FIXED ANGLE INVERSE SCATTERING FOR RIEMANNIAN METRICS

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Abstract. The fixed angle inverse scattering problem for a velocity consists in determining a sound speed, or a Riemannian metric up to diffeomorphism, from measurements obtained by probing the medium with a single plane wave. This is a formally determined inverse problem that is open in general. In this article we consider the rigidity question of distinguishing a sound speed or a Riemannian metric from the Euclidean metric. We prove that a general smooth metric that is Euclidean outside a ball can be distinguished from the Euclidean metric. The methods involve distorted plane waves and a combination of geometric, topological and unique continuation arguments.

Key words. Inverse scattering, fixed angle scattering, distorted plane waves, eikonal equation solutions.

AMS subject classifications. 35R30

1. Introduction.

1.1. The statement of the problem. An acoustic medium occupying $\mathbb{R}^n$, $n > 1$, with non-constant sound speed, is probed by a **single** impulsive plane wave (hence exciting all frequencies), and the far-field medium response is measured in all directions for all frequencies. A longstanding open problem, called the fixed angle scattering inverse problem (for velocity), is the recovery of the sound speed of the medium from this far-field response. In some situations, the acoustic properties of the medium are modeled by a Riemannian metric and then the goal is the recovery of this Riemannian metric from the far field measurements corresponding to perhaps $n(n+1)/2$ incoming plane waves.

We consider time domain, near field measurement versions of these problems. It is not known whether these versions of the problems are equivalent to the frequency domain versions of these problems. For the related operator $\partial_t^2 - \Delta_x + q(x)$, such time domain inverse problems are equivalent to fixed angle inverse scattering problems in the frequency domain [26]. We show that the near field measurements distinguish between a constant velocity (or Euclidean metric) medium and a non-constant velocity (non-Euclidean metric) medium. This can be considered as a rigidity theorem analogous to a corresponding result for the boundary rigidity problem [12]. The significant aspect of our result is that the only condition imposed on the velocity/metric is that they are smooth and constant/Euclidean outside a compact set. We do not require a non-trapping condition or a 'diffeomorphism condition' on the velocity/metric. The results are obtained by combining topological, geometrical and PDE based arguments.

In [24], we have obtained similar results for the more difficult Lorentzian metric case. Here the data consists of near field measurements of solutions associated with the operator $\Box_h$ where $h(x,t)$ is a space and time dependent Lorentzian metric on $\mathbb{R}^n \times \mathbb{R}$.

In $\mathbb{R}^n$ with $n > 1$, e_i denotes the unit vector parallel to the positive x_i-axis, B

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denotes the origin centered open ball of radius 1, $B_r(p)$ denotes the p centered open ball of radius $r > 0$ and $\overline{B}_r(p)$ its closure. For a curve $s \to \gamma(s)$ in $\mathbb{R}^n$, $\dot{\gamma}(s)$ will denote its derivative.

For the velocity problem, the waves propagate as solutions of the homogeneous PDE associated with the operator $\rho(x)\partial_t^2 - \Delta_x$, corresponding to a medium with velocity $1/\sqrt{\rho}$, or the operator $\partial_t^2 - \Delta_g$ associated to a Riemannian metric $g(x)$ on $\mathbb{R}^n$. Recall that if $g = (g_{ij})$ and $g^{-1} = (g^{ij})$ then

$$\Delta_g := \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n} \partial_i \left(\sqrt{\det g}\, g^{ij}\, \partial_j \right)$$

is the Laplace-Beltrami operator associated with the Riemannian metric g. The operator

$$\mathcal{L} := \partial_t^2 - \frac{1}{m\sqrt{\det g}} \sum_{i,j=1}^{n} \partial_i \left(m\sqrt{\det g}\, g^{ij} \partial_j \right) \tag{1.1}$$

associated with a positive smooth function $m(x)$ and a smooth Riemann metric $g(x)$ captures both cases. If we take $m = 1$ we have $\mathcal{L} = \partial_t^2 - \Delta_g$ and if we take $g = \rho I$, $m = \rho^{(2-n)/2}$ then $\mathcal{L} = \partial_t^2 - \rho^{-1}\Delta_x$ which is equivalent to $\rho\partial_t^2 - \Delta_x$ for the homogeneous PDE. Note that $\mathcal{L}$ is self-adjoint with weight $m\sqrt{\det g}$.

We assume that $m(x) = 1$ and $g(x) = I$ for $|x| \geq 1 - \delta$ for some small positive δ. For a fixed unit vector ω in $\mathbb{R}^n$, let $U(x, t; \omega)$ be the solution of the initial value problem (IVP)

$$\mathcal{L}U(x, t; \omega) = 0, \qquad (x, t) \in \mathbb{R}^n \times \mathbb{R}, \tag{1.2a}$$

$$U(x, t; \omega) = H(t - x \cdot \omega), \qquad \text{for } t << 0. \tag{1.2b}$$

Here $H(s)$ is the Heaviside function and the initial value for U represents an impulsive incoming plane wave moving in the direction ω. *This is a well posed problem whose solution has a trace on $\partial B \times \mathbb{R}$ (see Proposition 2.2 and the remarks after it).* For a fixed unit vector ω in $\mathbb{R}^n$, for a finite set Ω of unit vectors in $\mathbb{R}^n$, and $T \in \mathbb{R}$, define the forward maps

$$\mathcal{F}_{\omega,T} : (m, g) \to U(\cdot, \cdot; \omega)|_{\partial B \times (-\infty, T]}, \tag{1.3}$$

$$\mathcal{F}_{\Omega,T} : (m, g) \to [\mathcal{F}_{\omega,T}(m, g)]_{\omega \in \Omega} \tag{1.4}$$

The fixed angle inverse scattering problem is the study of the injectivity, the stability and the inversion of $\mathcal{F}_{\Omega,T}$.

This problem is formally determined because $m(x), g(x)$ depend on n variables and $\mathcal{F}_{\Omega,T}(m, g)$ is also a function n parameters. Formally determined problems are more difficult than the overdetermined problems such as the Dirichlet to Neumann map inverse problem for the operator $\mathcal{L}$ where the data depends on $2n - 1$ variables. Later in this section, we give a detailed survey of the literature for our problem but we summarize the earlier results by saying that, as far as we know, the only significant past result for our inverse problems is for the ρ problem with $g = I$, where it was shown that $\mathcal{F}$ is injective and stable if ρ is restricted to a small enough neighborhood of 1, in the $C^k(\mathbb{R}^n)$ norm, for some k dependent on n, and T is large enough.

There are strong results for the fixed angle inverse scattering problem for the operator $\partial_t^2 - \Delta + q(x)$, a formally determined problem, but our inverse problem is more difficult because, in our problems, the waves move with non-constant speed so the solution $U(x,t;\omega)$ has a much more complicated structure than the solution $U(x,t;\omega)$ associated with the operator $\partial_t^2 - \Delta + q(x)$. That is why progress so far has been limited to the set of ρ close to 1.

1.2. The results.

DEFINITION 1.1. *The operator $\mathcal{L}$ defined by (1.1) is called* **admissible** *if $m(x)$ is a smooth positive function on $\mathbb{R}^n$, $g(x)$ is a smooth Riemannian metric on $\mathbb{R}^n$, and there is a small positive δ such that $m(x) = 1$ and $g(x) = I$ on $|x| \geq 1 - \delta$.*

For admissible $\mathcal{L}$ and for positive functions ρ with $\rho - 1$ compactly supported, we define the positive constants

$$\begin{aligned} \rho_{min} &:= \inf_{x \in \mathbb{R}^n} \rho(x), & \rho_{max} &:= \sup_{x \in \mathbb{R}^n} \rho(x), \\ m_{min} &:= \inf_{x \in \mathbb{R}^n} m(x), & m_{max} &:= \sup_{x \in \mathbb{R}^n} m(x), \\ g_{min} &:= \inf_{x \in \mathbb{R}^n,\ v \in \mathbb{R}^n,\ |v|=1} v^T g(x) v, & g_{max} &:= \sup_{x \in \mathbb{R}^n,\ v \in \mathbb{R}^n,\ |v|=1} v^T g(x) v. \end{aligned}$$

Note that $\rho_{min} \leq 1 \leq \rho_{max}$, $m_{min} \leq 1 \leq m_{max}$ and $g_{min} \leq 1 \leq g_{max}$.

We reserve the subscript 0 to denote objects associated with the operator $\mathcal{L}$ when it is $\partial_t^2 - \Delta$, that is when $m = 1, g = I, \rho = 1$. Hence $g_0 = I$, $\mathcal{L}_0 := \partial_t^2 - \Delta$ and we use U_0, Λ_0 (see Proposition 2.2) to denote the objects associated with $\mathcal{L}_0$.

Our first result is for the operator $\mathcal{L} = \partial_t^2 - \rho^{-1}\Delta$.

THEOREM 1.2 (Distinguishing ρ from 1). *Consider the admissible operator $\mathcal{L} = \partial_t^2 - \rho^{-1}\Delta$, a real number $T > 4\sqrt{\rho_{max}} - 1$ and ω a fixed unit vector in $\mathbb{R}^n$. If $\mathcal{F}_{\omega,T}(\rho) = \mathcal{F}_{\omega,T}(1)$ then $\rho = 1$.*

Our second result is for the operator $\mathcal{L} := \partial_t^2 - \Delta_g$.

THEOREM 1.3 (Distinguishing g from the Euclidean metric). *Consider the admissible operator $\mathcal{L} = \partial_t^2 - \Delta_g$, a real number $T > 4\sqrt{g_{max}} - 1$, and let*

$$\Omega := \{e_i : i = 1, \cdots, n\} \cup \{(e_i + e_j)/\sqrt{2} : i \neq j,\ i,j = 1, \cdots, n\}. \tag{1.5}$$

If $\mathcal{F}_{\Omega,T}(g) = \mathcal{F}_{\Omega,T}(g_0)$, there is diffeomorphism $\psi : \mathbb{R}^n \to \mathbb{R}^n$ with $\psi = Id$ outside B and $g = \psi^* g_0$.

We observe that the metric g is made of possibly $n(n+1)/2$ different functions, and correspondingly we employ measurements resulting from incoming waves traveling in $n(n+1)/2$ different directions. If $\psi : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism with $\psi = id$ outside B then one can verify that $\mathcal{F}_{\omega,T}(\psi^* g) = \mathcal{F}_{\omega,T}(g)$. Hence, one can hope to recover g only up to a diffeomorphism. There is no such non-uniqueness for the ρ problem.

