QUADRATURE DOMAINS FOR THE HELMHOLTZ EQUATION WITH APPLICATIONS TO NON-SCATTERING PHENOMENA

PU-ZHAO KOW, SIMON LARSON, MIKKO SALO, AND HENRIK SHAHGHOLIAN

ABSTRACT. In this paper, we introduce quadrature domains for the Helmholtz equation. We show existence results for such domains and implement the so-called partial balayage procedure. We also give an application to inverse scattering problems, and show that there are non-scattering domains for the Helmholtz equation at any positive frequency that have inward cusps.

CONTENTS

1. INTRODUCTION AND MAIN RESULTS

1.1. **Background.** This work is motivated by a problem in inverse scattering theory, but it raises questions of independent interest in the context of quadrature domains and free boundary problems. We recall that a bounded domain $D \subset \mathbb{R}^n$ is called a *quadrature domain* (for harmonic functions), corresponding to a measure μ with $\mathrm{supp}(\mu) \subset D$, if

$$(1.1) \qquad \int_D h(x) \, dx = \int h(x) \, d\mu(x)$$

for every harmonic function $h \in L^1(D)$. More generally, one can consider distributions $\mu \in \mathscr{E}'(D)$. In the most classical case one is interested in domains D for which μ is supported at finitely many points, so that (1.1) reduces to a quadrature identity for computing integrals of harmonic functions.

Quadrature domains can be viewed as a generalization of the mean value theorem (MVT) for harmonic functions. Indeed, we can rephrase the MVT for harmonic functions as follows:

$$B_r(a) \text{ is a quadrature domain with } \mu = \mathsf{m}(B_r(a))\delta_a,$$

---

where δ_a is the Dirac measure at a, m denotes the Lebesgue measure in $\mathbb{R}^n$ (i.e. $d\mathsf{m} = dx$) and $B_r(a)$ is the ball of radius r centered at a. In general, the boundary of a quadrature domain is a free boundary in an obstacle-type problem (see [PSU12]), and hence near any given point $z \in \partial D$ the domain D is either smooth or D^c has zero density at z. Various examples can be constructed via complex analysis, for example, the cardioid domain in Example 3.3 below. We refer to [Dav74], [Sak83], and [GS05] for further background.

The inverse scattering problems studied in [SS21] lead to a related concept, for solutions of the Helmholtz equation $(\Delta + k^2)u = 0$, where $k \geq 0$ is a frequency. This setting gives rise to various interesting questions. We are not aware of earlier work on quadrature domains for $k > 0$, and in this article we only give some first steps. In addition, we show that any quadrature domain is a *non-scattering domain* (cf. Definition 1.8) if it admits an incident wave that is positive on its boundary. In [SS21] it was observed that in the case $k = 0$ quadrature domains are non-scattering domains, and hence there are non-scattering domains having inward cusps. Corollary 1.9 below provides a similar result valid for all $k > 0$.

1.2. **Notation.** Here we gather recurring notation and definitions, with reference to relevant pages. We also mention here that all functions and measures will be real-valued unless stated otherwise.

m Lebesgue measure in $\mathbb{R}^n$, 2
$B_r(a)$ ball of radius r centered at a, 2
B_r ball of radius r centered at origin, 2
Definition of k-quadrature domain, 2
$\mathbb{D}$ unit disk, 4
J_α Bessel function of first kind, 3
$j_{\alpha,1}$ the first positive zero of the Bessel function J_α, 5
Y_α Bessel function of second kind, 6
$R(n, k) = c_n^{\mathrm{ref}} k^{-1}$ maximal length scale, 15
$\tilde{\Phi}_k = \tilde{\Phi}_{k,R(n,k)}$ a particular fundamental solution of the Helmholtz operator $-(\Delta + k^2)$, 15
$U_k^\mu = \tilde{\Phi}_k * \mu$ potential of a measure μ, 15
$\mathscr{F}_k(\mu)$ the class of admissible functions in an obstacle problem, 16
$c_{n,k,r}^{\mathrm{MVT}}$ constant related to mean value theorem, 37
$D(\mu)$ saturated set for $\nu = \mathrm{Bal}_k(\mu)$, 20
$\omega(\mu)$ a subset in $D(\mu)$, 22
$[D]$ saturation of D with respect to Lebesgue measure, 26

1.3. **Main results.** We begin with a definition generalizing (1.1).

Definition 1.1. Let $k > 0$. A bounded open set $D \subset \mathbb{R}^n$ (not necessarily connected) is called a *quadrature domain for* $(\Delta + k^2)$, or a *k-quadrature domain*, corresponding to a distribution $\mu \in \mathscr{E}'(D)$, if

$$\int_D w(x) \, dx = \langle \mu, w \rangle$$

for all $w \in L^1(D)$ satisfying $(\Delta + k^2)w = 0$ in D.

We remark that solutions of $(\Delta + k^2)w = 0$ are sometimes called *metaharmonic functions*, see e.g. [Kuz19, Section 4] or [Fri57] for a discussion. It is important that $\mathrm{supp}(\mu)$ has to be a

subset of D (see however [KM96, Lemma 2.8] for a discussion that weakens this assumption for harmonic functions). Indeed, that $\operatorname{supp}(\mu) \subset D$ implies that the distributional pairing $\langle \mu, w \rangle$ is well defined, because solutions of $(\Delta + k^2)w = 0$ are smooth in D. Furthermore, without this requirement the existence of a distribution satisfying the definition would be trivial, indeed one could choose $\mu = \chi_D$.

The first question is whether k-quadrature domains even exist for $k > 0$. This is indeed the case. In fact, balls are always k-quadrature domains. This is a consequence of a MVT for the Helmholtz equation which goes back to H. Weber [Web68, Web69] (see also [Kuz19], [Kuz21], or [CH89, p. 289]). The MVT takes the form

$$\int_{B_r(a)} w(x)\, dx = c_{n,k,r}^{\mathrm{MVT}} w(a)$$

whenever $w \in L^1(B_r(a))$ and $(\Delta + k^2)w = 0$ in $B_r(a)$. However, unlike for harmonic functions, the constant $c_{n,k,r}^{\mathrm{MVT}}$ has varying sign depending on k, r. In particular, the constant vanishes when $J_{n/2}(kr) = 0$ where J_α denotes the Bessel function of the first kind. More details are given in Appendix A. It follows that unions of disjoint balls are also k-quadrature domains corresponding to linear combinations of delta functions. Choosing two balls whose closures intersect at one point furnishes an example of a k-quadrature domain whose boundary is not smooth.

In order to make further progress we consider a PDE characterization of k-quadrature domains. One can show (see Proposition 2.1) that D is a k-quadrature domain corresponding to $\mu \in \mathscr{E}'(D)$ if and only if there is a distribution $u \in \mathscr{D}'(\mathbb{R}^n)$ satisfying

$$(1.2) \qquad \begin{cases} (\Delta + k^2)u = \chi_D - \mu & \text{in } \mathbb{R}^n, \\ u = |\nabla u| = 0 & \text{in } \mathbb{R}^n \setminus D. \end{cases}$$

Note that by elliptic regularity the distribution u solving $(\Delta + k^2)u = \chi_D$ near ∂D must be C^1 near ∂D, and thus the condition that u and ∇u vanish in $\mathbb{R}^n \setminus D$ (instead of $\mathbb{R}^n \setminus \overline{D}$) makes sense. The following result is a local version of the above fact, characterizing domains D that are k-quadrature domains for some distribution μ. However, there is no reason to expect that μ could be chosen to have support at finitely many points.

Theorem 1.2. *Let $k > 0$, and let D be a bounded open set in $\mathbb{R}^n$. Then D is a k-quadrature domain for some $\mu \in \mathscr{E}'(D)$ if and only if there is a neighborhood U of ∂D in $\mathbb{R}^n$ and a distribution $u \in \mathscr{D}'(U)$ satisfying*

$$(1.3) \qquad \begin{cases} (\Delta + k^2)u = \chi_D & \text{in } U, \\ u = |\nabla u| = 0 & \text{in } U \setminus D. \end{cases}$$

Moreover, if D is a k-quadrature domain for some $\mu \in \mathscr{E}'(D)$, then D is also a k-quadrature domain for some measure $\tilde{\mu}$ having smooth density with respect to Lebesgue measure.

Remark 1.3. If u is as in Theorem 1.2, then clearly

$$(1.4) \qquad \begin{cases} \Delta u = f\chi_D & \text{in } U, \\ u = |\nabla u| = 0 & \text{in } U \setminus D, \end{cases}$$

with $f = 1 - k^2 u$. Extending u from a neighborhood of ∂D into some distribution in $\mathbb{R}^n$ with $u = |\nabla u| = 0$ in $\mathbb{R}^n \setminus D$ shows that we have an analogue of (1.2) with $k = 0$ and with χ_D replaced by $f\chi_D$. Thus any k-quadrature domain is a weighted 0-quadrature domain. Since

the weight f is positive on ∂D, free boundary regularity results for weighted 0-quadrature domains apply also to k-quadrature domains. In particular, such a domain has locally either smooth boundary or its complement is thin in the sense of minimal diameter (see [PSU12, page 109]). We also remark that when $k = 0$ the equation (1.2) is related to harmonic continuation of potentials, see [Isa90] for further information.

Theorem 1.2 has an immediate consequence showing that domains with real-analytic boundary are k-quadrature domains.

Corollary 1.4. *If $k > 0$, then any bounded open set $D \subset \mathbb{R}^n$ with real-analytic boundary is a k-quadrature domain.*

Proof. Since ∂D is real-analytic, we can use the Cauchy–Kowalevski theorem to find a real-analytic function u near ∂D satisfying

$$\begin{cases} (\Delta + k^2)u = 1 \text{ near } \partial D, \\ \quad u|_{\partial D} = \partial_\nu u|_{\partial D} = 0, \end{cases}$$

where ∂_ν denotes the derivative in the normal direction to ∂D. We redefine u to be zero outside D. One can directly check that u and ∇u are Lipschitz continuous across ∂D. Hence u will be $C^{1,1}$ near ∂D and will satisfy the condition in Theorem 1.2. This proves that D is a k-quadrature domain. $\qquad\square$

The next result gives further examples of k-quadrature domains in two dimensions.

Theorem 1.5. *Let $k > 0$, and let $\mathbb{D}$ be the unit disc in $\mathbb{R}^2 \cong \mathbb{C}$. Suppose that $D = \varphi(\mathbb{D})$ where φ is a complex analytic function in a neighbourhood of $\overline{\mathbb{D}}$ such that $\varphi \colon \mathbb{D} \to D$ is bijective. Then D is a k-quadrature domain.*

Domains D as in Theorem 1.5 include cardioid type domains and domains with double points. Examples and further properties of these domains are given in the end of Section 3.

We also study k-quadrature domains from the potential theoretic point of view. More precisely, we construct some k-quadrature domains by using *partial balayage*, that is, given a non-negative compactly supported Radon measure μ, we construct a measure ν by distributing the mass of μ more uniformly. By investigating the structure of ν, we then construct a k-quadrature domain D with respect to μ. For the case when $k = 0$ this procedure is classical, see e.g. [GR18, Gus90, Gus04, Sak83]. In this paper, we give similar results for $k > 0$ and many of our results and proofs follow those in the case of $k = 0$ as presented in [Gus90, Gus04]. In this direction, our main goal is to prove the following theorem:

Theorem 1.6 (see also Theorem 7.1)**.** *Let μ be a positive measure supported in a ball of radius $\epsilon > 0$. There exists a constant $c_n > 0$ depending only on the dimension such that if*

$$(1.5) \qquad\qquad 0 < k < \frac{c_n}{\mu(\mathbb{R}^n)^{1/n}} \quad and \quad \epsilon < c_n \mu(\mathbb{R}^n)^{1/n},$$

then there exists an open connected set D with real-analytic boundary which is a k-quadrature domain for μ. Moreover, for each $w \in L^1(D) \cap L^1(\mu)$ satisfying $(\Delta + k^2)w \geq 0$ in D we have

$$(1.6) \qquad\qquad \int_D w(x)\, dx \geq \int w(x)\, d\mu(x).$$

Remark 1.7. The assumption $w \in L^1(\mu)$ is in order to ensure that the right-hand side of (1.6) is well defined.

Finally we consider the relation of k-quadrature domains to the inverse problem of determining the shape of a penetrable obstacle from a single measurement, as discussed in [SS21]. See [CCH16, CK19, Yaf10] for more details about scattering problems. Let $D \subset \mathbb{R}^n$ be a bounded open set, and let $h \in L^\infty(D)$ satisfy $|h| \geq c > 0$ a.e. near ∂D (such a function h is called a *contrast* for D). The pair (D, h) describes a penetrable obstacle D with contrast h.

We now probe the penetrable obstacle (D, h) by some incident field u_0 at frequency $k > 0$. The incident field is a solution of

$$(\Delta + k^2)u_0 = 0 \quad \text{in } \mathbb{R}^n.$$

Let u_{sc} be the corresponding scattered field. That is, the unique function u_{sc} so that the total field $u_{\text{tot}} = u_0 + u_{\text{sc}}$ satisfies

(1.7)
$$\begin{cases} (\Delta + k^2 + h\chi_D)u_{\text{tot}} = 0 & \text{in } \mathbb{R}^n, \\ u_{\text{sc}} \text{ satisfies the Sommerfeld radiation condition} & \text{at } |x| \to \infty. \end{cases}$$

Here we recall that a solution u of $(\Delta + k^2)u = 0$ in $\mathbb{R}^n \setminus \overline{B_R}$ (for some $R > 0$) satisfies the Sommerfeld radiation condition if

$$\lim_{|x| \to \infty} |x|^{\frac{n-1}{2}} (\partial_r u - iku) = 0, \quad \text{uniformly in all directions } \hat{x} = \frac{x}{|x|} \in \mathcal{S}^{n-1},$$

where ∂_r denotes the radial derivative. Solutions satisfying the Sommerfeld radiation condition are also called outgoing. The functions u_0, u_{sc} and u_{tot} are allowed to be complex.

