $\sigma > 0$ be a small parameter and define $z = z_\sigma = (0, -\sigma)$, so that one may find $c = c(\varepsilon_0) < 1$ with $B(z, c\sigma) \subseteq \mathbf{R}^n \smallsetminus \overline{\Omega}$ for σ small. We also have $\varepsilon = \varepsilon(\varepsilon_0) < 1$ so that

$$C_\varepsilon(z) \cap \{|x - z| > 2\sigma\} \subseteq C_{\varepsilon_0}(0).$$

In fact one may choose ε so that this holds for $\sigma = 1$, and the same ε works for all σ by scaling. We also require that for $x \in C_\varepsilon(z)$ one has

$$\Big|\eta' \cdot \Big(\frac{x - z}{|x - z|}\Big)'\Big| \leq \frac{1}{2}\Big|\eta_n \Big(\frac{x - z}{|x - z|}\Big)_n\Big|.$$

To obtain this we decrease ε so that $\varepsilon \leq 1 - \big(\frac{M}{M+1}\big)^{1/2}$ where $M = \frac{4|\eta'|^2}{\eta_n^2}$. This final ε will thus depend only on ε_0 and η. As a last remark, we note that

$$|C_\varepsilon(z) \cap \partial B(z, r)| = \gamma |\partial B(z, r)|$$

where $|\cdot|$ is $(n-1)$-dimensional surface measure and $\gamma = \gamma(\varepsilon) > 0$ is fixed.

Now extend W_1 and W_2 to Hölder continuous vector fields in $\mathbf{R}^n$, and choose R so that $\Omega \subseteq B(0, R/2)$. We use Theorem 1.8 to find solutions $u = u_0 + u_1$ to $\Delta u + W_1 \cdot \nabla u = 0$ in $B(z,R) \smallsetminus \{z\}$ and $v = v_0 + v_1$ to $\Delta v - \nabla \cdot (W_2 v) = 0$ in $B(z,R) \smallsetminus \{z\}$, so that

$$\begin{aligned} u_0(x) &= |x-z|^{2-n}, \quad |u_1(x)| + |x-z||\nabla u_1(x)| \leq C|x-z|^{2-n+\beta}, \\ v_0(x) &= |x-z|^{-n}(\eta \cdot (x-z)), \; |v_1(x)| + |x-z||\nabla v_1(x)| \leq C|x-z|^{1-n+\beta} \end{aligned}$$

where $\beta > 0$. Write $W = W_1 - W_2$ and use (5.7) with these u and v to obtain

$$\begin{aligned} -\int_{B(z,r)\cap\Omega} \eta \cdot (\nabla u_0) v_0 \, dx &= \int_{B(z,r)\cap\Omega} \eta \cdot ((\nabla u_0) v_1 + (\nabla u_1) v) \, dx \\ &+ \int_{B(z,r)\cap\Omega} (W(x) - W(0)) \cdot (\nabla u) v \, dx + \int_{\Omega \smallsetminus B(z,r)} W \cdot (\nabla u) v \, dx. \quad (5.26) \end{aligned}$$

Here $r = r(\sigma) = \sigma^{1/2}$. We write (5.26) as $I = I_1 + I_2 + I_3$ and want to show that I blows up as $\sigma \to 0$ at a faster rate than $I_1 + I_2 + I_3$.

We have

$$I = (n-2) \int_{B(z,r)\cap\Omega} |x-z|^{-2n} [\eta \cdot (x-z)]^2 \, dx.$$

The integrand is nonnegative so reducing the integration set makes the integral smaller. We define the set

$$E_\sigma = C_\varepsilon(z) \cap \{2\sigma < |x-z| < r\}$$

and note that by the considerations above E_σ is contained in $B(z,r) \cap \Omega$ when r is small. For $x \in E_\sigma$ we have

$$\Big|\eta \cdot \frac{x-z}{|x-z|}\Big| \geq \Big|\eta_n \Big(\frac{x-z}{|x-z|}\Big)_n\Big| - \Big|\eta' \cdot \Big(\frac{x-z}{|x-z|}\Big)'\Big| \geq \frac{1}{2}|\eta_n|(1-\varepsilon)$$

and

$$\begin{aligned} \int_{B(z,r)\cap\Omega} |x-z|^{-2n} [\eta \cdot (x-z)]^2 \, dx &\geq \int_{E_\sigma} |x-z|^{2-2n} \Big[\eta \cdot \frac{x-z}{|x-z|}\Big]^2 \, dx \\ \geq \frac{\eta_n^2 (1-\varepsilon)^2}{4} \int_{2\sigma}^{r} s^{2-2n} \gamma |\partial B(z,s)| \, ds &= C(n, \varepsilon_0, \eta)((2\sigma)^{2-n} - r^{2-n}). \end{aligned}$$

Using the choice $r = \sigma^{1/2}$ this gives $I \geq C\sigma^{2-n}$ when σ is small.