Our results are a small step towards perhaps proving the injectivity of $\mathcal{F}_{\Omega,T}$ (up to a diffeomorphism), but a significant step since we make no assumptions about ρ being close to 1, or g being close to the Euclidean metric, and we do not require a non-trapping condition or the absence of conjugate points or caustics.

1.3. History. In the literature, for the fixed angle inverse scattering problem, the medium is probed by a single incoming wave and the far field data is measured in all directions and at all frequencies. We chose to work with time domain data since it permits the use of powerful ideas and tools available for formally determined inverse problems for hyperbolic PDEs in the time domain. In [26], we showed that, for the operator $\square + q(x)$ with $q(x)$ smooth and compactly supported, the frequency domain version of the problem and the time domain version are equivalent problems. We have not studied the equivalence of the time and frequency domain data for the **admissible** operators $\rho\partial_t^2 - \Delta$ or $\partial_t^2 - \Delta_g$.

The $n = 1$ version of the problem for the operator $\rho(x)\partial_t^2 - \Delta_x$ received considerable attention from the 1950s to 1980s. Here $\rho(x)$ is a smooth positive function on $\mathbb{R}$ with $\rho - 1$ supported in $[0, X]$, $U(x,t)$ is the solution of the IVP

$$\begin{aligned} (\rho(x)\partial_t^2 - \partial_x^2)U(x,t) &= 0, \qquad (x,t) \in \mathbb{R} \times \mathbb{R} \\ U(x,t) &= H(t-x), \qquad (x,t) \in \mathbb{R} \times (-\infty, 0), \end{aligned}$$

and the forward map is the reflection data map

$$\mathcal{F} : \rho \to U(0, \cdot)|_{(-\infty,T]},$$

for some large T. Even though the problem is in one space dimension, it is still a difficult problem to tackle directly since the medium has non-constant speed (and is the unknown). Using the travel time change of variables

$$s(x) = \int_0^x \sqrt{\rho(\xi)}\, d\xi, \qquad x \geq 0,$$

the original one dimensional inverse problem is transformed to an inverse problem for the constant speed operator $\partial_t^2 - \partial_s^2 - \sigma(s)\partial_s$, where

$$\sigma(s) = (1/\sqrt{\rho})'(x(s)).$$

The new forward map is the reflection data map

$$\mathcal{G} : \sigma \to V(0, \cdot)|_{(-\infty,T]}$$

for some large T, where $V(s,t)$ is the solution of the IVP

$$\begin{aligned} (\partial_t^2 - \partial_s^2 - \sigma(s)\partial_s)V(s,t) &= 0, \qquad (s,t) \in \mathbb{R} \times \mathbb{R}, \\ V(s,t) &= H(t-s), \qquad (s,t) \in \mathbb{R} \times (-\infty, 0). \end{aligned}$$

The injectivity, stability, the range and the inversion of $\mathcal{G}$ was resolved in the period 1950s to 1980s through the work of several researchers (see [7] for the results and a survey), however subtle issues arise when studying the behavior of the map $\rho(\cdot) \to \sigma(\cdot)$, needed to translate the results for the operator $\partial_t^2 - \partial_s^2 - \sigma(s)\partial_s$ to results for the operator $\rho(x)\partial_t^2 - \partial_x^2$.

For the $n > 1$ case, as far as we know, the only results addressing the injectivity or stability of $\mathcal{F}$ are for ρ close to 1. Under that assumption, the φ defined in (2.6) is a diffeomorphism and U has a much simpler structure; further there are other simplifications because ρ is close to 1. The first stability result, which is for ρ close to 1, is due to Romanov in [28]. Even though the result in [28] deals only with ρ close to 1, the problem is still non-trivial as one must deal with solutions whose smooth parts are defined on different regions and one needs estimates on the difference of the smooth parts. [28] devised an important idea to tackle this issue. Later, using ideas different from the one used in [28], a similar result but with weaker norms was proved in [21].

Romanov analyzed the problem for the operator $\rho\partial_t^2 - \Delta$ also when $\rho(y,z)$ (with $y \in \mathbb{R}^{n-1}$ and $z \in \mathbb{R}$) is analytic in y and the corresponding φ is a diffeomorphism. In section 3.3 of [28], under the analyticity assumption, he shows a provable reconstruction method for recovering ρ over $0 \leq z \leq z_0$ for some (possibly small) z_0, from $\mathcal{F}(\rho)$. He combines ideas from the one dimensional case with power series expansions to obtain the result. The long statement of the result may be found in Section 3.3 of [28]. Techniques for the one dimensional problem may be adapted to obtain results also for the problem where $\rho(y,z)$ is discretized in y as done in [18], [17].

There has been much more progress on studying fixed angle inverse scattering problems for operators with constant speeds. For the operator $\partial_t^2 - \Delta^2 + q(x)$, using an adaption of the Bukhgeim-Klibanov method introduced in [8] (see [5] for an exposition), it was shown in [27], [26] that the map $q \to [F_+(q), F_-(q)]$ is injective with a stable inverse. Here $F_+(q), F_-(q)$ are the fixed angle scattering data for the incoming waves $H(t - x \cdot \omega)$ and $H(t + x \cdot \omega)$ respectively. In [23], these ideas were adapted to prove similar results for the fixed angle scattering problem of recovering the vector field $a(x)$ and the function $q(x)$ from the data associated with the operator $\partial_t^2 - \Delta + a(x) \cdot \nabla + q(x)$. These ideas were adapted even to the non-constant speed case in [22] to prove similar results for the problem of recovering $q(x)$ from the data for the operator $\partial_t^2 - \Delta_g + q$, where g is a known Riemannian metric on $\mathbb{R}^n$ satisfying certain symmetry conditions and has a global convex function. This result does not address the recovery of the Riemannian metric g from the fixed angle scattering data.

There are other formally determined inverse problems for the operator $\rho\partial_t^2 - \Delta$, arising from other source receiver combinations than the one associated with $\mathcal{F}$. [30] studies the backscattering problem for this operator and shows that the backscattering data operator is injective if ρ is close to 1. They reprove this result in [32] as an application of a generalization of the inverse function theorem. There are also results for the Bukhgeim-Klibanov type formally determined inverse problems where the source is an internal source exciting the whole medium initially. Results for these types of problems were the first results for multidimensional formally determined inverse problems for hyperbolic operators. We state one such problem and result from [15] about the recovery of a 'velocity' because the problems are difficult and the ideas may be relevant for further studies of our problem. The article [33] also has a result for this Bukhgeim-Klibanov type velocity inversion problem.

Suppose $p(x)$ is a smooth positive function on $\overline{B}$ and $u(x,t)$ is the solution of the IBVP

$$\begin{aligned} u_{tt} - \nabla \cdot (p\,\nabla u) &= 0, && \text{on } B \times [0,T], \\ u(x,0) = f(x),\ u_t(x,0) &= 0, && x \in B, \\ u(x,t) &= g(x,t), && (x,t) \in \partial B \times [0,T]. \end{aligned}$$

Here the smooth functions f and g (satisfying the boundary matching condition) are fixed and may be regarded as the source for the inverse problem. Note that $1/\sqrt{p(x)}$ is the unknown wave speed associated with the operator. Under the strong assumption on the source f that

$$(x - x_0) \cdot (\nabla f)(x) > 0, \qquad \forall x \in \overline{B},$$

for some x_0 not in $\overline{B}$, in [15], it is shown that the map (for large T)

$$\mathcal{G} : p \to \partial_\nu u|_{\partial B \times [0,T]}$$

is injective if p is restricted to the set of functions which obey a condition which essentially guarantees that, for some $\beta > 0$, the function $|x - x_0|^2 - \beta t^2$ is a pseudo-convex function for the operator $\partial_t^2 - \nabla \cdot p \nabla$.

For the operators $\rho(x)\partial_t^2 - \Delta$ (and the operator $\partial_t^2 - \Delta_g$), there is also considerable work on overdetermined inverse problems such as the injectivity, stability and inversion of the map

$$\mathcal{H} : \rho \to \Lambda_\rho$$

where Λ_ρ is the Dirichlet to Neumann map for the operator $\rho(x)\partial_t^2 - \Delta$ over some space time domain. The Boundary Control method of Belishev introduced in [3] (see [4] or [16] for an exposition), and subsequent methods motivated by it, are very effective in showing the injectivity of $\mathcal{H}$ and providing reconstruction algorithms, even for problems with non-constant velocity. There are also some weak stability results (see [6]) associated with this method. We found intriguing the unexpected results (Theorems 3.1, 3.3) in [2] which suggest that the range of $\mathcal{H}$ is discrete in, what one would consider to be, the appropriate topology for the set of Λ_ρ. The article [34] explains the surprising phenomena in [2]. We also remark that knowledge of the map Λ_ρ implies knowledge of the scattering relation of the sound speed ρ, and this connection has been exploited, for example in [31], [34], to study wave equation inverse problems. We also suggest [19] which studies an inverse problem for the operator $\rho\partial_t^2 - \Delta$, for various source-receiver combinations so that a certain product of solutions is dense in some function spaces. Some of the results in [19] seem to be for formally determined problems. We also mention [1], where the recovery of a Riemannian metric from the minimal areas enclosed by certain curves is investigated.

1.4. Organization. This article is organized as follows. Section 1 is the introduction. Section 2 states and proves the important geometrical results needed for the proofs of our main results -Theorems 1.2, 1.3. Section 2 also contains the short proof of the well-posedness of the forward problem. Section 3 contains the proofs of our main results. In the Appendix, we state and prove a uniqueness theorem for an IBVP for distributional solutions of the wave equation in exterior domains. This is needed in the proofs of Theorems 1.2, 1.3.

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2. The Lagrangian manifold and the well-posedness of the IVP. We recall some standard material about null bicharacteristics for $\mathcal{L}$, geodesics for a Riemannian manifold, and address the well-posedness of the IVP (1.2a), (1.2b). The definitions of the terms used below may be found in [9] and [20].

Fix a unit vector ω. Here the $\mathcal{L}$ defined by (1.1) is *assumed to be an admissible operator.* We define some of the objects associated with the solution U of (1.2a), (1.2b). These will be useful in the proofs of Theorems 1.2, 1.3. We associate with $\mathcal{L}$ the Riemannian metric g.