The single measurement inverse problem is to determine some properties of the obstacle D from knowledge of the scattered wave $u_{\text{sc}}(x)$ when $|x|$ is large. If $D = \emptyset$, then $u_{\text{sc}} \equiv 0$, and a related question is to ask whether some nontrivial domain D admits some h and u_0 so that $u_{\text{sc}} = 0$ for large x. Such a penetrable obstacle (D, h) would be invisible when probed by the incident wave u_0 and would look like empty space. Domains D having this property for some h and u_0 will be called non-scattering domains.

Definition 1.8. We say that a bounded open set $D \subset \mathbb{R}^n$ is a *non-scattering domain* if there is some $h \in L^\infty(D)$ with $|h| \geq c > 0$ a.e. near ∂D and some solution u_0 of $(\Delta + k^2)u_0 = 0$ in $\mathbb{R}^n$ such that the corresponding scattered wave u_{sc} satisfies $u_{\text{sc}}|_{\mathbb{R}^n \setminus \overline{B_R}} = 0$ for some $R > 0$.

The following result states that k-quadrature domains are also non-scattering domains, at least if there is some incident wave u_0 that is positive on ∂D. By the results in [SS21] such an incident wave u_0 exists at least when

- D is a C^1 domain (Lipschitz if $n = 2, 3$) so that $\mathbb{R}^n \setminus \overline{D}$ is connected and k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D; or
- D is contained in a ball of radius $< k^{-1}j_{\frac{n-2}{2},1}$ where $j_{\frac{n-2}{2},1}$ is the first positive zero of the Bessel function $J_{\frac{n-2}{2}}$.

By combining Theorem 1.2 and [SS21, Remark 2.4] we deduce the following corollary.

Corollary 1.9. *Let $D \subset \mathbb{R}^n$ be a k-quadrature domain, and assume that there exists u_0 solving $(\Delta + k^2)u_0 = 0$ in $\mathbb{R}^n$ with $u_0|_{\partial D} > 0$. Then D is a non-scattering domain (for the incident wave u_0 and for some contrast h).*

From Theorem 1.5 and Corollary 1.9 we see that there exist non-scattering domains with inward cusps for any $k > 0$, extending the corresponding result for $k = 0$ in [SS21]. In contrast, domains having suitable corner points cannot be non-scattering domains for any $k > 0$, i.e.

"corners always scatter". This line of research was initiated in [BPS14] and various further results were obtained in [Blå18, BL17, BL21, CV21, CX21, EH15, PSV17].

1.4. **Organization.** We prove Theorems 1.2 and 1.5 in §2, respectively §3. In §4, we introduce an obstacle problem, and define the partial balayage in terms of the maximizer of such an obstacle problem. We then study the structure of partial balayage in §5 and §6. Using these properties, we prove Theorem 1.6 in §7. Finally, we provide some details about a real-valued fundamental solution relevant to our construction, some results related to maximum principles, the mean value theorem (MVT), and conformal images of $\mathbb{D}$ in Appendix A.

2. PDE CHARACTERIZATION OF QUADRATURE DOMAINS

In this section we will prove Theorem 1.2 from the introduction. We begin with a global PDE characterization of k-quadrature domains.

Proposition 2.1. *Let $k > 0$, and let $D \subset \mathbb{R}^n$ be a bounded open set. Then D is a k-quadrature domain corresponding to $\mu \in \mathscr{E}'(D)$ if and only if there is a distribution $u \in \mathscr{D}'(\mathbb{R}^n)$ satisfying*

$$(2.1) \qquad \begin{cases} (\Delta + k^2)u = \chi_D - \mu & in \ \mathbb{R}^n, \\ \quad u = |\nabla u| = 0 & in \ \mathbb{R}^n \setminus D. \end{cases}$$

Note that even though u is only assumed to be in $\mathscr{D}'(\mathbb{R}^n)$, the equation $(\Delta + k^2)u = \chi_D$ near ∂D and elliptic regularity imply that u is C^1 near ∂D and hence the condition that $u = |\nabla u| = 0$ in $\mathbb{R}^n \setminus D$ is meaningful.

Example 2.2. (When μ is a Dirac mass.) For the case when $D = B_R$ with $R > 0$, and the measure is a constant multiple of the Dirac mass, we can find an explicit solution $u = u_{k,R}$ of (2.1) in terms of Bessel functions. The general radially symmetric solution of $(\Delta + k^2)u = 1$ in $\mathbb{R}^n \setminus \{0\}$ is

$$u_k(x) = \frac{1}{k^2} + c_1 |x|^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(k|x|) + c_2 |x|^{1-\frac{n}{2}} Y_{\frac{n}{2}-1}(k|x|).$$

Given a radius $R > 0$ there is a unique choice of the constants c_1, c_2 so that $u_{k,R} := u_k \chi_{B_R} \in C^{1,1}(\mathbb{R}^n \setminus \{0\})$, namely

$$c_1 = \frac{\pi R^{\frac{n}{2}} Y_{\frac{n}{2}}(kR)}{2k} \quad \text{and} \quad c_2 = -\frac{\pi R^{\frac{n}{2}} J_{\frac{n}{2}}(kR)}{2k}.$$

With these choices of coefficients $u_{k,R}$ satisfies

$$(\Delta + k^2)u_{k,R} = \chi_{B_R} - k^{-\frac{n}{2}}(2\pi R)^{\frac{n}{2}} J_{\frac{n}{2}}(kR)\,\delta \quad \text{in } \mathbb{R}^n,$$

$$u_{k,R}|_{\mathbb{R}^n \setminus \overline{B_R}} = 0,$$

which gives an example of Proposition 2.1 with $\mu = k^{-\frac{n}{2}}(2\pi R)^{\frac{n}{2}} J_{\frac{n}{2}}(kR) \delta$.

Example 2.3. (When $\mu \equiv 0$.) Let D be a bounded simply connected Lipschitz domain in $\mathbb{R}^n$. The well-known Pompeiu problem [Pom29, Wil76, Zal92] asks whether the existence of a nonzero continuous function on $\mathbb{R}^n$ whose integral vanishes on all congruent copies of D implies that D is a ball.

The problem can be reformulated in terms of free boundary problems, or in the context of this paper, in terms of null k-quadrature domains (i.e., $\mu \equiv 0$). Indeed the assumption in Pompeiu problem is equivalent to the existence of a function v solving the free boundary problem

$$(2.2) \qquad \Delta v + \lambda v = \chi_D \text{ in } \mathbb{R}^n, \qquad v = 0 \text{ outside } D,$$

for some $\lambda \in \mathbb{C} \setminus \{0\}$; see [Wil76]. If we further assume $\partial D = \partial^* D$, where $\partial^* D$ is the outer boundary of D, then λ in (2.2) is necessarily positive, see [Wil81, Theorem 1]. If the bounded open set D satisfies the assumption in the Pompeiu problem and its boundary ∂D is Lipschitz and homeomorphic to a sphere (e.g. D is a convex set), then ∂D is analytic [Wil81]. See also [BK82, BST73, Den12] for some related results. The so-far unanswered question is: whether D has to be a ball?

The fact that balls (with appropriate radii depending on $k > 0$) solve this problem is evident from the following simple procedure: take the function $u(x) = |x|^{\frac{2-n}{2}} J_{\frac{n-2}{2}}(k|x|)$ that solves $\Delta u + k^2 u = 0$ in $\mathbb{R}^n$, add a constant to u so that one of the local minima (say $|x| = R$) of u reaches the level zero, and then redefine the function to be zero outside B_R. After multiplying by a suitable constant, this function obviously solves the free boundary formulation of the Pompeiu problem.

An interesting observation is that the solution to the free boundary formulation of the Pompeiu problem thus constructed may change sign. If on the other hand we choose R to be the smallest radius for which u takes a minimum, then the construction leads to a non-negative solution.

The above discussion also gives an indication of the failure of the application of the classical moving-plane technique for this problem.

We will require the following Runge approximation type result, see e.g. [AH96, Chapter 11] for related results. We will follow the argument in [Sak84, Lemma 5.1].

Proposition 2.4. *Let $k \geq 0$, and let $D \subset \mathbb{R}^n$ be a bounded open set. Let Ψ_k be any fundamental solution of $-(\Delta + k^2)$ and let $\Omega \supset \overline{D}$ be any open set in $\mathbb{R}^n$. Then the linear span of*

$$F = \{\partial^\alpha \Psi_k(z - \cdot)|_D \ : \ z \in \Omega \setminus D, \ |\alpha| \leq 1\}$$

is dense in

$$H_k L^1(D) = \{w \in L^1(D) \ : \ (\Delta + k^2)w = 0 \text{ in } D\}$$

with respect to the $L^1(D)$ topology.

Remark 2.5. If the domain D has sufficiently regular boundary it suffices to take $\alpha = 0$ in F. However, for domains like the slit disk one needs to consider also $\partial^\alpha \Psi_k(z - \cdot)|_D$ for all $|\alpha| = 1$ and $z \in \Omega \setminus D$ below (note that these functions are all in $L^1(D)$). We shall later require a version of this result for sub-solutions (see Proposition 7.6).

Proof of Proposition 2.4. By the Hahn–Banach theorem, it is enough to show that any bounded linear functional ℓ on $L^1(D)$ that satisfies $\ell|_F = 0$ also satisfies $\ell|_{H_k L^1(D)} = 0$. Since the dual of $L^1(D)$ is $L^\infty(D)$, there is a function $f \in L^\infty(D)$ with

$$\ell(w) = \int_D fw \, dx, \qquad w \in L^1(D).$$

We extend f by zero to $\mathbb{R}^n$ and consider the function

$$u(z) = -(\Psi_k * f)(z) \quad \text{for all } z \in \Omega.$$

By the assumption $\ell|_F = 0$, the function u satisfies

$$\begin{cases} (\Delta + k^2)u = f & \text{in } \Omega, \\ u = |\nabla u| = 0 & \text{in } \Omega \setminus D. \end{cases}$$

We now consider the zero extension of u, still denoted by u, which satisfies

$$\begin{cases} (\Delta + k^2)u = f & \text{in } \mathbb{R}^n, \\ u = |\nabla u| = 0 & \text{in } \mathbb{R}^n \setminus D. \end{cases}$$

Note that since $f \in L^\infty$, we have $u \in C^{1,\alpha}$ for any $\alpha < 1$. In order to show that $\ell|_{H_k L^1(D)} = 0$, we take some $w \in H_k L^1(D)$ and compute

$$\ell(w) = \int_D fw \, dx = \int_D ((\Delta + k^2)u)w \, dx.$$

We claim that one can integrate by parts and use the condition $(\Delta + k^2)w = 0$ to conclude that

$$(2.3) \qquad \int_D ((\Delta + k^2)u)w \, dx = 0.$$

This implies that $\ell|_{H_k L^1(D)} = 0$ and proves the result. However, the proof of (2.3) is somewhat delicate due to the failure of Calderón–Zygmund estimates when $p = \infty$. In the case $k = 0$, (2.3) follows from [Sak84, Lemma 5.1]. We will verify that the same argument works for $k > 0$.

First observe that u solves

$$\Delta u = f - k^2 u \text{ in } \mathbb{R}^n.$$

Since f and u are L^∞, it follows from [GT01, Theorem 3.9] that ∇u satisfies

$$|\nabla u(x) - \nabla u(y)| \le C|x - y| \log(1/|x - y|), \qquad x, y \in \overline{D}, \ |x - y| < e^{-2}.$$

Using the condition $u = |\nabla u| = 0$ in $\mathbb{R}^n \setminus D$, this implies that uniformly for $x \in D$ near ∂D one has

$$u(x) = O(\delta(x)^2 \log(1/\delta(x))),$$
$$\nabla u(x) = O(\delta(x) \log(1/\delta(x))),$$

where $\delta(x) = \text{dist}(x, \partial D)$.

As in [Sak84, Lemma 5.1] we introduce the sequence $(\omega_j)_{j=1}^\infty$ of Ahlfors-Bers mollifiers [Ahl64, Ber65] that satisfy $\omega_j \in C^\infty(\mathbb{R}^n)$, $0 \le \omega_j \le 1$, $\omega_j = 0$ near ∂D, $\omega_j = 1$ outside a neighborhood of ∂D, $\omega_j(x) \to 1$ for $x \notin \partial D$, and

$$|\partial^\alpha \omega_j(x)| \le C_\alpha j^{-1} \delta(x)^{-|\alpha|} (\log 1/\delta(x))^{-1} \text{ for } x \notin \partial D,$$

see [Hed73, Lemma 4]. We now begin the proof of (2.3). One has

$$\int_D ((\Delta + k^2)u)w \, dx = \lim_{j \to \infty} \int_D ((\Delta + k^2)u)\omega_j w \, dx$$
$$= \lim_{j \to \infty} \int_D \left[(\Delta + k^2)(\omega_j u) - 2\nabla\omega_j \cdot \nabla u - (\Delta\omega_j)u \right] w \, dx.$$

Using the estimates for u and ω_j, the limits corresponding to the last two terms inside the brackets are zero. Moreover, since w is smooth near $\mathrm{supp}(\omega_j)$, we have

$$\int_D (\Delta + k^2)(\omega_j u)w \, dx = \int_D \omega_j u(\Delta + k^2)w \, dx = 0.$$

This concludes the proof of (2.3). $\qquad\qquad\square$

Proof of Proposition 2.1. Let Ψ_k be any fundamental solution of $-(\Delta + k^2)$, i.e. $\Psi_k \in \mathscr{D}'(\mathbb{R}^n)$ solves $-(\Delta + k^2)\Psi_k = \delta_0$ in $\mathbb{R}^n$. In particular, Ψ_k is smooth away from the origin. If D is a k-quadrature domain corresponding to μ, then

$$(2.4) \qquad \int_D \partial^\alpha \Psi_k(z - x) \, dx = \langle \mu, \partial^\alpha \Psi_k(z - \cdot) \rangle$$

whenever $z \in \mathbb{R}^n \setminus D$ and $|\alpha| \leq 1$. Let $u = -\Psi_k * (\chi_D - \mu)$, which is well defined since $\chi_D - \mu$ is a compactly supported distribution. We see that $(\Delta + k^2)u = \chi_D - \mu$ and $u = |\nabla u| = 0$ in $\mathbb{R}^n \setminus D$ as required.