For the right hand side of (5.26), first we have

$$|I_1| \leq C \int_{c\sigma < |x-z| < r} |x-z|^{2-2n+\beta} \, dx \leq C\sigma^{2-n+\beta}$$

for σ small. For I_2 note $|x| \leq (1+\frac{1}{c})|x-z| \leq Cr$ on $\Omega \cap B(z,r)$, so that $|W(x)-W(0)| = o(1)$ as $\sigma \to 0$ by the continuity of W at 0. Then

$$|I_2| \leq o(1) \int_{c\sigma<|x-z|<r} |x-z|^{2-2n}\, dx = \sigma^{2-n} o(1)$$

as $\sigma \to 0$. Finally

$$|I_3| \leq C \int_{r<|x-z|<R} |x-z|^{2-2n}\, dx \leq C\sigma^{\frac{2-n}{2}}.$$

Now multiplying (5.26) by σ^{n-2} and letting $\sigma \to 0$ gives a contradiction. □

Again global uniqueness is obtained from the boundary result and Theorem 1.5.

Theorem 1.10. Let $\Omega \subseteq \mathbf{R}^n$ be a bounded open set with Lipschitz boundary, and suppose $n \geq 3$. If W_1 and W_2 are two Lipschitz continuous vector fields in Ω, then $\Lambda_{W_1} = \Lambda_{W_2}$ implies $W_1 = W_2$ in Ω.

Proof. By Theorem 1.9, W_1 and W_2 have Lipschitz continuous extensions to a larger ball so that they coincide outside Ω, and an analogue of Lemma 4.2 shows that $\Lambda_{W_1} = \Lambda_{W_2}$ in this ball. Therefore, we may assume that Ω is a ball and $W_1 = W_2 = 0$ on $\partial\Omega$.

If $W \in W^{1,\infty}(\Omega; \mathbf{R}^n)$ define $q(W) = \frac{|W|^2}{4} + \frac{\nabla \cdot W}{2}$. It follows from (4.1) that

$$L_{W/2i,q(W)} = -\Delta - W \cdot \nabla$$

and

$$\langle \Lambda_{W/2i,q(W)} f, g \rangle = \int_\Omega (\nabla u_f \cdot \nabla e_g - W \cdot (\nabla u_f) e_g)\, dx + \frac{1}{2} \int_{\partial\Omega} (W \cdot \nu) f g \, dS$$

where $u_f \in H^1(\Omega)$ solves $(\Delta + W \cdot \nabla) u_f = 0$ in Ω, $u_f = f$ on $\partial\Omega$, and $e_g \in H^1(\Omega)$ satisfies $e_g = g$ on $\partial\Omega$. This shows that

$$\Lambda_{W/2i,q(W)} f = \Lambda_W f + \frac{1}{2}(W \cdot \nu)|_{\partial\Omega} f.$$

From $\Lambda_{W_1} = \Lambda_{W_2}$ and $W_1|_{\partial\Omega} = W_2|_{\partial\Omega}$ we have $\Lambda_{W_1/2i,q(W_1)} = \Lambda_{W_2/2i,q(W_2)}$. Then Theorem 1.5 implies $\operatorname{curl} W_1 = \operatorname{curl} W_2$ in Ω, and since Ω is a ball we have $W_2 = W_1 + \nabla p$ where $p \in W^{2,\infty}(\Omega; \mathbf{R}^n)$. Here $\nabla p = 0$ near $\partial\Omega$, so by substracting a constant we may assume that $p = 0$ on $\partial\Omega$.

From Theorem 1.5 we also have that the potentials $\frac{|W_j|^2}{4} + \frac{\nabla \cdot W_j}{2}$ must be the same. Using $W_2 = W_1 + \nabla p$, this implies that

$$\Delta p + W_1 \cdot \nabla p + \frac{1}{2}|\nabla p|^2 = 0 \quad \text{in } \Omega.$$

Since also $p|_{\partial\Omega} = 0$, the maximum principle for quasilinear elliptic equations ([18, 10.1]) implies that $p = 0$. Hence $W_1 = W_2$ in Ω. □

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