2.1. Notation. For a Riemannian metric g on $\mathbb{R}^n$ and $v, w \in T_x(\mathbb{R}^n)$, define

$$\langle v, w\rangle = v^T g(x) w, \qquad \|v\| = \sqrt{v^T g(x) v}.$$

We reserve $v\cdot w$ and $|v|$ for the Euclidean inner product and norm on $\mathbb{R}^n$. For $x \in \mathbb{R}^n$, $v \in T_x(\mathbb{R}^n)$, $t \to \gamma_{x,v}(t)$ will denote the geodesic which starts at x, with velocity v, at time $t = 0$.

For a unit vector ω in $\mathbb{R}^n$, define the hyperplanes

$$\Sigma_{-,\omega} := \{x \in \mathbb{R}^n : x\cdot\omega = -1\}, \qquad \Sigma_{+,\omega} := \{x \in \mathbb{R}^n : x\cdot\omega = 1\}.$$

A generic point in $\Sigma_{-,\omega}$ will be denoted by a. When the context is clear, to avoid cumbersome notation, we do not write the dependence on ω; so $\Sigma_{+,\omega}$ and $U(x, t; \omega)$ may be written as Σ_+ and $U(x, t)$.

If K is a subset of $\mathbb{R}^{n+1}$ and Λ a subset of $T^*(\mathbb{R}^{n+1})$, we define

$$\Lambda|_K = \Lambda \cap \{(x, t; \xi, \tau) \in T^*(\mathbb{R}^n \times \mathbb{R}) : (x, t) \in K\}.$$

A similar meaning will be given to $\Lambda|_K$ if K is a subset of $\mathbb{R}^n$.

2.2. Bicharacteristics, geodesics, and the first arrival time function.

2.2.1. Bicharacteristics and geodesics. The principal symbol of $\mathcal{L}$ is

$$p(x, t; \xi, \tau) = -\tau^2 + \xi^T(g(x))^{-1}\xi, \qquad (x, t; \xi, \tau) \in T^*(\mathbb{R}^n \times \mathbb{R})$$

and the null bicharacteristics associated with the solution $U(x, t; \omega)$ of the IVP (1.2a), (1.2b) are the solutions of the IVP

$$\begin{aligned}
\frac{dx_k}{ds} &= \frac{\partial p}{\partial \xi_k} = 2(g(x)^{-1}\xi)_k, & \frac{dt}{ds} &= \frac{\partial p}{\partial \tau} = -2\tau, & k &= 1, \cdots, n,\\
\frac{d\xi_k}{ds} &= -\frac{\partial p}{\partial x_k} = -\xi^T\partial_{x_k}(g(x)^{-1})\xi, & \frac{d\tau}{ds} &= -\frac{\partial p}{\partial t} = 0, & k &= 1, \cdots, n,
\end{aligned}$$

with the initial conditions

$$x(0) = a, \quad t(0) = -1, \quad \xi(0) = -\tau_0\omega, \quad \tau(0) = \tau_0,$$

for some unit vector $\omega \in \mathbb{R}^n$, $a \in \Sigma_{-,\omega}$ and real $\tau_0 \neq 0$. One sees that $p(x(s), t(s); \xi(s), \tau(s))$ is constant along solutions of this system of ODEs and 0 when $s = 0$, so these solutions are null bicharacteristics of $\mathcal{L}$.

Now $g(x)g^{-1}(x) = I_n$, so

$$\partial_k(g^{-1}) = -g^{-1}\partial_k(g)\, g^{-1}.$$

Further, τ is constant along the solution and $\tau_0 \neq 0$, hence dt/ds is never zero. So these solutions may be reparametrized with respect to t and the relevant null bicharacteristics are the solutions $t \to (x(t, a, \tau), \xi(a, t, \tau))$ of the IVP (here $\cdot$ means d/dt).

$$\text{(2.1a)} \qquad \dot{x} = -\frac{1}{\tau}g^{-1}\xi, \qquad \dot{\xi}_k = -\frac{1}{2\tau}(g^{-1}\xi)^T\partial_{x_k}(g)(g^{-1}\xi), \qquad k = 1, \cdots, n,$$

$$\text{(2.1b)} \quad x(t = -1, a, \tau) = a, \qquad \xi(t = -1, a, \tau) = -\tau\omega.$$

Note that $x(t, a, \tau)$ is independent of τ and $\xi(t, a, \tau) = \tau \xi(t, a, 1)$.

Since the solutions are null bicharacteristics, we have $\tau^2 = \xi^T g^{-1}(x)\xi$ and τ is constant, hence $|\xi(t, a, \tau)|$ is bounded for any fixed a, τ. Hence, from (2.1a), $\dot{x}(t, a, \tau)$ is bounded for a fixed a, τ. So solutions of the IVP (2.1a), (2.1b) exist for all $t \in \mathbb{R}$.

From (2.1a) we see that

$$\xi = -\tau g(x)\dot{x}, \tag{2.2}$$

hence solutions $[x(t, a, \tau), t; \xi(t, a, \tau), \tau]$ of the IVP (2.1a), (2.1b), when projected onto $\mathbb{R}^n$, are the solutions of the IVP

$$\frac{d}{dt}(g\dot{x})_k = \frac{1}{2}\dot{x}^T \partial_{x_k}(g)\dot{x}, \qquad k = 1, \cdots, n, \tag{2.3a}$$

$$x(t = -1, a) = a, \quad \dot{x}(t = -1, a) = \omega, \tag{2.3b}$$

Actually, there is a one-to-one correspondence because solutions of (2.3a), (2.3b) generate solutions of (2.1a), (2.1b) through the relation (2.2).

Equation (2.3a) is an ODE satisfied by the critical points of the energy functional $E(\gamma)$ associated with the Riemannian metric g, where

$$E(\gamma) = \int_c^d \|\dot{\gamma}(t)\|^2 \, dt,$$

and $\gamma : [c, d] \to \mathbb{R}^n$ varies over continuous piece-wise smooth curves with fixed end points. Further, this equation is also the equation satisfied by the geodesics of g (the zero acceleration curves) - see Proposition 39 in Chapter 10 of [25]. Hence the projection of the solutions of (2.1a), (2.1b) onto $\mathbb{R}^n$ sets up a one-to-one correspondence between the bicharacteristics with the initial condition (2.1b) and the geodesics with the initial condition (2.3b).

These special bicharacteristics/geodesics start on $\Sigma_{-,\omega}$, at $t = -1$, with velocity ω, and we label these special null bicharacteristics/geodesics as **ω-bicharacteristics/ω-geodesics**. For future use we observe that the ω-geodesics, with the t parametrization, are unit speed curves because

$$\|\dot{x}\|^2 = \dot{x}^T g(x)\dot{x} = \frac{1}{\tau^2}\xi^T g^{-1}(x)\xi = 1.$$

2.2.2. The length functional and the distance metric. We use the definitions and the results in Chapters 5 and 6 of [20]. An **admissible curve** γ on $\mathbb{R}^n$ is a continuous map $\gamma : [c, d] \to \mathbb{R}^n$ with a partition $c = c_0 < c_1 < \cdots < c_m = d$ such that, for each $i = 1, \cdots, m$, $\gamma(r)$ is smooth on $[c_{i-1}, c_i]$ with one sided derivatives of all orders at c_{i-1}, c_i, and $\dot{\gamma}(r)$ **is never zero on** $[c_{i-1}, c_i]$. The length, in the Riemannian metric, of an admissible curve γ is

$$L(\gamma) := \int_c^d \|\dot{\gamma}(r)\| \, dr;$$

note that $L(\gamma)$ is invariant under a smooth increasing reparametrization of γ.

As in [20], one can define admissible variations of an admissible curve γ and seek critical points of $\mathcal{L}$ for admissible variations but with fixed end points. Further, from Corollary 6.7 in [20], every fixed end critical point of L is a smooth curve and, when reparametrized to have constant speed, is a geodesic of g. Conversely, every geodesic is a critical point of L; note that geodesics have constant speed (in the Riemannian sense) and any affine reparametrization still results in a geodesic.

For points $p, q \in \mathbb{R}^n$, define the Riemannian distance

$$d(p,q) := \inf\{L(\gamma) : \gamma \text{ an admissible curve in } \mathbb{R}^n \text{ from } p \text{ to } q\}.$$

From Chapter 6 of [20] we know that $\mathbb{R}^n$ is a metric space with the metric $d(p,q)$. Further, using the line segment joining p to q, we note that

$$\sqrt{g_{min}}\,|q-p| \leq d(p,q) \leq \sqrt{g_{max}}\,|q-p|, \qquad p,q \in \mathbb{R}^n, \tag{2.4}$$

so $(\mathbb{R}^n, d)$ is a complete metric space. Hence, by Corollary 6.15 in [20] of the Hopf-Rinow theorem, for any $p, q \in \mathbb{R}^n$, there is a geodesic (so a smooth curve) γ joining p to q such that

$$L(\gamma) = d(p,q).$$

2.2.3. The time of first arrival function. Now we restrict attention to ω-geodesics. Note that the ω-geodesics, at time $t=-1$, are at the point $a \in \Sigma_{-,\omega}$, have velocity ω - so are orthogonal to $\Sigma_{-,\omega}$, and have unit speed for all t.

Since $\mathcal{L}$ is admissible, $\mathcal{L} = \Box$ on the region $|x| \geq 1-\delta$, so in particular on the region $x \cdot \omega \leq -1$. With some effort one can show (we do not use it) that the solution $U(x,t)$, of the IVP (1.2a), (1.2b), is zero if t is smaller than the time of first arrival, at x, of any of the ω-geodesics. This suggests the definition of a candidate for the time of first arrival function.