Conversely, suppose that $u \in \mathscr{D}'(\mathbb{R}^n)$ is as in the statement. We easily obtain the quadrature identity for functions w that solve $(\Delta + k^2)w = 0$ near $\overline{D}$, since by taking a cutoff $\psi \in C_c^\infty(\mathbb{R}^n)$ with $\psi = 1$ near $\overline{D}$ we have

$$\int_D w \, dx - \langle w, \mu \rangle = \langle \chi_D - \mu, \psi w \rangle = \langle -(\Delta + k^2)u, \psi w \rangle = \langle u, -(\Delta + k^2)(\psi w) \rangle = 0,$$

using that the derivatives of ψ vanish near $\mathrm{supp}(u)$.

For solutions $w \in L^1(D)$ we need another argument. Since u is compactly supported, by the properties of convolution for distributions we have

$$u = \delta_0 * u = -(\Delta + k^2)\Psi_k * u = -\Psi_k * (\Delta + k^2)u = -\Psi_k * (\chi_D - \mu).$$

Using that $u = |\nabla u| = 0$ in $\mathbb{R}^n \setminus D$, we have

$$(2.5) \qquad \int_D \partial^\alpha \Psi_k(z - x) \, dx = \langle \mu, \partial^\alpha \Psi_k(z - \cdot) \rangle$$

for all $z \in \mathbb{R}^n \setminus D$ and $|\alpha| \leq 1$. Now let $w \in L^1(D)$ solve $(\Delta + k^2)w = 0$ in D, and use Proposition 2.4 to find a sequence $w_j \in \mathrm{span}\{\partial^\alpha \Psi_k(z - \cdot)|_D : z \in \mathbb{R}^n \setminus D, |\alpha| \leq 1\}$ with $w_j \to w \in L^1(D)$. In particular, for any $j \geq 1$ we have

$$(2.6) \qquad \int_D w_j \, dx = \langle \mu, w_j \rangle.$$

Since $\mu \in \mathscr{E}'(D)$, there is a compact set $K \subset D$ and an integer $m \geq 0$ such that

$$|\langle \mu, \varphi \rangle| \leq C\|\varphi\|_{C^m(K)}, \qquad \varphi \in C^\infty(D).$$

By elliptic regularity and Sobolev embedding, any $v \in L^1(D)$ with $(\Delta + k^2)v \in H^{s-2}(D)$ satisfies $v \in C^m(K)$ when $s > m + n/2$. By the closed graph theorem this yields the estimate

$$\|v\|_{C^m(K)} \leq C(\|v\|_{L^1(D)} + \|(\Delta + k^2)v\|_{H^{s-2}(D)}).$$

Applying this estimate to $v = w_j - w$ gives

(2.7) $$\|w_j - w\|_{C^m(K)} \leq C\|w_j - w\|_{L^1(D)}.$$

Thus we may take limits as $j \to \infty$ in (2.6) and obtain that

$$\int_D w\,dx = \langle \mu, w \rangle.$$

This shows that D is a k-quadrature domain for μ. □

Proof of Theorem 1.2. If D is a k-quadrature domain corresponding to μ, then taking a neighborhood U of ∂D that is disjoint from $\mathrm{supp}(\mu)$ and restricting the distribution from Proposition 2.1 to U gives the required $u \in \mathscr{D}'(U)$ satisfying (1.3).

Conversely, assume that $u \in \mathscr{D}'(U)$ satisfies (1.3). Let $\psi \in C_c^\infty(U)$ satisfy $0 \leq \psi \leq 1$ and $\psi = 1$ near ∂D, and define $\tilde{u} = \psi u \in \mathscr{D}'(\mathbb{R}^n)$. Also define

(2.8) $$\tilde{\mu} := \chi_D - (\Delta + k^2)\tilde{u}.$$

Then $\tilde{u}$ satisfies

$$\begin{cases} (\Delta + k^2)\tilde{u} = \chi_D - \tilde{\mu} & \text{in } \mathbb{R}^n, \\ \tilde{u} = \nabla\tilde{u} = 0 & \text{in } \mathbb{R}^n \setminus D. \end{cases}$$

Moreover, $\tilde{\mu} \in \mathscr{D}'(\mathbb{R}^n)$ satisfies $\mathrm{supp}(\tilde{\mu}) \subset D$ by the assumption on u. Then D is a k-quadrature domain by Proposition 2.1. By elliptic regularity u is smooth in $U \cap D$, and thus $\tilde{\mu}$ coincides with a smooth function in D. Since one also has $\mathrm{supp}(\tilde{\mu}) \subset D$, it follows that $\tilde{\mu}$ has a smooth density with respect to Lebesgue measure. □

3. QUADRATURE DOMAINS WITH CUSPS

This section contains the proof of Theorem 1.5. The proof will employ the following simple fact regarding the vanishing order of solutions. In this section all functions are allowed to be complex valued.

Lemma 3.1. *Let v be a C^∞ function near some $x_0 \in \mathbb{R}^n$ satisfying*

$$\begin{cases} \Delta v = O(|x - x_0|^m) \text{ near } x_0, \\ v|_S = \partial_\nu v|_S = 0, \end{cases}$$

where $m \geq 0$ is an integer, S is a smooth hypersurface through x_0, and ∂_ν denotes the derivative in the normal direction to S. Then one has $v = O(|x - x_0|^{m+2})$, and more precisely

$$v = \sum_{|\alpha| = m+2} v_\alpha(x)(x - x_0)^\alpha$$

where v_α are smooth near x_0.

Proof. After a rigid motion, we may assume that $x_0 = 0$ and the normal of S satisfies $\nu(0) = e_n$. We use the Taylor formula and write v as

$$v = \sum_{j=0}^{m+1} P_j + R, \qquad R = \sum_{|\alpha| = m+2} v_\alpha(x)(x - x_0)^\alpha,$$

where each P_j is a homogeneous polynomial of degree j and each v_α is smooth. Using the assumption $v|_S = \partial_\nu v|_S = 0$ we have $P_0 = P_1 = 0$. Moreover, the assumption for Δv implies that

$$\sum_{j=2}^{m+1} \Delta P_j = O(|x|^m).$$

Since the left hand side is a polynomial of degree $m-1$, it follows that we must have $\Delta P_j = 0$ for $2 \le j \le m+1$.

Suppose that γ is a smooth curve on S with $\gamma(0) = 0$ and $\dot\gamma(0) = \omega$ where $\omega \perp e_n$ and $|\omega| = 1$. Since $v|_S = \partial_\nu v|_S = 0$, we have

$$(3.1) \qquad 0 = v(\gamma(t)) = \sum_{j=2}^{m+1} |\gamma(t)|^j P_j(\gamma(t)/|\gamma(t)|) + O(|\gamma(t)|^{m+2}),$$

$$(3.2) \qquad 0 = \partial_\nu v(\gamma(t)) = \sum_{j=2}^{m+1} |\gamma(t)|^{j-1} \nu(\gamma(t)) \cdot \nabla P_j(\gamma(t)/|\gamma(t)|) + O(|\gamma(t)|^{m+1}).$$

Since $\gamma(t) = t\omega + O(t^2)$, we have $\gamma(t)/|\gamma(t)| \to \omega$ as $t \to 0$. If one would have $P_2(\omega) \ne 0$, then multiplying (3.1) by t^{-2} would lead to a contradiction as $t \to 0$. Similarly $\partial_n P_2(\omega) \ne 0$ would lead to a contradiction with (3.2). Thus $P_2(\omega) = \partial_n P_2(\omega) = 0$. Varying ω implies that

$$P_2|_{x_n=0} = \partial_n P_2|_{x_n=0} = 0.$$

But since $\Delta P_2 = 0$, unique continuation implies that $P_2 \equiv 0$. Iterating this argument shows that $P_2 \equiv \ldots \equiv P_{m+1} = 0$ as required.[1] $\qquad\square$

We are now ready to prove Theorem 1.5. As we shall illustrate in Examples 3.3–3.5 the domain D may have inward cusps and the map φ is not necessarily injective on $\partial\mathbb{D}$, which introduces some technicalities in the argument.

Proof of Theorem 1.5. Let $D = \varphi(\mathbb{D})$ where φ is analytic near $\overline{\mathbb{D}}$ and injective in $\mathbb{D}$. Note that D is an open set by the open mapping theorem for analytic functions [Rud87, Theorem 10.32]. We claim:

$$(3.3) \qquad\qquad \text{if } z_j \in \mathbb{D} \text{ and } d(\varphi(z_j), \partial D) \to 0, \text{ then } d(z_j, \partial\mathbb{D}) \to 0.$$

To see (3.3), we argue by contradiction and assume that $d(\varphi(z_j), \partial D) \to 0$ but there is $\epsilon > 0$ and a subsequence (z_{j_k}) with $d(z_{j_k}, \partial\mathbb{D}) \ge \epsilon$. After passing to another subsequence also denoted by (z_{j_k}), we have $z_{j_k} \to z \in \mathbb{D}$ with $d(z, \partial\mathbb{D}) \ge \epsilon$. However, since $d(\varphi(z_j), \partial D) \to 0$ we must have $d(\varphi(z), \partial D) = 0$. This contradicts the fact that $\varphi(\mathbb{D}) = D$, proving (3.3).

Next we show that

$$(3.4) \qquad\qquad\qquad \varphi(\partial\mathbb{D}) = \partial D.$$

We begin the proof of (3.4) by taking $z \in \partial\mathbb{D}$ and showing that $\varphi(z) \in \partial D$. By continuity $\varphi(z) \in \overline{D}$. If one had $\varphi(z) \in D$, then since φ is bijective $\mathbb{D} \to D$ there would be some $z' \in \mathbb{D}$ with $\varphi(z') = \varphi(z)$. For any $\epsilon < |z - z'|/2$ we consider the open sets $\varphi(B_\epsilon(z'))$ and $\varphi(B_\epsilon(z) \cap \mathbb{D})$. The point $\varphi(z)$ is contained in the interior of the first set and in the closure of the second, in

---

[1]Lemma 3.1 can also be proved by a simple blow-up argument. Starting with a quadratic blow-up one obtains P_2 in the limit and S becomes a hyperplane $\{x_n = 0\}$, along with the zero Cauchy-data for P_2. This implies $P_2 \equiv 0$. Repeating this argument, by a cubic scaling we obtain $P_3 \equiv 0$. Iterating this argument we have $P_j \equiv 0$ for all $j \le m+1$.

particular the two sets are not disjoint. This contradicts the assumption that φ is injective, and thus proves that $\varphi(\partial\mathbb{D}) \subset \partial D$. For the converse inclusion, if $x \in \partial D$ and $x_j \to x$ where $x_j \in D$, then $x_j = \varphi(z_j)$ for some $z_j \in \mathbb{D}$. After passing to a subsequence we may assume that $z_{j_k} \to z \in \overline{\mathbb{D}}$, and by (3.3) one must have $z \in \partial\mathbb{D}$. Thus $x = \lim \varphi(z_j) = \varphi(z)$, proving (3.4).

By Theorem 1.2, the result will follow if we can find a distribution u near ∂D solving

$$(3.5) \qquad \begin{cases} (\Delta + k^2)u = \chi_D & \text{near } \partial D, \\ u = |\nabla u| = 0 & \text{outside } D. \end{cases}$$

By the chain rule, the equation $(\Delta + k^2)u = 1$ in some set $\varphi(U_1)$, with $U_1 \subset \mathbb{D}$ open, is equivalent to the equation

$$(\Delta + k^2|\varphi'|^2)(u \circ \varphi) = |\varphi'|^2 \text{ in } U_1.$$

Since $|\varphi'|^2 = \partial\varphi\overline{\partial\varphi}$ is real-analytic near $\partial\mathbb{D}$, the Cauchy–Kowalevski theorem implies that there exists a neighborhood U of $\partial\mathbb{D}$ and a function $\hat{u}$ which is real-analytic in U such that

$$(3.6) \qquad \begin{cases} (\Delta + k^2|\varphi'|^2)\hat{u} = |\varphi'|^2 & \text{in } U, \\ \hat{u} = \partial_\nu\hat{u} = 0 & \text{on } \partial\mathbb{D}. \end{cases}$$

By (3.4) and the open mapping theorem we know that $V = \varphi(U)$ is an open neighborhood of ∂D. We define

$$u(x) := \begin{cases} \hat{u}(\varphi^{-1}(x)) & \text{for all } x \in V \cap D, \\ 0 & \text{for all } x \in V \setminus D. \end{cases}$$

The function u is defined piecewise and it satisfies (3.5) away from ∂D. If we can prove that $u \in C^{1,1}(V)$, then u will satisfy (3.5) also near ∂D and the proof of the theorem will be concluded. Note that by the inverse function theorem, u is smooth in $V \cap D$. We would like to show that u is continuous up to ∂D. If $x \in \partial D$ and $x_j \in V \cap D$ satisfy $x_j \to x$, then $x_j = \varphi(z_j)$ for some $z_j \in \mathbb{D}$. Then $d(\varphi(z_j), \partial D) \to 0$, and (3.3) ensures that $d(z_j, \partial\mathbb{D}) \to 0$. It follows that

$$u(x_j) = \hat{u}(z_j) \to 0$$

since $\hat{u}$ is Lipschitz near $\partial\mathbb{D}$ and $\hat{u}|_{\partial\mathbb{D}} = 0$. This shows that $u \in C^0(V)$.