Definition 2.1. *The* time of first arrival function $\alpha : \mathbb{R}^n \to \mathbb{R}$ *is defined as*

$$\alpha(x) := \begin{cases} x \cdot \omega & \text{if } x \cdot \omega < -1, \\ -1 + \inf\{d(x,a) : a \in \Sigma_-\} & \text{if } x \cdot \omega \geq -1. \end{cases}$$

Note that

$$\alpha(x) = x \cdot \omega, \qquad \text{if } x \cdot \omega \leq -1 + \delta.$$

Since the metric $d(\cdot,\cdot)$ on $\mathbb{R}^n$ is topologically equivalent to the Euclidean metric on $\mathbb{R}^n$ (because of (2.4)), the usual continuity and compactness argument shows that, for points x in the region $x \cdot \omega \geq -1$, the infimum is attained in the definition of $\alpha(x)$. Hence, given an x with $x \cdot \omega \geq -1$, there is an $a_x \in \Sigma_-$ such that

$$\alpha(x) = d(x, a_x).$$

Hence, as discussed earlier, there is a geodesic γ joining a_x to x such that

$$\alpha(x) = -1 + d(x, a_x) = -1 + L(\gamma).$$

Since the geodesic γ is a Riemannian distance minimizing curve from Σ_- to x, it is orthogonal to Σ_- in the Euclidean/Riemannian sense (they are the same near Σ_-), hence its velocity there is parallel to ω and points in the positive ω direction. So if we reparametrize γ to have unit speed and start on a_x at time $t=-1$ then γ will be an ω-geodesic with

$$\gamma(t) = \gamma_{a_x,\omega}(t+1).$$

Summarizing, for every x in the region $x \cdot \omega \geq -1$, there is a segment of an ω-geodesic γ, from Σ_- to x such that

$$\alpha(x) = -1 + L(\gamma).$$

We note that α is Lipschitz continuous, hence differentiable a.e., because using the distance minimizing definition of $\alpha(x)$ one can see that for $x, y \in \mathbb{R}^n$, if γ is a line segment from x to y then

$$\alpha(x) \leq \alpha(y) + L(\gamma) \leq \alpha(y) + \sqrt{g_{max}}\,|y - x|,$$
$$\alpha(y) \leq \alpha(x) + L(\gamma) \leq \alpha(x) + \sqrt{g_{max}}\,|y - x|.$$

Hence

$$|\alpha(y) - \alpha(x)| \leq \sqrt{g_{max}}\,|y - x|.$$

2.3. The well-posedness of the IVP. Define the maps $\Phi : \mathbb{R} \times \Sigma_- \times (\mathbb{R} \backslash \{0\}) \to T^*(\mathbb{R}^n \times \mathbb{R})$ with

$$\Phi(t, a, \tau) = [x(t, a),\, t; \xi(t, a, \tau),\, \tau], \tag{2.5}$$

and, its projection onto $\mathbb{R}^n$, the map $\varphi : \mathbb{R} \times \Sigma_- \to \mathbb{R}^n$ with

$$\varphi(t, a) = x(t, a) = \gamma_{a,\omega}(t + 1). \tag{2.6}$$

Let Λ be the range of Φ. So Λ is a subset of $T^*(\mathbb{R}^n \times \mathbb{R})$ with

$$\Lambda := \{[x(t, a, \tau), t; \xi(t, a, \tau), \tau] : t \in \mathbb{R},\ a \in \Sigma_-,\ \tau \in \mathbb{R},\ \tau \neq 0\}. \tag{2.7}$$

From the material in Chapter 4 of [9] we can conclude that Λ is a conic Lagrangian submanifold of $T^*(\mathbb{R}^n \times \mathbb{R})$.

We state the well-posedness result for the IVP (1.2a), (1.2b).

Proposition 2.2 (The forward problem). *For the admissible $\mathcal{L}$ defined by (1.1), the IVP (1.2a), (1.2b) has a unique distributional solution U with WF$(U) = \Lambda$.*

Note that $\mathrm{WF}(U) = \Lambda$ and not just $\mathrm{WF}(U) \subset \Lambda$; this will be critical in the proofs of the theorems below.

Proof. The claim about the wave front set of U is a consequence of Hörmander's theorem on the propagation of singularities (see Theorem 6.1 in [10]) and that all the null bicharacteristics of $\mathcal{L}$ enter the region $t \ll 0$. The existence of U may be proved by an explicit construction of U as a Lagrangian distribution, following the method in Chapter 4 of [9]. To prove uniqueness, let V be a distributional solution of (1.2a) with $V = 0$ for $t \ll 0$. An application of Hörmander's propagation of singularities proves that V must be smooth. An application of energy estimates in conical regions of the form $\sqrt{g_{min}}|x - x_0| \leq (t - t_0)$, for a fixed (x_0, t_0), leads to $V = 0$ on $\mathbb{R}^n \times (-\infty, T)$. □

Since $WF(U) = \Lambda$ and Λ does not intersect the normal bundle of $\partial B \times \mathbb{R}$ (that is the singular directions of U are not normal to $\partial B \times \mathbb{R}$), U has a trace as a distribution on $\partial B \times \mathbb{R}$, hence the $\mathcal{F}_{\Omega,T}$ defined by (1.3) is well defined.

Λ is the subset of $T^*(\mathbb{R}^n \times \mathbb{R})$ traced out by the ω-bicharacteristics. The properties of Λ, Φ and φ give us detailed structural information about the solution of the IVP (1.2a), (1.2b). We do not state their additional important properties in this article as they are not needed here. However, these will be important for future attempts to study the injectivity of $\mathcal{F}_{\Omega,T}$.

2.4. The diffeomorphism condition. The next proposition is crucial in the proofs of Theorems 1.2, 1.3. It gives a useful condition guaranteeing that φ (defined in (2.6)) restricted to $\Sigma_- \times (-\infty, T)$ is a diffeomorphism. The proposition is a little stronger than what we need for the proofs of the theorems, and parts (a) - (c) are of independent interest.

PROPOSITION 2.3. *Suppose g is a smooth Riemannian metric with $g = I$ on $|x| \geq 1-\delta$ for some small positive δ, ω is a unit vector in $\mathbb{R}^n$, and $T > 2\sqrt{g_{max}} - 1$. If φ restricted to the set $\varphi^{-1}(\mathbb{R}^n \setminus B) \cap \{t < T\}$ is injective, the following holds.*

(a) Every ω-geodesic crosses Σ_+ before time T and never crosses it again.

(b) The set $W_T := \varphi((-\infty, T) \times \Sigma_-)$ is an open subset of $\mathbb{R}^n$ and contains the region $x \cdot \omega < -1 + (T+1)/\sqrt{g_{max}}$.

(c) φ restricted to $(-\infty, T) \times \Sigma_-$ is a diffeomorphism onto W_T.

(d) $\alpha(x)$ is a smooth function on W_T and $\|\nabla_g \alpha\| = 1$ on W_T, where $\nabla_g \alpha = g^{-1}\nabla\alpha$. Further $(\nabla_g\alpha)(x)$ is the velocity of the ω-geodesic through x.

(e) We have $\pi(\Lambda|_{W_T}) = \{(x, t = \alpha(x)) : x \in W_T\}$ where π is the projection $(x,t;\xi,\tau) \to (x,t)$.

Proof. Recall that

$$\varphi(t,a) = \gamma_{a,\omega}(t+1), \qquad t \in \mathbb{R},\ a \in \Sigma_-.$$

(a) First we show that every ω-geodesic reaches Σ_+ before time T.
Let x_* be a point in the region $x \cdot \omega < -1 + (T+1)/\sqrt{g_{max}}$. From the properties of $d(\cdot,\cdot)$ mentioned above, there is an $a_* \in \Sigma_-$ such that

$$d(a_*, x_*) = \min_{a \in \Sigma_-} d(a, x_*)$$

and the segment of the ω-geodesic $t \to \gamma_{a_*,\omega}(t+1)$ from a_* to x_* has length $d(a_*, x_*)$. Further, the geodesic has unit speed and, by (2.4), this minimum distance is at most $\sqrt{g_{max}}(x_* \cdot \omega + 1)$ so the geodesic reaches x_* at least by time

$$t = \sqrt{g_{max}}(x_* \cdot \omega + 1) - 1 < T$$

Hence the region $x \cdot \omega < -1 + (T+1)/\sqrt{g_{max}}$ lies in W_T. In particular, since $T > 2\sqrt{g_{max}} - 1$, W_T contains the region $x \cdot \omega \leq 2$.
Now Σ_+ lies in $W_T \setminus B$ and φ is injective on

$$\varphi^{-1}(\mathbb{R}^n \setminus B) \cap \{t < T\} = \varphi^{-1}(W_T \setminus B) \cap \{t < T\}.$$

Thus, for every point $p \in \Sigma_+$, there is exactly one ω-geodesic which reaches p before time T - actually it reaches by time $2\sqrt{g_{max}} - 1$. Further, noting that if $a \in \Sigma_-$ and $|a + \omega| > 1$, the ω-geodesics $t \to \gamma_{a,\omega}(t+1)$ are lines parallel to ω, the injectivity hypothesis implies any ω-geodesic reaching $\Sigma_+ \cap \overline{B}_1(\omega)$ before time T must have originated at a point $a \in \Sigma_- \cap \overline{B}_1(-\omega)$. So if we define

$$K = \varphi^{-1}(\Sigma_+ \cap \overline{B}_1(\omega)) \cap \{-1 \leq t \leq 2\sqrt{g_{max}} - 1\},$$

then K is closed and contained in $\overline{B}_1(-\omega)$ hence compact, and the map $F : K \to \Sigma_+ \cap \overline{B}_1(\omega)$ with

$$F(t,a) = \varphi(t,a) = \gamma_{a,\omega}(t+1)$$

is a continuous bijection, hence a homeomorphism.
Let $B_1(p)$ denote the p centered open ball of radius 1 in $\mathbb{R}^n$. So there are continuous maps $A : \Sigma_+ \cap \overline{B}_1(\omega) \to \Sigma_-$ and $\tau : \Sigma_+ \cap \overline{B}_1(\omega) \to \mathbb{R}$ with

$$\varphi(\tau(x), A(x)) = x, \qquad x \in \Sigma_+ \cap \overline{B}_1(\omega),$$

that is

$$\gamma_{A(x),\omega}(\tau(x)+1) = x, \qquad x \in \Sigma_+ \cap \overline{B}_1(\omega).$$

We show that A is surjective, which will prove that all ω-geodesics reach Σ_+ before time T.

First we observe that if γ is an ω-geodesic which reaches Σ_+ then it will never cross Σ_+ again. This is so because if γ reaches Σ_+ at time t_* then $\dot{\gamma}(t_*) \cdot \omega > 0$ because if $\dot{\gamma}(t_*) \cdot \omega \leq 0$ then running γ backwards from time t_* we will never reach Σ_- since all geodesics are straight lines outside B.