Next we show that u is C^1 up to ∂D. Let $x \in \partial D$ and $x_j \in D$ with $x_j \to x$. It is enough to show that for any $\epsilon > 0$ there is j_0 such that $|\nabla u(x_j)| \le \epsilon$ for $j \ge j_0$. Now $x_j = \varphi(z_j)$ where $z_j \in \mathbb{D}$, and by the chain rule one has

$$\partial u(\varphi(z)) = \frac{\partial\hat{u}(z)}{\varphi'(z)}, \qquad \overline{\partial}u(\varphi(z)) = \frac{\overline{\partial}\hat{u}(z)}{\overline{\varphi'(z)}}.$$

Thus

$$(3.7) \qquad |\nabla u(\varphi(z))| = \frac{|\nabla\hat{u}(z)|}{|\varphi'(z)|}.$$

Using (3.3) we know that $d(z_j, \partial\mathbb{D}) \to 0$, and thus $\nabla\hat{u}(z_j) \to 0$ since $\nabla\hat{u}|_{\partial\mathbb{D}} = 0$. However, $\varphi'(z_j)$ may also converge to zero and this requires some care. We start by observing that there are only finitely many points $z_0 \in \partial\mathbb{D}$ with $\varphi'(z_0) = 0$, and near any such z_0 one can write

$$\varphi(z) = \varphi(z_0) + (z - z_0)^2 g(z),$$

for some analytic function g. Since $\varphi \colon \mathbb{D} \to D$ is bijective it follows that $\varphi''(z_0) \neq 0$ and hence $g(z_0) \neq 0$ (see Remark 3.2). Thus $|\varphi'(z)|^2 = O(|z - z_0|^2)$. Using Lemma 3.1 we know that

$$(3.8) \qquad \partial^\alpha \hat{u}(z) = O(|z - z_0|^{4-|\alpha|}) \text{ for } |\alpha| \leq 4 \text{ and for all } z \text{ near } z_0.$$

By (3.7), for $z \in \mathbb{D}$ near z_0 we have

$$|\nabla u(\varphi(z))| \leq C \frac{|z - z_0|^3}{|z - z_0|} \leq C|z - z_0|^2.$$

Thus there is $\delta > 0$ such that

$$|\nabla u(\varphi(z))| \leq \epsilon \text{ when } z \in W := \bigcup_{z_0 \in \partial\mathbb{D}, \varphi'(z_0)=0} (B(z_0, \delta) \cap \mathbb{D}).$$

We have $\partial\mathbb{D} \subset W \cup W'$ where W' is some open set with $|\varphi'(z)| \geq c > 0$ for $z \in W'$. We already know that $|\nabla u(\varphi(z_j))| \leq \epsilon$ when $z_j \in W$, and for $z_j \in W'$ the expression (3.7) gives that

$$|\nabla u(\varphi(z_j))| \leq \frac{1}{c}|\nabla \hat{u}(z_j)|$$

which becomes $\leq \epsilon$ when $j \geq j_0$ for some sufficiently large j_0 by (3.3). This concludes the proof that $u \in C^1(V)$.

Finally, we use the chain rule again and observe that for $z \in \mathbb{D}$ one has

$$|\nabla^2 u(\varphi(z))| \leq C\left(\frac{|\nabla^2 \hat{u}(z)|}{|\varphi'(z)|^2} + \frac{|\nabla \hat{u}(z)||\varphi''(z)|}{|\varphi'(z)|^3} \right).$$

As before, the worst case is when z is close to some $z_0 \in \partial\mathbb{D}$ with $\varphi'(z_0) = 0$. By (3.8), for z near z_0 one has

$$|\nabla^2 u(\varphi(z))| \leq C\left(\frac{|\nabla^2 \hat{u}(z)|}{|z - z_0|^2} + \frac{|\nabla \hat{u}(z)|}{|z - z_0|^3} \right) \leq C.$$

It follows that ∇u is Lipschitz continuous in V. In fact it is Lipschitz in $V \cap D$ and $V \setminus D$, and if $x \in V \cap D$ and $y \in V \setminus D$ we let y_1 be a closest point to x in $\mathbb{R}^n \setminus D$ (so that $y_1 \in \partial D$) and observe that

$$|\nabla u(x) - \nabla u(y)| = |\nabla u(x) - \nabla u(y_1)| \leq C|x - y_1| \leq C|x - y|.$$

This proves that $u \in C^{1,1}(V)$, and therefore concludes the proof of Theorem 1.5. $\qquad \square$

Remark 3.2. Let $D = \varphi(\mathbb{D})$ where φ is an analytic function near $\overline{\mathbb{D}}$ which is injective in $\mathbb{D}$. In this remark we clarify what ∂D looks like. Recall from (3.4) that $\varphi(\partial\mathbb{D}) = \partial D$. We may divide the boundary points in three categories.

    (i) (Smooth points) If $x_0 \in \partial D$ is of the form $x_0 = \varphi(z_0)$ for a unique $z_0 \in \partial\mathbb{D}$ and $\varphi'(z_0) \neq 0$, then by the inverse function theorem D near x_0 is given by the region above the graph of a real-analytic function.

    (ii) (Inward cusp points) If $x_0 \in \partial D$ is of the form $x_0 = \varphi(z_0)$ for some $z_0 \in \partial\mathbb{D}$ with $\varphi'(z_0) = 0$, then $\varphi''(z_0) \neq 0$ since if φ vanished to higher order the bijectivity in $\mathbb{D}$ would fail in the same way that it does for $z \mapsto z^m$, $m > 2$, around $z = 0$ (the image of an arbitrary half-plane covers $\mathbb{C} \setminus \{0\}$ more than once). Thus φ behaves near z_0 like $z \mapsto z^2$ which produces an inward cusp.

(iii) (Double points) If $x_0 \in \partial D$ satisfies $x_0 = \varphi(z_1) = \varphi(z_2)$ for two distinct $z_1, z_2 \in \partial \mathbb{D}$, then by the bijectivity $\varphi'(z_1) \neq 0$ and $\varphi'(z_2) \neq 0$ and there exists an $r > 0$ small enough so that $\partial D \cap B_r(x_0)$ is the union of two analytic arcs whose intersection is $\{x_0\}$ where the arcs touch (by injectivity they do not cross).

Moreover, there are only finitely many points which fail to be in category (i).

This classification of the points on the boundary of D is rather classical. Remark 3.2 is also related to Sakai's regularity theorem, see [Sak91, Theorem 5.2] as well as [LM16, Section 3.2].

Example 3.3 (Figure 3.1). Let $\varphi(z) = z + \frac{1}{2}z^2$ and $D = \varphi(\mathbb{D})$. Then D is a cardioid whose boundary is smooth except at the point $\varphi(-1) = -1/2$ where it has an inward cusp. It is clear that φ satsifies the conditions of Theorem 1.5. Similarly, if $\varphi(z) = z + \frac{1}{m}z^m$ for integer $m \geq 2$ then D has $m - 1$ inward cusps.

Example 3.4 (Figure 3.2). Let $\varphi(z) = z - \frac{2\sqrt{2}}{3}z^2 + \frac{1}{3}z^3$ and $D = \varphi(\mathbb{D})$ (see e.g. [LM16, equation (1.9)]). Then the corresponding domain D is not a Jordan domain and furthermore its boundary has inward cusps. By Theorem 1.5, the domain D is a k-quadrature domain.

Example 3.5 (Figure 3.3). Let $\varphi(z) = (z-1)^2 - (1-\frac{i}{2})(z-1)^3$ and $D = \varphi(\mathbb{D})$. The domain D looks similar to a cardioid, but with a curved cusp. It is also a k-quadrature domain by Theorem 1.5.
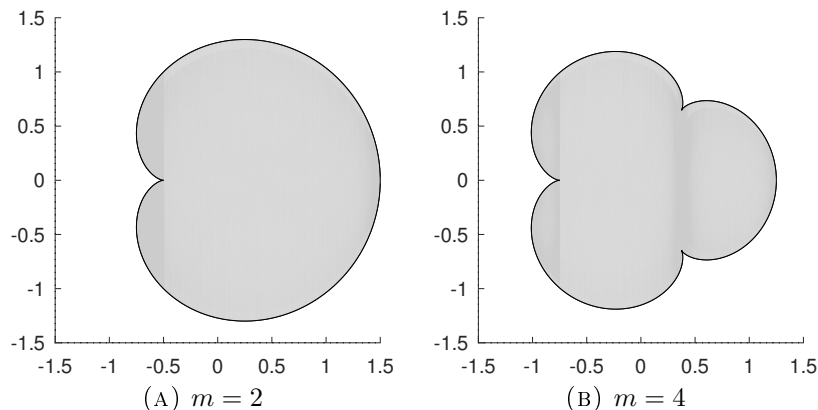

(A) $m = 2$ (B) $m = 4$

FIGURE 3.1.   Plot of Example 3.3 (GNU Octave)

(A) Global view

(B) Near points of interest
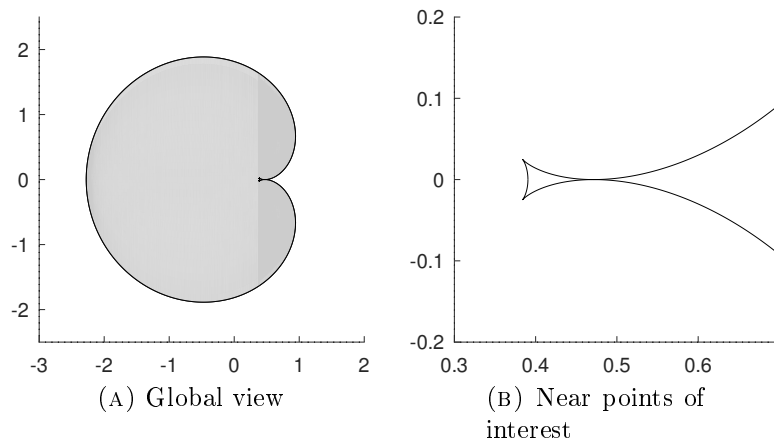
FIGURE 3.2. Plot of Example 3.4 (GNU Octave)

(A) Global view

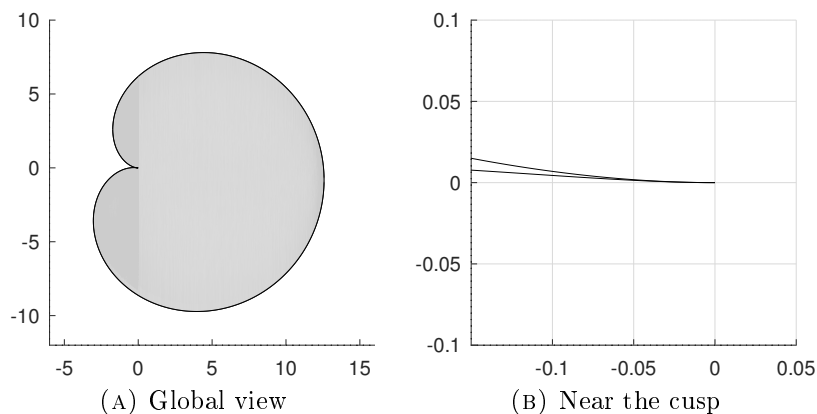(B) Near the cusp

FIGURE 3.3. Plot of Example 3.5 (GNU Octave)

4. PARTIAL BALAYAGE VIA AN OBSTACLE PROBLEM

In this section we define the partial balayage measure $\mathrm{Bal}_k(\mu)$ with respect to $\Delta + k^2$ for certain sufficiently concentrated measures μ when $k > 0$ is small. For simplicity we will assume that μ is concentrated near the origin, but by translation invariance any other point would do. There are several ways of defining partial balayage, and we will proceed via an obstacle problem (see e.g. [Gus04, Definition 3.2] for $k = 0$).

Let $0 < R(n,k) < j_{\frac{n-2}{2},1}k^{-1}$ and let $\tilde{\Phi}_k = \tilde{\Phi}_{k,R(n,k)}$ be the fundamental solution given in Proposition A.1. Here and in what follows we write $U_k^\mu := \tilde{\Phi}_k * \mu$ for any Radon measure μ. In the special case where $\mu = \chi_\Omega \mathsf{m}$ for some open set Ω we simply write $U_k^\Omega := \tilde{\Phi}_k * \chi_\Omega$.

We now restrict ourselves to measures μ having a bounded density with respect to Lebesgue measure. Slightly abusing notation we write $\mu \in L^\infty(B_{R(n,k)})$ to mean that μ has the form

$$\mu = f\mathsf{m},$$

where $f \in L^\infty(\mathbb{R}^n)$ satisfies $f = 0$ outside $B_{R(n,k)}$. Under this assumption, by elliptic regularity (see e.g. [GT01, Theorem 9.11])

$$U_k^\mu \in \bigcap_{1 < p < \infty} W_{\mathrm{loc}}^{2,p}(\mathbb{R}^n) \subset \bigcap_{0 < \alpha < 1} C^{1,\alpha}(\overline{B_{R(n,k)}}).$$

For $k > 0$ and $\mu \in L^\infty(B_{R(n,k)})$ define

(4.1) $$\mathscr{F}_k(\mu) = \left\{ \left. v \in H^1(B_{R(n,k)}) \; \right| \; \begin{array}{l} (\Delta + k^2)v \geq -1 \text{ in } B_{R(n,k)} \\ v \leq U_k^\mu \text{ in } B_{R(n,k)} \\ v = U_k^\mu \text{ on } \partial B_{R(n,k)} \end{array} \right\}.$$

Lemma 4.1. Let $0 < \gamma < R(n,k) < j_{\frac{n-2}{2},1} k^{-1}$ and $\mu \in L^\infty(B_\gamma)$. Assume that there exists $r > 0$ such that

(4.2) $$r < R(n,k) - \gamma \quad \text{and} \quad c_{n,k,r}^{\mathrm{MVT}} \geq \mu_+(\mathbb{R}^n),$$

where $c_{n,k,r}^{\mathrm{MVT}}$ is the constant appearing in the mean value theorem for the Helmholtz equation (see (A.11)). Then $\mathscr{F}_k(\mu)$ contains an element $\tilde{u}_k$ which equals U_k^μ in $B_{R(n,k)} \setminus B_{\gamma+r}$ (note that $\gamma + r < R(n,k)$).

Proof. Let

(4.3) $$\tilde{u}_k := U_k^{\mu_+} * h_r - U_k^{\mu_-} \quad \text{where} \quad h_r := \frac{1}{c_{n,k,r}^{\mathrm{MVT}}} \chi_{B_r}.$$

Using the mean value theorem in Proposition A.5, we have

$$U_k^{\mu_+ * h_r}(x) \leq U_k^{\mu_+}(x) \text{ for all } x \in \mathbb{R}^n \text{ with equality if } \mu_+(B_r(x)) = 0,$$

which implies

$$\tilde{u}_k \leq U_k^\mu \text{ in } \mathbb{R}^n \quad \text{and} \quad \tilde{u}_k = U_k^\mu \text{ in } \mathbb{R}^n \setminus B_{\gamma+r}.$$

Finally we note that

$$(\Delta + k^2)\tilde{u}_k(x) = (-\mu_+ * h_r + \mu_-)(x) \geq -\mu_+ * h_r(x) = -\frac{\mu_+(B_r(x))}{c_{n,k,r}^{\mathrm{MVT}}} \geq -\frac{\mu_+(\mathbb{R}^n)}{c_{n,k,r}^{\mathrm{MVT}}} \geq -1,$$

which shows that $\tilde{u}_k \in \mathscr{F}_k(\mu)$. $\qquad\square$

We now choose some parameters in Lemma 4.1. It is easy to see that the second inequality in (4.2) is equivalent to

(4.4) $$k \leq \frac{(2\pi\epsilon)^{\frac{1}{2}} J_{\frac{n}{2}}(\epsilon)^{\frac{1}{n}}}{\mu_+(\mathbb{R})^{\frac{1}{n}}}$$

provided $\epsilon = rk$. By choosing

(4.5) $$R(n,k) = \frac{1}{2} j_{\frac{n-2}{2},1} k^{-1},$$

we see that the condition $r > 0$ is guaranteed by

$$k \leq \frac{j_{\frac{n-2}{2},1}}{4\gamma}.$$

Consequently, we see that

$$\epsilon = rk = \frac{\frac{1}{2} j_{\frac{n-2}{2},1} - \gamma k}{2} \geq \frac{j_{\frac{n-2}{2},1}}{8}$$

Since $t \mapsto t^{\frac{1}{2}} J_{\frac{n}{2}}(t)^{\frac{1}{n}}$ is strictly increasing on $[0, j_{\frac{n-2}{2},1}]$, then we know that (4.4) can be achieved by the assumption

$$k \leq \frac{(\frac{\pi j_{\frac{n-2}{2},1}}{4})^{\frac{1}{2}} J_{\frac{n}{2}}(\frac{j_{\frac{n-2}{2},1}}{8})^{\frac{1}{n}}}{\mu_+(\mathbb{R})^{\frac{1}{n}}}$$

If we define

$$c_n := \min\left\{ \left(\frac{\pi j_{\frac{n-2}{2},1}}{4}\right)^{\frac{1}{2}} J_{\frac{n}{2}}\left(\frac{j_{\frac{n-2}{2},1}}{8}\right)^{\frac{1}{n}}, \frac{j_{\frac{n-2}{2},1}}{4} \right\}$$

then we conclude the following lemma:

Lemma 4.2. *Fix any $\gamma > 0$ and $\mu \in L^\infty(B_\gamma)$. There exists a positive constant c_n (depending only on the dimension n) such that if*

$$(4.6) \qquad\qquad 0 < k \leq c_n \min\{\gamma^{-1}, \mu_+(\mathbb{R}^n)^{-\frac{1}{n}}\}$$

then $\mathscr{F}_k(\mu)$ contains an element $\tilde{u}_k$ with

$$(4.7) \qquad\qquad \tilde{u}_k = U_k^\mu \quad near \ \mathbb{R}^n \setminus B_{R(n,k)},$$

where $R(n,k)$ is given in (4.5).

The following proposition characterizes partial balayage in terms of the solution of our obstacle problem.

Proposition 4.3. *Let μ and $k > 0$ be as in Lemma 4.2. Then there exists a largest element V_k^μ in $\mathscr{F}_k(\mu)$. In addition, the element V_k^μ satisfies*

$$(4.8) \qquad\qquad \langle 1 + (\Delta + k^2) V_k^\mu, V_k^\mu - U_k^\mu \rangle = 0,$$

where $\langle \cdot, \cdot \rangle$ is the $H^{-1}(B_{R(n,k)}) \times H_0^1(B_{R(n,k)})$ duality pairing.

Remark 4.4. Note that Lemma 4.2 implies that there exists $\tilde{u}_k \in \mathscr{F}_k(\mu)$ satisfying $\tilde{u}_k = U_k^\mu$ near $\partial B_{R(n,k)}$. Therefore, if V_k^μ is that largest element in $\mathscr{F}_k(\mu)$ then

$$(4.9) \qquad\qquad V_k^\mu = U_k^\mu \quad near \ \partial B_{R(n,k)}.$$

Therefore, we can extend V_k^μ to the whole $\mathbb{R}^n$, by defining $V_k^\mu := U_k^\mu$ outside $B_{R(n,k)}$.

The proof of the proposition is based on variational arguments. In particular, we shall need the following elementary lemma several times in the proof.

Lemma 4.5. *Fix $k > 0$ and $0 < R < j_{\frac{n-2}{2},1} k^{-1}$. Let $a \colon H_0^1(B_R) \times H_0^1(B_R) \to \mathbb{R}$ be the symmetric bilinear form defined by*

$$(4.10) \qquad\qquad a(u_1, u_2) := \int_{B_R} (\nabla u_1 \cdot \nabla u_2 - k^2 u_1 u_2) \, d\mathsf{m}.$$

Then a is continuous, positive, and coercive.

Proof. That a is a continuous is clear from the definition. To prove that the form is coercive and positive we observe that by assumption k^2 is strictly smaller than the first eigenvalue of Dirichlet Laplacian on B_R (which is exactly $j_{\frac{n-2}{2},1}^2 R^{-2}$). Therefore,

$$a(u, u) = \int_{B_R} (|\nabla u|^2 - k^2 |u|^2) \, d\mathsf{m} \geq (1 - k^2 R^2 j_{\frac{n-2}{2},1}^{-2}) \int_{B_R} |\nabla u|^2 \, d\mathsf{m}.$$

This concludes the proof. $\qquad\square$

Proof of Proposition 4.3. Let $\varphi \in H^1(B_{R(n,k)})$ be the unique solution to

$$(4.11) \qquad \begin{cases} (\Delta + k^2)\varphi = -1 & \text{in } B_{R(n,k)}, \\ \varphi = U_k^\mu & \text{on } \partial B_{R(n,k)}, \end{cases}$$

and define

$$\tilde{\mathscr{F}}_k(\mu) = \left\{ \, w = \varphi - v \mid v \in \mathscr{F}_k(\mu) \, \right\} = \left\{ \, w \in H_0^1(B_{R(n,k)}) \, \left| \, \begin{matrix} (\Delta + k^2)w \leq 0 \text{ in } B_{R(n,k)} \\ w \geq \varphi - U_k^\mu \text{ in } B_{R(n,k)} \end{matrix} \, \right. \right\}.$$

We claim that there exists a smallest element u_* of $\tilde{\mathscr{F}}_k(\mu)$. If this is the case then

$$(4.12) \qquad V_k^\mu := \varphi - u_* \quad \text{in } B_{R(n,k)}$$

is the largest element of $\mathscr{F}_k(\mu)$.

To see that there exists a smallest element in $\tilde{\mathscr{F}}_k(\mu)$ we argue as follows. Let a be the bilinear form defined in (4.10) with $R = R(n,k)$. By Lemma 4.5 a is symmetric, continuous, and coercive. Define the constraint set

$$(4.13) \qquad \tilde{\mathcal{K}}_k := \left\{ \, u \in H_0^1(B_{R(n,k)}) \mid u \geq \varphi - U_k^\mu \, \right\}.$$

Note that $\varphi - U_k^\mu \in H_0^1(B_{R(n,k)})$ by definition of φ, thus $\tilde{\mathcal{K}}_k$ is nonempty. Since $\tilde{\mathcal{K}}_k$ is a nonempty closed convex subset of $H_0^1(B_{R(n,k)})$, Stampacchia's theorem [Bre11, Theorem 5.6] implies that there exists a unique $u_* \in \tilde{\mathcal{K}}_k$ that minimizes the functional

$$(4.14) \qquad u \mapsto a(u, u)$$

and $u_* \in \tilde{\mathcal{K}}_k$ can also be characterized by

$$(4.15) \qquad a(u_*, u - u_*) \geq 0 \quad \text{for all } u \in \tilde{\mathcal{K}}_k.$$

Plugging in $u = u_* + \phi$ with non-negative $\phi \in C_c^\infty(B_{R(n,k)})$ into (4.15), the definition of a in (4.10) implies that

$$(4.16) \qquad (\Delta + k^2)u_* \leq 0 \text{ in } B_{R(n,k)}.$$

In particular, we conclude that $u_* \in \tilde{\mathscr{F}}_k(\mu)$. Finally, by arguing as in the proof of [KS00, Theorem II.6.4], one can prove that $u_* \leq v$ in $B_{R(n,k)}$ for all $v \in \tilde{\mathscr{F}}_k(\mu)$. Consequently, we have found the desired smallest element in $\tilde{\mathscr{F}}_k(\mu)$.

Choosing $u = \varphi - U_k^\mu \in H_0^1(B_{R(n,k)})$ in (4.15), we have

$$(4.17) \qquad \langle (\Delta + k^2)u_*, \varphi - U_k^\mu - u_* \rangle \leq 0.$$

Since $u_* \in \tilde{\mathcal{K}}_k$ we know that $u_* \geq \varphi - U_k^\mu$. Along with (4.16) and (4.17) this inequality implies

$$(4.18) \qquad \langle (\Delta + k^2)u_*, \varphi - U_k^\mu - u_* \rangle = 0.$$

Combining (4.18) with (4.12), as well as (4.11), we obtain

$$0 = \langle (\Delta + k^2)(\varphi - V_k^\mu), V_k^\mu - U_k^\mu \rangle = -\langle 1 + (\Delta + k^2)V_k^\mu, V_k^\mu - U_k^\mu \rangle,$$

which shows that V_k^μ satisfies (4.8). $\qquad \square$

We are now ready to define the partial balayage for Helmholtz operator.

Definition 4.6. Let μ and $k > 0$ be as in Lemma 4.2. The *partial balayage* of μ is defined by

$$(4.19) \qquad \text{Bal}_k(\mu) := -(\Delta + k^2)V_k^\mu \quad \text{in distribution sense,}$$

where V_k^μ is given by Proposition 4.3.

We have the following basic properties of the partial balayage measure and the corresponding potential.

Lemma 4.7. *Let μ and $k > 0$ be as in Lemma 4.2. Then*

$$(4.20a) \qquad\qquad \mathrm{Bal}_k(\mu) \le 1 \quad in \; \mathbb{R}^n,$$

$$(4.20b) \qquad\qquad U_k^{\mathrm{Bal}_k(\mu)} \equiv V_k^\mu \quad in \; \mathbb{R}^n.$$

We also have

$$(4.20c) \qquad\qquad U_k^{\mathrm{Bal}_k(\mu)} \le U_k^\mu \quad in \; B_{R(n,k)}$$

and

$$(4.20d) \qquad\qquad U_k^{\mathrm{Bal}_k(\mu)} = U_k^\mu \quad in \; a \; neighborhood \; of \; \mathbb{R}^n \setminus B_{R(n,k)}.$$

Proof. If we can show (4.20b), then (4.20a) and (4.20c) are immediate consequence of Proposition 4.3 and the definition of $\mathscr{F}_k(\mu)$, while (4.20d) is an immediate consequence of Remark 4.4. It remains to prove (4.20b). Write $u = U_k^\mu - V_k^\mu$, so $-(\Delta + k^2)u = \mu - \mathrm{Bal}_k(\mu)$. Note that u has compact support by Remark 4.4. Thus

$$u = \tilde{\Phi}_k * (-(\Delta + k^2)u) = \tilde{\Phi}_k * (\mu - \mathrm{Bal}_k(\mu)) = U_k^\mu - U_k^{\mathrm{Bal}_k(\mu)}.$$

This proves that $U_k^{\mathrm{Bal}_k(\mu)} = V_k^\mu$. $\qquad\qquad\qquad\square$