Next we show that A is injective. Suppose $A(x') = A(x'')$ for some $x', x'' \in \Sigma_+ \cap \overline{B}_1(\omega)$ with $x' \neq x''$. Let $a = A(x') = A(x'')$ and $t' = \tau(x'),\ t''(\tau(x'')$. Then $a \in \Sigma_-$, $0 < t', t'' < T$ and

$$x' = \varphi(a, t'), \qquad x'' = \varphi(a, t'').$$

Since φ is injective, we must have $t' \neq t''$. So there is an ω-geodesic which crosses Σ_+ at two different times, which contradicts the assertion in the previous paragraph. Hence A is injective.

If $\mathcal{A}$ is the range of A, then $A : \Sigma_+ \cap \overline{B}_1(\omega) \to \mathcal{A}$ is a continuous bijection, hence a homeomorphism since $\Sigma_+ \cap \overline{B}_1(\omega)$ is compact. Hence $\mathcal{A}$ has the same (trivial) homology as $\Sigma_+ \cap \overline{B}_1(\omega)$. It is clear that

$$A(x) = x - 2\omega, \qquad \text{if } x \in \Sigma_+ \cap \left(\overline{B}_1(0) \setminus B_{1-\delta}(0)\right),$$

so $\mathcal{A}$ contains an annulus, that is

$$\Sigma_- \cap [B_1(-\omega) \setminus B_{1-\delta}(-\omega)] \subset \mathcal{A} \subset \Sigma_-.$$

Hence, if any point of $\Sigma_- \cap B_1(-\omega)$ were missing from $\mathcal{A}$, the $(n-1)$-th homology group of $\mathcal{A}$ would be non-zero. So $\mathcal{A} = \overline{B}_1(-\omega) \cap \Sigma_-$.

Since we already know the behavior of the ω-gedoesics starting outside $\Sigma_- \cap B_1(-\omega)$, we conclude that all ω-geodesics cross Σ_+ before time T and never cross it again.

(b) We prove (b) and (c) together.

We have already shown above that W_T contains the region $x \cdot \omega < -1 + (T + 1)/\sqrt{g_{max}}$.

From (a) we note that if γ is an ω-geodesic and it crosses Σ_+ at x, then γ minimizes the distance of x from Σ_- and there are no other ω-geodesics through x.

We first show that φ restricted to $(-\infty, T) \times \Sigma_-$ is injective. Suppose $p \in B$ and there are two ω-geodesics γ_1, γ_2 through p. Suppose γ_i originates at a_i in Σ_- and crosses Σ_+ at x_i, $i = 1, 2$. The lengths of the segments of γ_1, γ_2 from a_1, a_2 to p must be same otherwise we would be able to create a curve from one of the a_i to the other x_j whose length is shorter than the shortest distance from Σ_- to x_j. With this equality, consider the curve γ consisting of the part of γ_1 from a_1 to p followed by the part of γ_2 from p to x_2. Hence we have a curve from Σ_- to x_2 which has the same length as the shortest distance from Σ_- to x_2. Hence γ is a distance minimizing curve from Σ_- to x_2 so, by Corollary 26 in Chapter 10 of [25], γ must be a geodesic which is normal to Σ_-. In particular γ must be smooth. Hence γ_1 and γ_2 must have the same velocity at p, hence $\gamma_1 = \gamma_2$. A similar but simpler argument works if there is an ω-geodesic which goes through p at two different times. Finally, because of the injectivity hypothesis, it is clear that, for p outside B, there is at most one ω-geodesic reaching p before time T. Hence φ restricted to $(-\infty, T) \times \Sigma_-$ is injective.

Next, we show that φ restricted to $(-\infty, T) \times \Sigma_-$ is a local diffeomorphism. If φ' is not invertible then, from Proposition 30 in Chapter 10 of [25], there is an

ω-geodesic γ and a point p (reached before time T) on γ which is a focal point of the ω-geodesics. Let q be a point on γ, reached after p, but before time T, which lies in the region $x \cdot \omega \geq 1$. Then, from Proposition 48 in Chapter 10 of [25], applied to $P = \{t = -1\} \times \Sigma_-$, there is a curve μ from Σ_- to q with speed less than 1. Hence the length of μ will be less than the distance between Σ_- and q because the ω-geodesics are unit speed curves; a contradiction. Therefore φ' is invertible on $(-\infty, T) \times \Sigma_-$.

So $W_T = \varphi((-\infty, T) \times \Sigma_-)$ is an open subset of $\mathbb{R}^n$ and $\varphi : (-\infty, T) \times \Sigma_- \to W_T$ is a local diffeomorphism and a bijection. Hence φ is a diffeomorphism.

(d) The inverse of the restricted φ in (c) is the map

$$W_T \ni x \to (\tau(x), A(x)) \in (-\infty, T) \times \Sigma_-,$$

for some smooth functions $A(x), \tau(x)$. Since $\varphi(a, t) = \gamma_{a,\omega}(t + 1)$ and the ω-geodesics are unit speed curves, one sees that $\alpha(\varphi(a, t)) = t$, so $\alpha(x) = \tau(x)$ hence smooth.

Again

$$\alpha(\gamma_{a,\omega}(t)) = t,$$

so if we construct, as in the proof of Theorem 2 in Section 3.2 of [11], the solution $\psi(x)$ of the first order non-linear PDE $(\nabla\psi)^T g(x)\,(\nabla\psi) = 1$ with $\psi(x) = x \cdot \omega$ on $x \cdot \omega = -1$, one sees that $\psi(x) = \alpha(x)$ and $(\nabla_g \alpha)(x) := g(x)^{-1}(\nabla\alpha)(x)$ is the speed of the ω-geodesic through x. Hence (d) has been proved.

(e) Noting that $\alpha(\varphi(t, a)) = t$, we have

$$\pi(\Lambda|_{W_T}) = \{(\varphi(t, a), t) : (t, a) \in (-\infty, T) \times \Sigma_-\} = \{(\alpha(x), x) : x \in W_T\}. \qquad \square$$

The following proposition has the same conclusion as Proposition 2.3 but a simpler proof of (a) because of the stronger hypothesis. This stronger hypothesis holds in the proofs of Theorems 1.2 and 1.3.

Proposition 2.4. *Suppose g is a smooth Riemannian metric with $g = g_0$ on $|x| \geq 1 - \delta$ for some small positive δ, and ω is a unit vector in $\mathbb{R}^n$. If $T > 2\sqrt{g_{max}} - 1$ and*

$$\Lambda_g|_{(\mathbb{R}^n \setminus B) \times (-\infty, T)} = \Lambda_0|_{(\mathbb{R}^n \setminus B) \times (-\infty, T)} \tag{2.8}$$

then (a)-(e) of Proposition 2.3 hold.

Remark. It will be clear from the proof that the hypothesis of Proposition 2.4 implies the hypothesis of Proposition 2.3, hence (a)-(e) of Proposition 2.3 will hold. What we gain from the stronger hypothesis of Proposition 2.4 is a simpler proof of (a).

Proof. The proof of (a) is more direct than the proof of (a) in Proposition 2.3. The proofs of (b) - (e) are the same as in Proposition 2.3, so we will not repeat them.

From the hypothesis we have that

$$\Lambda_g|_{(\mathbb{R}^n \setminus B) \times (-\infty, T)} = \{(x, x \cdot \omega; -\tau\omega, \tau) : x \cdot \omega < T,\ x \notin B, \tau \in \mathbb{R}\}.$$

So

$$\varphi((-\infty, T) \times \Sigma_-) \setminus B = \{x \in \mathbb{R}^n : x \cdot \omega < T,\ x \notin B\},$$

and through each point x in this region there is an ω-geodesic which reaches x at time $x \cdot \omega$ and, at that instant, has velocity ω (through the relation $\xi = -\tau g(x)\,\dot{x}$). Since the position and the velocity of a geodesic at a specific time determine the geodesic, through each point x in the above region there is a unique ω-geodesic and it reaches x at time $x \cdot \omega$ and has velocity ω at that time. Hence φ is injective on the region $\varphi^{-1}(\mathbb{R}^n \setminus B) \cap \{t < T\}$ and satisfies the hypothesis of Proposition 2.3.

From the previous paragraph we can conclude that for each $x \in \Sigma_+$, there is a unique $a_x \in \Sigma_-$ such that the ω-geodesic $t \to \gamma_{a_x,\omega}(t+1)$ reaches x before time T - in fact it reaches x at time $x \cdot \omega = 1$ and with velocity ω. Hence

$$a_x = \gamma_{x,\omega}(-2).$$

and we have the smooth map $A : \Sigma_+ \to \Sigma_-$ with

$$A(x) = \gamma_{x,\omega}(-2) = a_x, \qquad x \in \Sigma_+.$$

We claim that A is a diffeomorphism.

A is injective because if $A(x') = A(x'') = a$ for some $x', x'' \in \Sigma_+$ and $a \in \Sigma_-$ then, from our work in the previous paragraph,

$$x' = \gamma_{a,\omega}(2) = x''.$$

Next we show that A' is invertible at each point of Σ_+. Let $x_* \in \Sigma_+$ and (t_*, a_*) the unique point in $(-\infty, T) \times \Sigma_-$ with $\gamma_{a_*,\omega}(t_* + 1) = x_*$ - also $t_* = 1$ and $\dot{\gamma}_{a_*,\omega}(t_* + 1) = \omega$. Hence the equation

$$\gamma_{a,\omega}(t+1) \cdot \omega = 1$$

has a solution (t_*, a_*) and, since

$$\dot{\gamma}_{a_*,\omega}(t_* + 1) \cdot \omega = \omega \cdot \omega = 1 \neq 0,$$

by the implicit function theorem, we have a smooth map $a \to \tau(a)$ defined in some neighborhood N (in Σ_-) of a_* with $\tau(a_*) = t_*$ and

$$\gamma_{a,\omega}(\tau(a) + 1) \cdot \omega = 1, \qquad a \in N.$$

So we have a smooth map $\psi : N \to \Sigma_+$ with

$$\psi(a) = \gamma_{a,\omega}(\tau(a) + 1) = \gamma_{a,\omega}(2)$$

such that $\psi \circ A$ is the identity map on N, hence $A'(x_*)$ is invertible.