Remark 4.8 (Partial balayage and energy minimization). Let μ and $k > 0$ be as in Lemma 4.2. Using Proposition 5.1, we know that $\nu := \mathrm{Bal}_k(\mu) \in L^\infty(B_{R(n,k)})$. We define the following bilinear form:

$$(\mu_1, \mu_2)_{e,k} := \iint_{B_{R(n,k)} \times B_{R(n,k)}} \tilde{\Phi}_k(x - y) \, d\mu_1(y) \, d\mu_2(x) = \int_{B_{R(n,k)}} U_k^{\mu_1}(x) \, d\mu_2(x)$$

for all $\mu_1, \mu_2 \in L^\infty(B_{R(n,k)})$. Using Lemma 4.7, we can write (4.8) as $(\nu - \mu, \mathsf{m} - \nu)_{e,k} = 0$. Accordingly, for each $\sigma \in L^\infty(B_{R(n,k)})$ with $\sigma \le \mathsf{m}$, we see that

$$(4.21) \qquad\qquad (\nu - \mu, \sigma - \nu)_{e,k} = (\nu - \mu, \sigma - \mathsf{m})_{e,k} \ge 0,$$

where the inequality follows from Lemma 4.7. By writing the "energy" $E_k(\lambda) := (\lambda, \lambda)_{e,k}$, we see that

$$(\nu - \mu, \sigma - \nu)_{e,k} = -E_k(\nu - \mu) + (\nu - \mu, \sigma - \mu)_{e,k},$$

thus from (4.21), we have

$$(4.22) \qquad E_k(\nu - \mu) \le (\nu - \mu, \sigma - \mu)_{e,k} \quad \text{for all } \sigma \in L^\infty(B_{R(n,k)}) \text{ with } \sigma \le \mathsf{m}.$$

When $U_k^{\mu_1}, U_k^{\mu_2} \in H_0^1(B_{R(n,k)})$, then we can compute

$$(\mu_1, \mu_2)_{e,k} = -\int_{B_{R(n,k)}} U_k^{\mu_1}(\Delta + k^2)U_k^{\mu_2} \, dx = a(U_k^{\mu_1}, U_k^{\mu_2}),$$

$$E_k(\mu_1) = a(U_k^{\mu_1}, U_k^{\mu_1}) \ge 0,$$

where $a(\cdot, \cdot)$ is the (real) inner product given by Lemma 4.5. Thus the notion of E_k as an energy functional makes sense. Using this observation and the Cauchy-Schwarz inequality, if

we restrict σ in (4.22) to those functions satisfying $U_k^\sigma - U_k^\mu \in H_0^1(B_{R(n,k)})$, then we have

$$
\begin{aligned}
E_k(\nu - \mu) &\le a(U_k^{\nu-\mu}, U_k^{\sigma-\mu}) \\
&\le (a(U_k^\nu - U_k^\mu, U_k^\nu - U_k^\mu))^{\frac{1}{2}} (a(U_k^\sigma - U_k^\mu, U_k^\sigma - U_k^\mu))^{\frac{1}{2}} \\
&\equiv (E_k(\nu - \mu))^{\frac{1}{2}} (E_k(\sigma - \mu))^{\frac{1}{2}}.
\end{aligned}
$$

Therefore, the partial balayage ν minimizes the energy in the following sense:

$$(4.23) \qquad E_k(\nu - \mu) \le E_k(\sigma - \mu) \quad \text{for all } \sigma \in L^\infty(B_{R(n,k)}) \text{ with } U_k^\sigma \in \mathscr{F}_k(\mu),$$

where $\mathscr{F}_k(\mu)$ is given by (4.1). Here we also refer to [Trè75, Section 30] for a related discussion.

The following observation will be very useful in our construction of k-quadrature domains.

Lemma 4.9. *Let μ and $k > 0$ be as in Lemma 4.2. If*

$$(4.24) \qquad \mathrm{Bal}_k(\mu) = \chi_D \mathsf{m} \quad \text{for some open set } D$$

then

$$(4.25a) \qquad U_k^D \le U_k^\mu,$$

$$(4.25b) \qquad U_k^D = U_k^\mu \quad in \ B_{R(n,k)} \setminus D.$$

Proof. We already proved (4.25a) in Lemma 4.7. Set $\nu = \mathrm{Bal}_k(\mu)$ and rewrite (4.8) as

$$0 = \int_{B_{R(n,k)}} (U_k^\mu - U_k^D)(1 - \chi_D) \, dx = \int_{B_{R(n,k)} \setminus D} (U_k^\mu - U_k^D) \, dx.$$

Combining this equality with (4.25a), we conclude (4.25b). $\qquad \square$

5. Structure of partial balayage

In this section we prove the following proposition which provides information concerning the structure of $\mathrm{Bal}_k(\mu)$. This will in particular be useful when we later on wish to construct k-quadrature domains.

Proposition 5.1 (Structure of partial balayage). *Let μ and $k > 0$ be as in Lemma 4.2 and let $\nu := \mathrm{Bal}_k(\mu)$. Then*

$$(5.1) \qquad \min\{\mu, 1\} \le \nu \le 1 \quad in \ B_{R(n,k)}.$$

Furthermore, if we define $D(\mu) := \overline{B_{R(n,k)}} \setminus \mathrm{supp}(\mathsf{m} - \nu)$, then

$$(5.2) \qquad \overline{D(\mu)} \subset B_{R(n,k)} \quad and \quad \nu = \chi_{D(\mu)} \mathsf{m} + \chi_{B_{R(n,k)} \setminus D(\mu)} \mu.$$

Remark 5.2. The set $D(\mu)$ is called the saturated set for $\nu = \mathrm{Bal}_k(\mu)$, which is the largest open set in $B_{R(n,k)}$ such that $\nu = \mathsf{m}$. The corresponding result for $k = 0$ can be found in [Gus90, Theorem 2.3(c)]. We refer also to [GS09, Gus04, Sjö07], and in particular [Gus04, Figure 3] for a visualization.

Proof of Proposition 5.1. **Step 1: A minimization problem.** Let $\xi \in H_0^1(B_{R(n,k)})$ be the unique solution to $-(\Delta + k^2)\xi = (1 - \mu)_+$, and consider the constraint set

$$\hat{\mathcal{K}}_k = \left\{ \, w \in H_0^1(B_{R(n,k)}) \mid w \ge \xi - u_* \, \right\},$$

where $u_* \in H_0^1(B_{R(n,k)})$ is the function appearing in Proposition 4.3. We recall that u_* minimizes the functional $a(u, u)$ among all functions in $\tilde{\mathcal{K}}_k$, where a is the bilinear form defined in (4.10) and $\tilde{\mathcal{K}}_k$ was defined in (4.13). Note that $\hat{\mathcal{K}}_k$ is nonempty since $\xi - u_* \in \hat{\mathcal{K}}_k$.

By Lemma 4.5 and Stampacchia's Theorem (see [Bre11, Theorem 5.6]) there exists a unique $w_* \in \hat{\mathcal{K}}_k$ which minimizes the functional $w \mapsto a(w, w)$ among $w \in \hat{\mathcal{K}}_k$. Moreover, the minimizer w_* is characterized by the property

$$(5.3) \qquad a(w_*, w - w_*) = \langle -(\Delta + k^2)w_*, w - w_* \rangle \geq 0 \quad \text{for all } w \in \hat{\mathcal{K}}_k.$$

Step 2: Complementarity formulation. Since $w_* \in \hat{\mathcal{K}}_k$, we can in (5.3) restrict w to those satisfying $w \geq w_*$. The definition of a implies that

$$(5.4) \qquad\qquad\qquad (\Delta + k^2)w_* \leq 0 \quad \text{in } B_{R(n,k)}.$$

Choosing $w = \xi - u_*$ in (5.3),

$$\langle (\Delta + k^2)w_*, \xi - u_* - w_* \rangle \leq 0,$$

which along with (5.4) and the fact that $w_* \geq \xi - u_*$ implies

$$(5.5) \qquad\qquad\qquad \langle (\Delta + k^2)w_*, \xi - u_* - w_* \rangle = 0.$$

In fact, if $w_* \in \hat{\mathcal{K}}_k$ satisfies (5.4) and (5.5), then

$$\langle (\Delta + k^2)w_*, w - w_* \rangle$$
$$= \langle (\Delta + k^2)w_*, w - (\xi - u_*) \rangle + \langle (\Delta + k^2)w_*, \xi - u_* - w_* \rangle \leq 0,$$

for all $w \in \hat{\mathcal{K}}_k$. Hence the minimizer $w_* \in \hat{\mathcal{K}}_k$ can also be characterized by the complementarity problem (5.4) and (5.5).

Step 3: An energy inequality. We can rewrite (5.5) as

$$(5.6) \qquad\qquad \langle (\Delta + k^2)w_*, \xi - w_* \rangle = \langle (\Delta + k^2)w_*, u_* \rangle.$$

The inequalities $(\Delta + k^2)\xi = -(1 - \mu)_+ \leq 0$ and $w_* \geq \xi - u_*$ (i.e. $u_* \geq \xi - w_*$) thus imply that

$$(5.7) \qquad\qquad \langle (\Delta + k^2)\xi, \xi - w_* \rangle \geq \langle (\Delta + k^2)\xi, u_* \rangle.$$

Combining (5.6) and (5.7), one finds

$$a(\xi - w_*, \xi - w_*) = \langle -(\Delta + k^2)(\xi - w_*), \xi - w_* \rangle$$
$$\leq \langle -(\Delta + k^2)(\xi - w_*), u_* \rangle = a(\xi - w_*, u_*)$$
$$\leq a(\xi - w_*, \xi - w_*)^{\frac{1}{2}} a(u_*, u_*)^{\frac{1}{2}}.$$

By Lemma 4.5 the bilinear form a is positive, and thus we obtain the energy inequality

$$(5.8) \qquad\qquad\qquad a(\xi - w_*, \xi - w_*) \leq a(u_*, u_*).$$

Step 4: Verifying that $w_* = \xi - u_*$. If we can show that $\xi - w_* \in \tilde{\mathcal{K}}_k$, i.e. that it satisfies

$$(5.9) \qquad\qquad\qquad \xi - w_* \geq \varphi - U_k^\mu \quad \text{in } B_{R(n,k)}$$

where φ is the function in (4.11), then since u_* minimizes $a(u, u)$ among all $u \in \tilde{\mathcal{K}}_k$ the inequality in (5.8) implies that $\xi - w_* = u_*$ in $B_{R(n,k)}$, in other words $w_* = \xi - u_*$ in $B_{R(n,k)}$.

To prove (5.9) we argue as follows. Let

$$(5.10) \qquad\qquad \phi := \min\{w_*, \xi - (\varphi - U_k^\mu)\} \quad \text{in } B_{R(n,k)}.$$

By the definition of ϕ and Proposition A.3,

$$(5.11) \qquad \phi \leq w_*, \qquad \phi \in \hat{\mathcal{K}}_k, \quad \text{and} \quad -(\Delta + k^2)\phi \geq 0 \quad \text{in } B_{R(n,k)}.$$

Using (5.11) and the facts that $w_* \in H_0^1(B_{R(n,k)})$ and $-(\Delta + k^2)w_* \geq 0$ in $B_{R(n,k)}$, we have in terms of distributional pairings in $B_{R(n,k)}$ that

$$\begin{aligned} a(\phi, \phi) &= \langle -(\Delta + k^2)\phi, \phi \rangle \\ &\leq \langle -(\Delta + k^2)\phi, w_* \rangle = \langle \phi, -(\Delta + k^2)w_* \rangle \\ &\leq \langle w_*, -(\Delta + k^2)w_* \rangle = a(w_*, w_*). \end{aligned}$$

Since w_* was defined to be the unique minimizer of $a(w, w)$ among $w \in \hat{\mathcal{K}}_k$, we obtain

$$\phi = w_* \quad \text{in } B_{R(n,k)}.$$

By the definition of ϕ this can be equivalently stated as

$$w_* \leq \xi - (\varphi - U_k^\mu) \quad \text{in } B_{R(n,k)}.$$

After rearranging we deduce the desired inequality (5.9). By the discussion following (5.9) it holds that

$$w_* = \xi - u_* \quad \text{in } B_{R(n,k)}.$$

Step 5: Proving (5.1). By Step 4, (5.4), and the definition of ξ,

$$(5.12) \qquad (\Delta + k^2)u_* \geq (\Delta + k^2)\xi = -(1 - \mu)_+.$$

From (4.12) and (4.20b) we deduce that

$$(5.13) \qquad u_* = \varphi - U_k^\nu.$$

Combining (5.12) and (5.13), we obtain that in $B_{R(n,k)}$

$$\nu - 1 \geq -(1 - \mu)_+ = -\max\{1 - \mu, 0\} = \min\{\mu - 1, 0\}$$
$$\iff \min\{1, \mu\} \leq \nu.$$

By the definition of partial balayage $\nu \leq 1$ in $B_{R(n,k)}$, and we have arrived at (5.1).

Step 6: Proving (5.2). By the Calderón–Zygmund inequality $U_k^\mu, U_k^\nu \in \bigcap_{p < \infty} W_{\text{loc}}^{2,p}(\mathbb{R}^n)$. Therefore the set

$$(5.14) \qquad \omega(\mu) := \left\{ \; x \in \overline{B_{R(n,k)}} \; \middle| \; U_k^\mu(x) > U_k^\nu(x) \; \right\}$$

is open relative to $\overline{B_{R(n,k)}}$. From (4.8) and Proposition 4.3, it follows that

$$0 \leq \int_{D_0(\mu)} (U_k^\mu - U_k^\nu) \, d(\mathsf{m} - \nu) \leq \int_{B_{R(n,k)}} (U_k^\mu - U_k^\nu) \, d(\mathsf{m} - \nu) = 0,$$

and hence

$$(5.15) \qquad \int_{D_0(\mu)} (U_k^\mu - U_k^\nu) \, d(\mathsf{m} - \nu) = 0.$$

From (5.15) and Proposition 4.3, we have found that $\nu = \mathsf{m}$ in $\omega(\mu)$. In particular, $\omega(\mu) \subset D(\mu)$ and

$$\begin{aligned} \overline{B_{R(n,k)}} \setminus D(\mu) \subset \overline{B_{R(n,k)}} \setminus \omega(\mu) &= \left\{ \; x \in \overline{B_{R(n,k)}} \; \middle| \; U_k^\mu(x) = U_k^\nu(x) \; \right\} \\ &\subset \left\{ \; x \in \overline{B_{R(n,k)}} \; \middle| \; \mu(x) = \nu(x) \; \right\}. \end{aligned}$$

Consequently,

(5.16) $$\left\{ \ x \in \overline{B_{R(n,k)}} \ \big| \ \nu(x) \neq \mu(x) \ \right\} \subset D(\mu) = \left\{ \ x \in \overline{B_{R(n,k)}} \ \big| \ \nu(x) = 1 \ \right\}.$$

From Lemma 4.7 we deduce that $\overline{D(\mu)} \subset B_{R(n,k)}$, and therefore (5.16) completes the proof of Proposition 5.1. □

Remark 5.3. Let μ and $k > 0$ be as in Lemma 4.2 and let $\nu := \mathrm{Bal}_k(\mu)$. We observe that the function $u := U_k^\mu - U_k^\nu \in C^1(\mathbb{R}^n)$ satisfies

$$u > 0 \text{ in } \omega(\mu) \quad \text{and} \quad u = 0 \text{ in } \mathbb{R}^n \setminus \omega(\mu).$$

Since u attains its minimum value at each point in $\mathbb{R}^n \setminus \omega(\mu)$, then we also have

$$\nabla u = 0 \quad \text{in } \mathbb{R}^n \setminus \omega(\mu).$$

If $\mathrm{supp}(\mu) \subset \omega(\mu)$, then $\nu = \chi_{\omega(\mu)}\mathsf{m}$ (because $\omega(\mu) \subset D(\mu)$ and (5.2)). Therefore we obtain (2.5) with $D = \omega(\mu)$. If $\mathrm{supp}(\mu) \subset \omega(\mu)$, arguing as in Proposition 2.1 we know that $\omega(\mu)$ is a k-quadrature domain.

We also have the following converse:

Lemma 5.4. *Let μ and $k > 0$ be as in Lemma 4.2. Suppose there is an open set D that satisfies $\overline{D} \subset B_{R(n,k)}$ with $\mathrm{supp}(\mu) \subset D$ and a distribution u that satisfies*

(5.17) $$\begin{cases} (\Delta + k^2)u = \chi_D - \mu & \text{in } B_{R(n,k)}, \\ u > 0 & \text{in } D, \\ u = 0 & \text{in } B_{R(n,k)} \setminus D, \end{cases}$$

then $\mathrm{Bal}_k(\mu) = \chi_D\mathsf{m}$ and $D = \omega(\mu) = D(\mu) = \{u > 0\}$.

Proof. Since u (with zero extension outside $B_{R(n,k)}$) is a compactly supported distribution, then again we have

$$u = \tilde{\Phi}_k * (-(\Delta + k^2)u) = U_k^\mu - U_k^D.$$

Since u is non-negative, then we know that $U_k^D \in \mathscr{F}_k(\mu)$, where $\mathscr{F}_k(\mu)$ is the collection of functions given in (4.1). For each $v \in \mathscr{F}_k(\mu)$, since $u = 0$ in $B_{R(n,k)} \setminus D$, we see that

$$w := U_k^D - v = U_k^D - U_k^\mu + U_k^\mu - v \geq 0 \quad \text{in } B_{R(n,k)} \setminus D.$$

On the other hand, we have $(\Delta + k^2)w = -1 - (\Delta + k^2)v \leq 0$ in D. Therefore using the maximum principle (Proposition A.2) we know that $w \geq 0$ in D as well. This shows that U_K^D is the largest element in $\mathscr{F}_k(\mu)$, therefore from (4.19) we have

$$\mathrm{Bal}_k(\mu) = -(\Delta + k^2)U_K^D = \chi_D\mathsf{m}.$$

Finally, from (5.17) we know that $D = \omega(\mu) = \{u > 0\}$. Since $\mathrm{supp}(\mu) \subset D = \omega(\mu) \subset D(\mu)$, from Proposition 5.1 we also know that

$$\mathrm{Bal}_k(\mu) = \chi_{D(\mu)}\mathsf{m},$$

which further shows that $D = \omega(\mu) = D(\mu) = \{u > 0\}$ □

6. Performing balayage in smaller steps

Fix $\gamma > 0$ and assume that $\mu_1, \mu_2 \in L^\infty(B_\gamma)$ are non-negative. By Proposition 4.3, there exists a positive constant c_n such that if

$$(6.1) \qquad 0 < k < c_n \min\{\gamma^{-1}, \mu_1(\mathbb{R}^n)^{-\frac{1}{n}}\},$$

then $U_k^{\mathrm{Bal}_k(\mu_1)}$ is the largest element in $\mathscr{F}_k(\mu_1)$ (defined as in (4.1)) and $U_k^{\mathrm{Bal}_k(\mu_1)} = U_k^{\mu_1}$ near $\partial B_{R(n,k)}$. Again Proposition 4.3 also implies that if

$$(6.2) \qquad 0 < k < c_n \min\{\gamma^{-1}, (\mu_1 + \mu_2)(\mathbb{R}^n)^{-\frac{1}{n}}\},$$

then $U_k^{\mathrm{Bal}_k(\mu_1+\mu_2)}$ is the largest element of $\mathscr{F}_k(\mu_1+\mu_2)$ and $U_k^{\mathrm{Bal}_k(\mu_1+\mu_2)} = U_k^{\mu_1+\mu_2}$ near $\partial B_{R(n,k)}$.

Finally, if we additionally assume that $\mathrm{supp}(\nu_1) \subset B_\gamma$ with $\nu_1 = \mathrm{Bal}_k(\mu_1)$, Proposition 4.3 implies that if

$$(6.3) \qquad 0 < k < c_n \min\{\gamma^{-1}, (\nu_1 + \mu_2)(\mathbb{R}^n)^{-\frac{1}{n}}\},$$

then $U_k^{\mathrm{Bal}_k(\nu_1+\mu_2)}$ is the largest element of $\mathscr{F}_k(\nu_1+\mu_2)$ and $U_k^{\mathrm{Bal}_k(\nu_1+\mu_2)} = U_k^{\nu_1+\mu_2}$ near $\partial B_{R(n,k)}$. Using Proposition 5.1 we observe that

$$\mathrm{supp}(\mu_1) \subset \mathrm{supp}(\mu_1 + \mu_2) \subset \mathrm{supp}(\nu_1 + \mu_2) \subset B_\gamma.$$

We are now ready to prove the following proposition:

Proposition 6.1. *Let $\gamma > 0$ and $\mu_1, \mu_2 \in L^\infty(B_\gamma)$ be non-negative and such that $\mathrm{supp}(\mathrm{Bal}_k(\mu_1)) \subset B_\gamma$. If*

$$0 < k < c_n \min\{\gamma^{-1}, (\mu_1 + \mu_2)(\mathbb{R}^n)^{-\frac{1}{n}}, (\mathrm{Bal}_k(\mu_1) + \mu_2)(\mathbb{R}^n)^{-\frac{1}{n}}\}$$

with c_n (depending only on the dimension n) as in Lemma 4.2, then

$$\mathrm{Bal}_k(\mathrm{Bal}_k(\mu_1) + \mu_2) = \mathrm{Bal}_k(\mu_1 + \mu_2).$$

Proof. Note that if k satisfies the inequality in the proposition then k satisfies the inequalities (6.1), (6.3), and (6.2).

Since $U_k^{\nu_1+\mu_2} = U_k^{\nu_1} + U_k^{\mu_2}$ and $U_k^{\nu_1} = U_k^{\mu_1}$ near $\partial B_{R(n,k)}$ it suffices to show that

$$(6.4) \qquad U_k^{\mathrm{Bal}_k(\nu_1+\mu_2)} = U_k^{\mathrm{Bal}_k(\mu_1+\mu_2)} \quad \text{in } B_{R(n,k)}.$$

Step 1: The implication "$\leq$" of (6.4). Using Lemma 4.7 we observe that

$$\begin{aligned} U_k^{\mathrm{Bal}_k(\nu_1+\mu_2)} &\leq U_k^{\nu_1+\mu_2} = U_k^{\nu_1} + U_k^{\mu_2} \\ &\leq U_k^{\mu_1} + U_k^{\mu_2} = U_k^{\mu_1+\mu_2} \quad \text{in } B_{R(n,k)} \end{aligned}$$

and

$$(\Delta + k^2) U_k^{\mathrm{Bal}_k(\nu_1+\mu_2)} \geq -1 \quad \text{in } B_{R(n,k)}.$$

Thus $U_k^{\mathrm{Bal}_k(\nu_1+\mu_2)} \in \mathscr{F}(\mu_1 + \mu_2)$. Since $U_k^{\mathrm{Bal}_k(\mu_1+\mu_2)}$ is the largest element in $\mathscr{F}(\mu_1 + \mu_2)$, we arrive at

$$(6.5) \qquad U_k^{\mathrm{Bal}_k(\nu_1+\mu_2)} \leq U_k^{\mathrm{Bal}_k(\mu_1+\mu_2)} \quad \text{in } B_{R(n,k)}.$$

Step 2: The implication "$\geq$" of (6.4). Observe that

$$U_k^{\mathrm{Bal}_k(\mu_1+\mu_2)} - U_k^{\mu_2} \leq U_k^{\mu_1+\mu_2} - U_k^{\mu_2} = U_k^{\mu_1} \quad \text{in } B_{R(n,k)}$$

and

$$(\Delta + k^2)(U_k^{\mathrm{Bal}_k(\mu_1+\mu_2)} - U_k^{\mu_2}) \geq -1 + \mu_2 \geq -1 \quad \text{in } B_{R(n,k)}.$$

Thus $U_k^{\mathrm{Bal}_k(\mu_1+\mu_2)} - U_k^{\mu_2} \in \mathscr{F}(\mu_1)$. Since $U_k^{\nu_1}$ is the largest element in $\mathscr{F}(\mu_1)$, it holds that

$$U_k^{\mathrm{Bal}_k(\mu_1+\mu_2)} - U_k^{\mu_2} \leq U_k^{\nu_1} \quad \text{in } B_{R(n,k)},$$

and hence

$$U_k^{\mathrm{Bal}_k(\mu_1+\mu_2)} \leq U_k^{\nu_1} + U_k^{\mu_2} = U_k^{\nu_1+\mu_2} \quad \text{in } B_{R(n,k)}.$$

Furthermore,

$$(\Delta + k^2)U_k^{\mathrm{Bal}_k(\mu_1+\mu_2)} \geq -1 \quad \text{in } B_{R(n,k)}.$$

Thus $U_k^{\mathrm{Bal}_k(\mu_1+\mu_2)} \in \mathscr{F}(\nu_1 + \mu_2)$. Since $U_k^{\mathrm{Bal}_k(\nu_1+\mu_2)}$ is the largest element in $\mathscr{F}(\nu_1 + \mu_2)$, it follows that

$$(6.6) \qquad U_k^{\mathrm{Bal}_k(\nu_1+\mu_2)} \geq U_k^{\mathrm{Bal}_k(\mu_1+\mu_2)} \quad \text{in } B_{R(n,k)}.$$

Step 3: Conclusion. Combining (6.5) and (6.6) implies (6.4) and completes the proof. $\square$

Remark 6.2. Suppose that the assumptions in Proposition 6.1 hold and we additionally assume that $\mathrm{supp}(\mu_1) \cup \mathrm{supp}(\mu_2) \subset B_{3\epsilon}$ with $0 < 10\epsilon < \frac{1}{2}j_{\frac{n-2}{2},1}k^{-1} = R(n,k)$. Using Lemma 4.1 (the assumptions of Proposition 6.1 imply that μ_1 satisfies the assumptions of Lemma 4.2, and so it satisfies the assumptions in Lemma 4.1) we know that

$$\omega(\mu_1) \subset \overline{B_{\frac{R(n,k)+3\epsilon}{2}}} \subset B_{\frac{13}{20}R(n,k)}.$$

Since $\mathrm{supp}(\mu_2) \subset B_{3\epsilon} \subset B_{\frac{3}{10}R(n,k)}$ and $\tilde{\Phi}_k > 0$ in $B_{R(n,k)}$, we know that

$$U_k^{\mu_2} > 0 \quad \text{in } B_{\frac{13}{20}R(n,k)}.$$

Then we have

$$\begin{aligned}
\omega(\mu_1) &= B_{\frac{13}{20}R(n,k)} \cap \{U_k^{\mu_1} > U_k^{\mathrm{Bal}_k(\mu_1)}\} \\
&= B_{\frac{13}{20}R(n,k)} \cap \{U_k^{\mu_1+\mu_2} > U_k^{\mathrm{Bal}_k(\mu_1)+\mu_2}\} \\
&\subset B_{\frac{13}{20}R(n,k)} \cap \{U_k^{\mu_1+\mu_2} > U_k^{\mathrm{Bal}_k(\mathrm{Bal}_k(\mu_1)+\mu_2)}\} \quad \text{(from the definition of } \mathrm{Bal}_k) \\
&= B_{\frac{13}{20}R(n,k)} \cap \{U_k^{\mu_1+\mu_2} > U_k^{\mathrm{Bal}_k(\mu_1+\mu_2)}\} \quad \text{(using Proposition 6.1)} \\
&\subset \omega(\mu_1 + \mu_2).
\end{aligned}$$

7. Construction of k-quadrature domains

In this section our aim is to prove the following theorem, which is slightly stronger than Theorem 1.6.

Theorem 7.1. *Let μ be a positive measure supported in B_ϵ for some $\epsilon > 0$. There exists a constant $c_n > 0$ depending only on the dimension such that if*

$$(7.1) \qquad 0 < k < \frac{c_n}{\mu(\mathbb{R}^n)^{1/n}} \quad \text{and} \quad \epsilon < c_n\mu(\mathbb{R}^n)^{1/n},$$

then there exists an open connected set D with real-analytic boundary satisfying $\overline{D} \subset B_{R(n,k)}$ which is a k-quadrature domain for μ.