We now prove the surjectivity of A. Since A is a local diffeomorphism the range of A is open. Also,

$$A(x) = x - 2\omega, \qquad x \in \Sigma_+, \ |x - \omega| \geq 1.$$

so the range of A is

$$A\left(\Sigma_+ \cap \bar{B}_1(\omega)\right) \cup \{a \in \Sigma_- : |a + \omega| \geq 1\}.$$

The first set is compact, hence closed, and the second set is closed. Hence the range of A is closed, open and non-empty so must be Σ_-. ∎

3. Proofs of the theorems. A major component of the proofs of Theorem 1.2 and Theorem 1.3 is the proposition below, which is proved for the general $\mathcal{L}$ defined by (1.1).

PROPOSITION 3.1. *Let $\mathcal{L}$ be the admissible operator defined by (1.1), ω a fixed unit vector and U the solution of the IVP (1.2a), (1.2b). If $T > 4\sqrt{g_{max}} - 1$ and*

$$U|_{\partial B\times(-\infty,T)} = U_0|_{\partial B\times(-\infty,T)},$$

(the time of first arrival function) $\alpha(x)$ is defined and smooth on the region $x\cdot\omega < T$, $\alpha(x) = x\cdot\omega$ outside B, and

$$\Delta_{m,g}\alpha := \frac{1}{m\sqrt{\det g}}\sum_{i,j}\partial_i(m\sqrt{\det g}\, g^{ij}\,\partial_j\alpha) = 0, \qquad \text{on the region } x\cdot\omega < T.$$

We postpone the proof of Proposition 3.1 to the end of this section and show how the theorems follow from this proposition.

3.1. Proof of Theorem 1.2. Here $\mathcal{L} = \partial_t^2 - \rho^{-1}\Delta_x$ which corresponds to $g = \rho I$ and $m = \rho^{(2-n)/2}$ in (1.1) and we use the α associated to this $\mathcal{L}$. We also note that $g_{max} = \rho_{\max}$.

Proposition 3.1 implies $\alpha(x)$ is defined and smooth on the region $x\cdot\omega < T$ and

$$\Delta\alpha = 0 \;\text{ on } x\cdot\omega < T$$

and the size of T implies that this region contains a neighborhood of $\overline{B}$. Now $\alpha(x) = x\cdot\omega$ outside B therefore, from the uniqueness of solutions of the BVP for Laplace's equation, we have

$$\alpha(x) = x\cdot\omega, \qquad \text{on } \overline{B}$$

Hence $\nabla_g\alpha = g^{-1}\nabla\alpha = \rho^{-1}\omega$, so $\|\nabla_g\alpha\|^2 = 1$ implies

$$1 = \langle\nabla_g\alpha, \nabla_g\alpha\rangle = \rho^{-1}, \qquad \text{on } \overline{B},$$

hence $\rho = 1$ on $\overline{B}$.

3.2. Proof of Theorem 1.3. Here we have $\mathcal{L} = \partial_t^2 - \Delta_g$ corresponding to $m = 1$, Ω the set of unit vectors defined by (1.5) and let

$$W := \{x\in\mathbb{R}^n : x\cdot\omega < T \;\text{ for each } \omega\in\Omega\};$$

note that W is an open set containing $\overline{B}$. From Proposition 3.1, for each $\omega\in\Omega$, we have the function $\alpha_\omega(x)$ (constructed in Section 2), defined and smooth at least on W, with

$$\Delta_g\alpha_\omega = 0, \qquad \text{on } W, \qquad \alpha_\omega(x) = x\cdot\omega \qquad x\in W\setminus B.$$

Extend α_ω smoothly to all of $\mathbb{R}^n$ with $\alpha_\omega(x) = x\cdot\omega$ for x outside B and, since $g = g_0$ outside B, we observe that

$$\Delta_g\alpha_\omega = 0 \;\text{ on } \mathbb{R}^n, \qquad \alpha_\omega(x) = x\cdot\omega \;\text{ for } x\in\mathbb{R}^n\setminus B. \tag{3.1}$$

We also note that (d) in Proposition 2.3 gives us

$$\|\nabla_g\alpha_\omega\| = 1 \;\text{ on } \mathbb{R}^n. \tag{3.2}$$

So (3.1) and (3.2) hold for each $\omega\in\Omega$.

For $i \neq j$,

$$\alpha_{(e_i+e_j)/\sqrt{2}}(x) = x \cdot (e_i + e_j)/\sqrt{2} = (\alpha_{e_i} + \alpha_{e_j})(x)/\sqrt{2}, \qquad \text{for } x \in \mathbb{R}^n \setminus B,$$

so from (3.1) and the uniqueness of solutions of the BVP for the operator Δ_g, we conclude that

$$\alpha_{(e_i+e_j)/\sqrt{2}} = \frac{\alpha_{e_i} + \alpha_{e_j}}{\sqrt{2}}, \qquad \text{on } \mathbb{R}^n.$$

Therefore (3.2) implies that

$$1 = \|\nabla_g \alpha_{(e_i+e_j)/\sqrt{2}}\|^2 = \frac{1}{2}\|\nabla_g \alpha_{e_i} + \nabla_g \alpha_{e_j}\|^2 = 1 + \langle \nabla_g \alpha_{e_i}, \nabla_g \alpha_{e_j} \rangle,$$

hence

$$\langle \nabla_g \alpha_{e_i}, \nabla_g \alpha_{e_j} \rangle = \delta_{ij}, \qquad i, j = 1, \cdots n. \tag{3.3}$$

Define the smooth map $\psi : \mathbb{R}^n \to \mathbb{R}^n$ with

$$\psi(x) = (\alpha_{e_1}(x), \cdots, \alpha_{e_n}(x)), \qquad x \in \mathbb{R}^n.$$

Then (3.3) implies $\psi'(x)$ is invertible for each $x \in \mathbb{R}^n$ and $\psi(x) = x$ for x outside B implies ψ is a proper map. So Hadamard's theorem (see Theorem 2.1 in [29]) implies ψ is a diffeomorphism. Further (3.3) may be rewritten as $\psi'(x) g(x)^{-1} \psi'(x)^T = I$ so $g = \psi^* g_0$.

3.3. Proof of Proposition 3.1. We note that $U_0 = H(t - x \cdot \omega)$ and

$$\begin{aligned}\Lambda_0 &= \{(a + (t+1)\omega, t; -\tau\omega, \tau) : a \in \Sigma_-, t \in \mathbb{R}, \tau \in \mathbb{R}\}, \\ &= \{(x \cdot \omega + 1, x; \tau, -\tau\omega : x \in \mathbb{R}^n,\ \tau \in \mathbb{R}\}.\end{aligned}$$

Define

$$V(x,t) = U(x,t) - U_0(x,t), \qquad (x,t) \in \mathbb{R}^n \times (-\infty, T).$$

Since $\mathcal{L} = \mathcal{L}_0$ on $\mathbb{R}^n \setminus B$ and $U = U_0$ on $\partial B \times (-\infty, T)$, we have

$$\begin{aligned}\Box V &= 0, && \text{on } (\mathbb{R}^n \setminus B)(-\infty, T), \\ V &= 0, && \text{on } (\mathbb{R}^n \setminus B) \times (-\infty, 0) \\ V &= 0, && \text{on } \partial B \times (-\infty, T).\end{aligned}$$

Noting that $\mathrm{WF}(V) \subset \Lambda_0 \cup \Lambda$, the hypotheses of Proposition A.1 are satisfed. Hence, by Proposition A.1, we have $V = 0$ on $(\mathbb{R}^n \setminus B) \times (-\infty, T)$, that is

$$U(x,t) = U_0(x,t), \qquad \text{on } (\mathbb{R}^n \setminus \overline{B}) \times (-\infty, T), \tag{3.4}$$

which implies

$$\begin{aligned}\Lambda \cap \{|x| \geq 1,\ t < T\} &= \Lambda_0 \cap \{|x| \geq 1,\ t < T\} \\ &= \{(x, x \cdot \omega; -\tau\omega, \tau) : |x| \geq 1, x \cdot \omega < T,\ \tau \in \mathbb{R}\}.\end{aligned}$$

Noting the relation $\xi = -\tau \dot{\gamma}$ (see (2.2)) between the velocity of ω-geodesics γ and the corresponding point in Λ and Λ_0, through every point x in $(\mathbb{R}^n \setminus B) \cap \{x \cdot \omega < T\}$, there is a unique ω-geodesic which reaches x at time $x \cdot \omega$ and its velocity that instant is ω; the uniqueness holds because the position and the velocity of a geodesic, at a given time, uniquely determines the geodesic. In particular,

$$\varphi((-\infty, T) \times \Sigma_-) \cap (\mathbb{R}^n \setminus B) = \{x \cdot \omega < T\} \cap (\mathbb{R}^n \setminus B). \tag{3.5}$$

Hence φ is injective on $\varphi^{-1}(\mathbb{R}^n \setminus B) \cap \{t < T\}$, so the hypothesis of Proposition 2.3 (and also Proposition 2.4) is satisfied. Therefore, from Proposition 2.3,

$$W_T := \varphi\left((-\infty, T) \times \Sigma_-\right)$$

is an open subset of $\mathbb{R}^n$ which contains the region $x \cdot \omega < -1 + (T+1)/\sqrt{g_{max}}$ (in particular the region $\overline{B}$) and $\varphi : (-\infty, T) \times \Sigma_- \to W_T$ is a diffeomorphism. Further $\alpha(x)$ is smooth on W_T. Combining the above with (3.5) and noting that

$$\alpha(\varphi(t, a)) = t,$$

we see that

$$W_T = \{x : x \cdot \omega < T\} \qquad \text{and} \qquad \alpha(x) = x \cdot \omega \ \text{ in } W_T \setminus B.$$

Since $T > 4\sqrt{g_{max}} - 1$ and $g_{max} \geq 1$, we see that W_T includes the region $x \cdot \omega \leq 3$.

From (3.4),

$$\begin{aligned} U &= 1 \qquad \text{on } (\mathbb{R}^n \setminus B) \times (-\infty, T) \cap \{t > \alpha(x)\} \\ U &= 0 \qquad \text{on } (\mathbb{R}^n \setminus B) \times (-\infty, T) \cap \{t < \alpha(x)\} \end{aligned}$$

We would like to show that

$$U(x, t) = H(t - \alpha(x)) \qquad \text{on } \overline{B} \times (-\infty, T).$$

This is done in several steps.

From Proposition 2.3, the projection of $\Lambda|_{W_T}$ onto the x, t plane is the smooth surface

$$S = \{(t = \alpha(x), x) : x \in W_T\}$$

Hence, from Proposition 2.2, U restricted to $W_T \times (-\infty, T)$ is smooth at all points not on S. Define the regions

$$Q_+ := \overline{B} \times (-\infty, T) \cap \{t > \alpha(x)\}, \qquad Q_- := \overline{B} \times (-\infty, T) \cap \{t < \alpha(x)\};$$

and their lateral boundaries

$$\Gamma_+ := (\partial B) \times (-\infty, T) \cap \{t > \alpha(x)\}, \qquad \Gamma_- := (\partial B) \times (-\infty, T) \cap \{t < \alpha(x)\};$$

then U is smooth on Q_+ and on Q_-.

We observe the **important fact** that

$$\mathcal{L}(1) = 0, \qquad \text{on } \mathbb{R}^n \times \mathbb{R}, \tag{3.6}$$

hence

$$\mathcal{L}(U - 1) = 0 \ \text{ on } \ Q_+; \qquad (U-1) = 0,\ \partial_\nu (U - 1) = 0 \quad \text{ on } \ \Gamma_+. \tag{3.7}$$

Since T is large enough, using unique continuation and energy estimates, we show that $U = 1$ on Q_+.

Let

$$t_* = 3\sqrt{g_{max}} - 1;$$

the level surfaces

$$\psi(x,t) := g_{max}|x|^2 - (t-t_*)^2 = c > 0$$

are non-characteristic because the principal symbol of $\mathcal{L}$ is $-\tau^2 + \xi^T g^{-1} \xi$ and

$$\frac{\psi_t^2}{\psi_x^T g^{-1} \psi_x} = \frac{1}{g_{max}{}^2} \frac{(t-t_*)^2}{x^T g(x)^{-1} x} \leq \frac{(t-t_*)^2}{g_{max}|x|^2} = \frac{(t-t_*)^2}{c + (t-t_*)^2} < 1.$$

The level surface $\psi(x,t) = g_{max}$ is outside the region $B \times \mathbb{R}$, for $0 < c \leq g_{max}$ the surfaces $\psi = c$ intersect the boundary of Q_+ only[1] on its lateral boundary. Further there is an open set N containing $B \setminus \{0\} \times \{t = t_*\}$ such that

$$N \subset \{(x,t) : 0 < \psi(x,t) < g_{max}\} \cap Q_+$$

The coefficients of $\mathcal{L}$ are independent of t, so applying the Robbiano-Tataru theorem (see Theorem 2.66 in [16]) to $U - 1$ over the region Q_+, there is unique continuation across $\psi = c > 0$ for $\mathcal{L}$. Hence

$$(U-1)(\cdot, t_*) = 0, \quad \partial_t(U-1)(\cdot, t_*) = 0, \qquad \text{on } \overline{B}.$$

Now we use a standard energy calculation to show that $U - 1 = 0$ on Q_+. For $\epsilon > 0$, define

$$Q_{+,\epsilon,*} := \{(x,t) : x \in \overline{B},\ \alpha(x) + \epsilon \leq t \leq t_*\},$$
$$S_\epsilon := \{(x, \alpha(x) + \epsilon) : x \in \overline{B}\}.$$

We have

$$\begin{aligned} \mathcal{L}(U-1) &= 0, && \text{on } Q_{+,\epsilon,*}, \\ (U-1)(\cdot, t_*) = 0, \quad (U-1)(\cdot, t_*) &= 0, && \text{on } B, \\ U - 1 &= 0, && \text{on } Q_{+,\epsilon,*} \cap (\partial B \times \mathbb{R}). \end{aligned}$$

and the identity (using the summation convention)

$$\begin{aligned} 2m\sqrt{\det g}\, \partial_t v\, \mathcal{L} v = \partial_t &\left(m\sqrt{\det g}\, (\partial_t v)^2 + m\sqrt{\det g}\, (\nabla v)^T g^{-1} (\nabla v) \right) \\ &- 2\partial_i \left(m\sqrt{\det g}\; \partial_t v\, g^{ij}\; \partial_j v \right). \end{aligned} \tag{3.8}$$

Therefore, using

$$v = U - 1,$$

[1] On $B \times \{t = T\}$ we have

$$\psi(x,t) \leq g_{max} - (T - t_*)^2 < g_{max} - g_{max} = 0.$$

Further, on B we have $\alpha(x) < -1 + 2\sqrt{g_{max}}$ hence $t_* - \alpha(x) > \sqrt{g_{max}}$, so

$$\psi(x,t) < g_{max} - (t_* - \alpha(x))^2 < 0, \qquad \text{on } (B \times \mathbb{R}) \cap \{t = \alpha(x)\}.$$

and integrating the relation $2\partial_t v\,\mathcal{L}v = 0$ over the region $Q_{+,\epsilon,*}$, the divergence theorem gives

$$0 = \int_{S_\epsilon} m\sqrt{\det g}\ \left[(\partial_t v)^2 + (\nabla v)^T g^{-1}\,(\nabla v) + 2\partial_t v\,\partial_i\alpha\, g^{-1}\,\partial_j v\right]\,dx$$
$$= \int_{S_\epsilon} m\sqrt{\det g}\ \left[(\partial_t v\,\nabla\alpha + \nabla v)^T\, g^{-1}\,(\nabla\alpha\,\partial_t v + \nabla v)\right]\,dx.$$

Hence $\partial_t v\,\nabla\alpha + \nabla v = 0$ on S_ϵ. Noting that

$$\nabla(v(x, \alpha(x) + \epsilon)) = \nabla v + \partial_t v\,\nabla\alpha$$

we conclude that $v = U - 1$ is constant on S_ϵ. Since $U - 1 = 0$ on the edge of S_ϵ, we conclude that $U - 1 = 0$ on S_ϵ, for every $\epsilon > 0$. Hence $U - 1 = 0$ on $Q_+ \cap \{t \le t_*\}$. Also, a simpler energy estimate argument can be used to show that $U - 1$ on $Q_+ \cap \{t_* \le t < T\}$. Hence

$$U = 1 \quad \text{on } Q_+. \tag{3.9}$$

Next

$$\mathcal{L}U = 0, \ \text{ on } Q_-, \qquad U = 0, \ \text{ on } \Gamma_-, \qquad U = 0 \ \text{ on } Q_- \cap \{t < -1\}$$

Using an argument similar to (but simpler than) the above energy estimate, one can show that

$$U = 0 \ \text{ on } Q_-. \tag{3.10}$$

Using (3.4), (3.9) and (3.10), we claim that

$$U(x,t) = H(t - \alpha(x)), \qquad (x,t) \in \mathbb{R}^n \times (-\infty, T). \tag{3.11}$$

We prove this claim. Let χ be a smooth function on $\mathbb{R}^n \times (-\infty, T)$ with

- $\chi = 1$ on a neighborhood of $\overline{B} \times [-1, T]$,
- $\chi = 0$ for $t \le -3$.

Define

$$\tilde{U} := \chi U;$$

then, using the notation ∂_0 for ∂_t and the summation convention, and that $U(x,t) = H(t - x\cdot\omega)$ outside $\overline{B} \times (-1, T)$, we have

$$\mathcal{L}(\tilde{U}) = [L, \chi]U = (c_{ij}(\partial_i\chi)\partial_j + (\mathcal{L}\chi))\,U = (c_{ij}(\partial_i\chi)\partial_j + (\mathcal{L}\chi))\,H(t - x\cdot\omega)$$
$$= a(x,t)\delta(t - x\cdot\omega) + b(x,t)H(t - x\cdot\omega)$$
$$= F(x,t)$$

where F is locally in H^{-1}. Since $\tilde{U}$ is the solution of the IVP

$$\mathcal{L}\tilde{U} = F, \qquad \text{on } \mathbb{R}^n \times (-\infty, T)$$
$$\tilde{U} = 0, \qquad \text{on } t \le -3,$$

from [14, Theorems 23.2.4 and 23.2.7] for IVP for hyperbolic PDEs, we conclude that $\tilde{U}$ is locally in L^2. Next $1 - \chi = 0$ on a neighborhood of $\overline{B} \times [-2, T)$, hence $(1 - \chi)U = (1 - \chi)H(t - x\cdot\omega)$. Therefore

$$U = \chi U + (1 - \chi)U = \tilde{U} + (1 - \chi)H(t - x\cdot\omega)$$

is locally in L^2. Therefore, using (3.4), (3.9) and (3.10), we conclude that (3.11) holds.

One may verify that

$$\mathcal{L}(H(t-\alpha(x))) = \left(1 - \nabla\alpha^T g^{-1}\nabla\alpha\right)\delta'(t-\alpha(x)) + (\mathcal{L}\alpha)\delta(t-\alpha(x)) = (\mathcal{L}\alpha)\delta(t-\alpha(x)),$$

hence (3.11) and (1.2a) imply that $\mathcal{L}\alpha = 0$ on $t = \alpha(x)$. Therefore

$$\frac{1}{m\sqrt{\det g}}\partial_i\left(m\sqrt{\det g}\, g^{ij}\,\partial_j\alpha\right) = 0, \qquad \text{on} \quad \mathbb{R}^n. \tag{3.12}$$

Appendix A. Uniqueness for an exterior IBVP problem. We prove a uniqueness result for distributional solutions of an exterior IBVP. An alternative proof based on propagation of singularities may be found in [24, Lemma 6.1].

Points $x \in \mathbb{R}^n$ will sometimes be written as $x = (y, z)$ with $y \in \mathbb{R}^{n-1}$ and $z \in \mathbb{R}$, and we define the hyperplane

$$H := \{(y, z) \in \mathbb{R}^n : z = 0\}.$$

Proposition A.1. *Suppose Ω is an open subset of $\mathbb{R}^n$ with $(\mathbb{R}^n \setminus B) \subset \Omega$ and $V(x,t)$ is a distribution on $\Omega \times (-\infty, T)$ such that*

$$WF(V) \subset \{(x, t; \xi, \tau) \in T^*(\Omega \times (-\infty, T)) : |\tau| \geq c|\xi|\},$$

for some $c > 0$. If

$$\Box V = 0 \quad \text{on } (\mathbb{R}^n \setminus \overline{B}) \times (-\infty, T),$$
$$V|_{\partial B \times (-\infty, T)} = 0, \qquad V = 0 \text{ on } (\Omega \setminus \overline{B}) \times (-\infty, S)$$

for some $S < T$, then

$$V = 0 \text{ on } (\mathbb{R}^n \setminus \overline{B}) \times (-\infty, T).$$

Note that since $WF(V)$ does not intersect the normal bundle of $\partial B \times (-\infty, T)$, the restriction of V to $\partial B \times (-\infty, T)$ is well defined by the manifold version of [13, Theorem 8.2.4] applied to the imbedding $i : \partial B \times (-\infty, T) \to \Omega \times (-\infty, T)$ with $i(x, t) = (x, t)$.

The proof of Proposition A.1 uses the following observation. Below, $V * \rho$ denotes convolution only in t.

Lemma A.2. *Suppose Ω is an open subset of $\mathbb{R}^n$ with $H \subset \Omega$ and $V(x,t)$ is a compactly supported distribution on $\Omega \times (a, b)$ such that*

$$WF(V) \subset \Gamma := \{(x, t; \xi, \tau) \in T^*(\Omega \times (a, b)) : |\tau| \geq c|\xi|\} \tag{A.1}$$

for some $c > 0$. If $\rho \in C_c^\infty(-\epsilon, \epsilon)$ for a small $\epsilon > 0$ then $V * \rho$ (convolution only in t) is smooth on $\Omega \times (a + \epsilon, b - \epsilon)$. Further, V has a trace on $H \times (a, b)$ and

$$(V * \rho)|_{H \times (a+\epsilon, b-\epsilon)} = \rho * (V|_{H \times (a,b)}) \tag{A.2}$$

with the RHS restricted to $H \times (a + \epsilon, b - \epsilon)$.

We postpone the proof of Lemma A.2 to the end of this section.

Proof of Proposition A.1. Choose a $\rho \in C_c^\infty(-1,1)$ with $\int_{\mathbb{R}} \rho = 1$ and, for any $\epsilon > 0$, define the function ρ_ϵ by $\rho_\epsilon(t) = \epsilon^{-1}\rho(\epsilon^{-1}t)$. Define $V_\epsilon = V * \rho_\epsilon$ on $\Omega \times (-\infty, T-\epsilon)$. We claim that V_ϵ is smooth and $V_\epsilon|_{\partial B \times (-\infty, T-\epsilon)} = 0$. These claims are a quick consequence of of Lemma A.2 because the claims are local in nature and a local diffeomorphism in x can straighten out pieces of the curved surface ∂B to a piece of H.

Hence, from the hypothesis, the smooth function V_ϵ on $\Omega \times (-\infty, T-\epsilon)$ is a solution of the IBVP

$$\begin{aligned} \square V_\epsilon = 0, &\qquad \text{on } (\mathbb{R}^n \setminus B) \times (-\infty, T-\epsilon) \\ V_\epsilon = 0, &\qquad \text{on } \partial B \times (-\infty, T-\epsilon) \\ V_\epsilon = 0, &\qquad \text{on } (\mathbb{R}^n \setminus B) \times (-\infty, S-\epsilon). \end{aligned}$$

Pick any (x_0, t_0) in $(\mathbb{R}^n \setminus \overline{B}) \times (-\infty, T-\epsilon)$ and define the conical region

$$K := \{(x,t) : |x - x_0| \leq t_0 - t\} \setminus (B \times (-\infty, t_0)).$$

Integrating over K, the relation

$$2\,\partial_t V_\epsilon \,\square V_\epsilon = \partial_t[(\partial_t V_\epsilon)^2 + |\nabla V_\epsilon|^2] - 2\,\nabla \cdot (\partial_t V_\epsilon \,\nabla V_\epsilon),$$

and noting that $\square V_\epsilon = 0$ on K, one can show that V_ϵ is constant on the surface

$$\{(x,t) : t_0 - t = |x - x_0|\} \cap \{(x,t) : |x| \geq 1\},$$

hence $V_\epsilon = 0$ on it. Varying (x_0, t_0) we see that $V_\epsilon = 0$ on $(\mathbb{R}^n \setminus B) \times (-\infty, T-\epsilon)$.

Finally, for any $\varphi \in C_c^\infty(U \times (-\infty, T))$, since $\varphi * \rho_\epsilon \to \varphi$ in the topology of $C_c^\infty(U \times (-\infty, T))$, we have $V_\epsilon \to V$ as distributions on $\Omega \times (-\infty, T')$, for any $T' < T$. So $V = 0$ on $(\Omega \setminus \overline{B}) \times (-\infty, T')$ for any $T' < T$.

Proof of Lemma A.2. The lemma follows from a standard argument.

Let

$$\Gamma_1 = \{(\xi, \tau) \neq 0 : |\tau| \leq c|\xi|/2\}, \qquad \Gamma_2 = \{(\xi, \tau) : |\tau| \geq c|\xi|/2\}.$$

Since V is compactly supported, there is an integer M and a constant A such that

$$|\hat{V}(\xi, \tau)| \leq A(1 + |\xi| + |\tau|)^M, \qquad \forall (\xi, \tau) \in \mathbb{R}^{n+1}. \tag{A.3}$$

From (A.1), $\Gamma_1 \cap \Gamma = \emptyset$, hence using the compactness of the support of V and of the set of unit vectors in Γ_1, and a partition of unity argument, for each positive integer N there is a B_N such that

$$|\hat{V}(\xi, \tau)| \leq B_N (1 + |\xi| + |\tau|)^{-N}, \qquad \forall (\xi, \tau) \in \Gamma_1.$$

Since ρ is smooth and compactly supported, for each positive integer N there is a C_N such that

$$|\hat{\rho}(\tau)| \leq C_N (1 + |\tau|)^{-N}, \qquad \forall \tau \in \mathbb{R}. \tag{A.4}$$

Hence, for any positive integer N, for $(\xi, \tau) \in \Gamma_1$ we have

$$|\widehat{V * \rho}(\xi, \tau)| = |\hat{V}(\xi, \tau)|\,|\hat{\rho}(\tau)| \leq B_N\, C_1 (1 + |\xi| + |\tau|)^{-N},$$

and for $(\xi, \tau) \in \Gamma_2$ we have

$$1 + |\xi| + |\tau| \le 1 + 2|\tau|/c + |\tau| \le C(1 + |\tau|)$$

for some $C > 0$, so for some constant D_N, we have

$$\begin{aligned} |\widehat{V * \rho}(\xi, \tau)| &= |\hat{V}(\xi, \tau)|\, |\hat{\rho}(\tau)| \\ &\le A\, C_N (1 + |\xi| + |\tau|)^M\, (1 + |\tau|)^{-N} \\ &\le D_N (1 + |\xi| + |\tau|)^M\, (1 + |\xi| + |\tau|)^{-N} \\ &= D_N (1 + |\xi| + |\tau|)^{M-N}. \end{aligned}$$

Hence $V * \rho$ is smooth.

Noting the condition (A.1) in the hypothesis, [13, Theorem 8.2.4] applied to the function $f : H \times (a, b) \to \Omega \times (a, b)$ with $f(y, t) = ((y, 0), t)$, implies that V has a trace on $H \times (a, b)$. Define

$$\Gamma' := f^*(\Gamma) = \{(y, t; \eta, \tau) \in T^*(H \times (a, b)) : |\tau| \ge c|\eta|\}.$$

Then from [13, Theorem 8.2.3, Theorem 8.2.4], there is a sequence $V_j \in C_c^\infty(\Omega \times (a, b))$ with $V_j \to V$ in $\mathcal{D}'_\Gamma(\Omega \times (a, b))$ and $V_j|_{H \times (a,b)} \to V|_{H \times (a,b)}$ in $\mathcal{D}'_{\Gamma'}(H \times (a, b))$. Further, since V is compactly supported, multiplying V_j and V by a compactly supported smooth function which is 1 on the support of V, one may assume that the V_j and V are supported in the same compact subset of $\Omega \times (a, b)$.

Since $\Gamma_1 \cap \Gamma = \emptyset$, using the compactness of the common support of V_j, V and the compactness of the set of unit vectors in Γ_1, and a partition of unity, for each positive integer N

$$\lim_{j \to \infty} \sup_{\Gamma_1} (|\xi| + |\tau|)^N \left| (\hat{V} - \hat{V}_j)(\xi, \tau) \right| = 0.$$

Now

$$|\widehat{V_j * \rho} - \widehat{V * \rho}| = |\hat{V}_j - \hat{V}|\, |\hat{\rho}| \le C |\hat{V}_j - \hat{V}|,$$

so[2] $V_j * \rho \to V * \rho$ in $\mathcal{D}'_{\Gamma_2}(\Omega \times (a + \epsilon, b - \epsilon))$. Hence, by [13, Theorem 8.2.4],

$$V_j * \rho|_{H \times (a+\epsilon, b-\epsilon)} \to V * \rho|_{H \times (a+\epsilon, b-\epsilon)} \qquad \text{in } \mathcal{D}'(H \times (a + \epsilon, b - \epsilon)).$$

Also, for any compactly supported $W \in \mathcal{D}'(H \times \mathbb{R})$ and $\varphi \in C_c^\infty(H \times \mathbb{R})$, one has

$$\langle W(y, t) * \rho(t), \varphi(y, t) \rangle = \langle W(y, t), \varphi(y, t) * \rho(-t) \rangle$$

and $\varphi(y, t) * \rho(-t) \in C_c^\infty(H \times \mathbb{R})$. This observation implies that

$$V_j|_{H \times (a,b)} * \rho \to V|_{H \times (a,b)} * \rho \qquad \text{in } \mathcal{D}'(H \times (a + \epsilon, b - \epsilon)).$$

Since

$$V_j|_{H \times (a,b)} * \rho = V_j * \rho|_{H \times (a+\epsilon, b-\epsilon)}$$

with the LHS restricted to $H \times (a + \epsilon, b - \epsilon)$, the second part of the lemma has been proved.

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[2] For an open subset X of $\mathbb{R}^m$ and a closed conical subset C of $T^*(X)$, convergence in $\mathcal{D}'_C(X)$ is defined (see [13, Definition 8.2.2]) via a localization (multiplying by a function in $C_c^\infty(X)$) and then a Fourier transform estimate. However, for distributions in $\mathcal{D}'_C(X)$ with support in fixed compact subset of X, convergence in $\mathcal{D}'_C(X)$ follows if we can establish the same estimate without the localization. This can be shown by imitating the argument in the proof of [13, Lemma 8.1.1].

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