Moreover, for each $w \in L^1(D) \cap L^1(\mu)$ satisfying $(\Delta + k^2)w \geq 0$ in D we have

$$(7.2) \qquad \int_D w(x)\,dx \geq \int w(x)\,d\mu(x).$$

In addition, the domain D is unique in the following sense: If $\tilde{D}$ is a domain which satisfies $\overline{\tilde{D}} \subset B_{R(n,k)}$ and (7.2), then $[\tilde{D}] = D$, where $[\tilde{D}]$ is the saturation of $\tilde{D}$ with respect to Lebesgue measure.

Remark 7.2. As we shall see in the proof the k-quadrature domain is constructed as the saturated set of the partial balayage of a measure obtained by averaging μ over a small ball. If $\mu \in L^\infty(B_{R(n,k)})$ then $\mathrm{Bal}_k(\mu)$ is well-defined and the k-quadrature domain we construct is precisely the saturated set $D(\mu)$. Under the assumptions of Theorem 7.1, in particular such a k-quadrature domain can be characterized by

$$D(\mu) = \omega(\mu) = \{u > 0\}$$

where u is a compactly supported distribution satisfying $(\Delta + k^2)u = \chi_{\{u>0\}} - \mu$ in $\mathbb{R}^n$. However, our definitions and results concerning $\mathrm{Bal}_k(\mu)$ and $D(\mu)$ as well as $\omega(\mu)$ are only valid if $\mu \in L^\infty(B_{R(n,k)})$, so for μ as in the statement of the theorem we need to take some care. For unbounded μ, we need to take extra care. In this case, such a k-quadrature domain can be characterized by

$$D(\mu * h) = \omega(\mu * h) = \{u > 0\}$$

where u is a compactly supported distribution satisfying $(\Delta + k^2)u = \chi_{\{u>0\}} - \mu * h$ in $\mathbb{R}^n$ and $h = h_{2\epsilon} = \frac{1}{c^{\mathrm{MVT}}_{n,k,\delta}}\chi_{B_\delta}$.

Remark 7.3. Here we recall that (see e.g. [Gus04]) if $\tilde{D} \subset \mathbb{R}^n$ then

$$[\tilde{D}] := \left\{\, x \in \mathbb{R}^n \,\middle|\, \mathsf{m}(B_r(x) \setminus \tilde{D}) = 0 \text{ for some } r > 0 \,\right\}.$$

It is easy to see that $\tilde{D} \subset [\tilde{D}] \subset \mathrm{int}\,\overline{\tilde{D}}$ and $\mathsf{m}([\tilde{D}] \setminus \tilde{D}) = 0$. In particular, if $\mathsf{m}(\partial\tilde{D})$ is zero, then $[\tilde{D}] = \mathrm{int}\,(\overline{\tilde{D}})$.

Before we turn to the proof of Theorem 7.1 we prove some preliminary results that we will need in our main argument.

The first is a simple lemma concerning the partial balayage of a multiple of Lebesgue measure restricted to a ball.

We now prove the following lemma:

Lemma 7.4. *Let $0 < r < r' < \frac{1}{2}j_{\frac{n-2}{2},1}k^{-1} = R(n,k)$. Then there exists a positive constant c_n (depending only on the dimension n) such that if*

$$(7.3) \qquad 0 < k \leq c_n \min\left\{\frac{1}{r}, \frac{J_{\frac{n}{2}}(kr)^{\frac{1}{n}}}{J_{\frac{n}{2}}(kr')^{\frac{1}{n}}(rr')^{\frac{1}{2}}}\right\},$$

then

$$(7.4) \qquad \mathrm{Bal}_k\left(\frac{c^{\mathrm{MVT}}_{n,k,r'}}{c^{\mathrm{MVT}}_{n,k,r}}\chi_{B_r}\mathsf{m}\right) = \chi_{B_{r'}}\mathsf{m} \quad and \quad \omega(\mu_{\mu_\kappa,r}) = D(\mu_{\mu_\kappa,r}) = B_{r'}.$$

Remark 7.5. Since $t \mapsto t^{\frac{n}{2}} J_{\frac{n}{2}}(t)$ is strictly increasing on $t \in [0, j_{\frac{n-2}{2},1}]$, then we see that

$$\frac{c_{n,k,r'}^{\text{MVT}}}{c_{n,k,r}^{\text{MVT}}} = \frac{(r')^{n/2} J_{\frac{n}{2}}(kr')}{r^{n/2} J_{\frac{n}{2}}(kr)} > 1.$$

Since $t \mapsto t^{-\frac{n}{2}} J_{\frac{n}{2}}(t)$ is a decreasing function on $[0, j_{\frac{n+2}{2},1}]$, we find that

$$\text{Bal}_k\left(\frac{c_{n,k,r'}^{\text{MVT}}}{c_{n,k,r}^{\text{MVT}}} \chi_{B_r} \mathsf{m}\right)(\mathbb{R}^n) - \frac{c_{n,k,r'}^{\text{MVT}}}{c_{n,k,r}^{\text{MVT}}} \chi_{B_r} \mathsf{m}(\mathbb{R}^n)$$

$$(7.5) \qquad = \mathsf{m}(B_1)\left((r')^n - \frac{(r')^{n/2} J_{\frac{n}{2}}(kr')}{r^{n/2} J_{\frac{n}{2}}(kr)} r^n\right)$$

$$= \mathsf{m}(B_1)(r')^{\frac{n}{2}} k^{-\frac{n}{2}} J_{\frac{n}{2}}(kr')\left(\frac{(kr')^{\frac{n}{2}}}{J_{\frac{n}{2}}(kr')} - \frac{(kr)^{\frac{n}{2}}}{J_{\frac{n}{2}}(kr)}\right) \geq 0.$$

Proof of Lemma 7.4. For each $x \in \mathbb{R}^n$, we see that the distribution $y \mapsto \tilde{\Phi}_k(x-y)$ is in $L^1_{\text{loc}}(\mathbb{R}^n)$ and satisfies $(\Delta + k^2)\tilde{\Phi}_k(x - \cdot) = -\delta_x \leq 0$ in $\mathbb{R}^n$. By applying the MVT in Proposition A.5, we have

$$\frac{1}{c_{n,k,r}^{\text{MVT}}} U_k^{B_r}(x) = \frac{1}{c_{n,k,r}^{\text{MVT}}} \int_{B_r} \tilde{\Phi}_k(x - y)\, dy \geq \frac{1}{c_{n,k,r'}^{\text{MVT}}} \int_{B_{r'}} \tilde{\Phi}_k(x - y)\, dy = \frac{1}{c_{n,k,r'}^{\text{MVT}}} U_k^{B_{r'}}(x)$$

for all $x \in \mathbb{R}^n$, and equality holds for all $x \in \mathbb{R}^n \setminus B_{r'}$. In other words $u := \frac{c_{n,k,r'}^{\text{MVT}}}{c_{n,k,r}^{\text{MVT}}} U_k^{B_r} - U_k^{B_{r'}} \in C^1(\mathbb{R}^n)$ satisfies

$$u \geq 0 \text{ in } \mathbb{R}^n \quad \text{and} \quad u = 0 \quad \text{in } \mathbb{R}^n \setminus B_{r'}.$$

Using Lemma 5.4 (the sufficient condition guaranteed by (7.3)) with the open set $D = \{u > 0\}$, we conclude (7.4). $\square$

The second result we require is an analogue of Proposition 2.4 but for sub-solutions of the Helmholtz equation. We again follow the argument in [Sak84, Lemma 5.1] which considered the case $k = 0$.

Proposition 7.6. *Let $k \geq 0$, and let $D \subset \mathbb{R}^n$ be a bounded open set. Let Ψ_k be any fundamental solution of $-(\Delta + k^2)$ and let $\Omega \supset \overline{D}$ be any open set in $\mathbb{R}^n$. Then the linear span with positive coefficients of*

$$F = \{\pm\partial^\alpha \Psi_k(z - \cdot)|_D : z \in \Omega \setminus D, |\alpha| \leq 1\} \cup \{-\Psi_k(z - \cdot)|_D : z \in D\}$$

is dense in

$$S_k L^1(D) = \{w \in L^1(D) : (\Delta + k^2)w \geq 0 \text{ in } D\}$$

with respect to the $L^1(D)$ topology.

Proof. We first show that if any bounded linear functional ℓ on $L^1(D)$ with $\ell|_F \geq 0$ also satisfies

$$(7.6) \qquad \ell|_{S_k L^1(D)} \geq 0,$$

then we have $\overline{G} = S_k L^1(D)$, where G is the linear span with positive coefficients of F. Suppose to the contrary that there exists $f_0 \in S_k L^1(D) \setminus \overline{G}$. Using the Hahn-Banach theorem (second

geometric form, see e.g. [Bre11, Theorem 1.7]), there exists a closed hyperplane $\{\ell_0 = \alpha\}$ that strictly separates the closed set $\overline{G}$ and the compact set $\{f_0\}$, thus we have

$$(7.7) \qquad\qquad \ell_0(f_0) < \alpha < \ell_0(f) \quad \text{for all } f \in G.$$

Since $\lambda \ell_0(f) = \ell_0(\lambda f) > \alpha$ for all $\lambda > 0$ and each fixed $f \in G$, we deduce that $\ell_0(f) \geq 0$ for all $f \in G$ and that $\alpha \leq 0$. By (7.6) and since $f_0 \in S_k L^1(D)$ we know that $\ell_0(f_0) \geq 0$. Combining this with (7.7) and the fact that $\alpha \leq 0$ gives a contradiction.

Now let ℓ be a bounded linear functional on $L^1(D)$ with $\ell|_F \geq 0$. We need to prove that $\ell|_{S_k L^1(D)} \geq 0$. Since the dual of $L^1(D)$ is $L^\infty(D)$, there is a function $f \in L^\infty(D)$ with

$$\ell(w) = \int_D f w \, dx, \quad w \in L^1(D).$$

We extend f by zero to $\mathbb{R}^n$ and consider the function

$$u(z) = -(\Psi_k * f)(z) \quad \text{for all } z \in \Omega.$$

By the assumption $\ell|_F \geq 0$, the function u satisfies

$$\begin{cases} (\Delta + k^2)u = f & \text{in } \Omega, \\ u = \nabla u = 0 & \text{in } \Omega \setminus D, \\ u \geq 0 & \text{in } D. \end{cases}$$

Our aim is to employ the same argument as in the proof of (2.3) to show that

$$\int_D ((\Delta + k^2)u)w \, dx \geq 0, \quad \text{for all } w \in S_k L^1(D),$$

which implies $\ell|_{S_k L^1(D)} \geq 0$. However, in order to carry out the the integration by parts which concluded that argument we used that solutions of the Helmholtz equation are smooth in the interior of D, this is not necessarily the case for sub-solutions. To circumvent this issue we use a classical mollification argument. Fix a non-negative $\psi \in C_c^\infty(\mathbb{R}^n)$ with support in B_1 and $\|\psi\|_{L^1(\mathbb{R}^n)} = 1$. For $\epsilon > 0$ set $\psi_\epsilon(x) = \epsilon^{-n}\psi(x/\epsilon)$. For $w \in S_k L^1(D)$ set $w_\epsilon = w * \psi_\epsilon$, which is well defined and C^∞ near any compact subset K of D if $\epsilon < \text{dist}(K, D^c)$. Then $w_\epsilon \to w$ in $L^1(K)$ as $\epsilon \to 0^+$. We claim that $(\Delta + k^2)w_\epsilon(x) \geq 0$ in K for $\epsilon < \text{dist}(K, D^c)$. Indeed, since $(\Delta + k^2)w(x) \geq 0$ in D this is in particular the case in $B_\epsilon(y)$ for any $y \in K$. The claim follows by differentiating under the integral sign and using the non-negativity of ψ. With this approximation in hand the argument can be completed as in the proof of (2.3) by appealing to the L^1 convergence of w_ϵ to w in the support of the cutoff function ω_j, and applying the integration by parts argument with w replaced by w_ϵ. $\qquad\square$

Finally we need the following result which is in the spirit of Proposition 2.1. This result can be interpreted as saying that D is a quadrature domain for sub-solutions w satisfying $(\Delta + k^2)w \geq 0$ in D (i.e. D is a quadrature domain for metasubharmonic functions).

Corollary 7.7. *Let $k > 0$, and let $D, \Omega \subset \mathbb{R}^n$ be bounded open sets such that $\overline{D} \subset \Omega$, and let $\mu \in L^\infty(D)$ be a non-negative measure with $\text{supp}(\mu) \subset D$. If*

$$(7.8a) \qquad\qquad U_k^D = U_k^\mu \quad \text{in } \Omega \setminus D,$$

$$(7.8b) \qquad\qquad U_k^D \leq U_k^\mu \quad \text{in } \Omega,$$

then for each $w \in L^1(D)$ satisfying $(\Delta + k^2)w \geq 0$ in D we know that

$$(7.9) \qquad \int_D w(x)\,dx \geq \int w(x)\,d\mu(x).$$

Proof. By equations (7.8a) and (7.8b) $U_k^\mu - U_k^D \geq 0$ with equality in $\Omega \setminus D$. By Calderón–Zygmund estimates $U_k^\mu, U_k^D \in C^1(\Omega)$. Since $U_k^\mu - U_k^D$ attains its minimum in $\Omega \setminus D$ it holds that $\nabla U_k^\mu = \nabla U_k^D$ in $\Omega \setminus D$. When combined with (7.8a) and (7.8b) we conclude that

$$(7.10a) \qquad \int_D \partial^\alpha \tilde{\Phi}_k(z - x)\,dx = \int_D \partial^\alpha \tilde{\Phi}_k(z - x)\,d\mu(x) \quad \text{for all } z \in B_{R(n,k)} \setminus D, |\alpha| \leq 1,$$

$$(7.10b) \qquad \int_D \tilde{\Phi}_k(z - x)\,dx \leq \int_D \tilde{\Phi}_k(z - x)\,d\mu(x) \quad \text{for all } z \in D.$$

Let w be the function as in the statement of the lemma, and use Proposition 7.6 to find a sequence

$$w_j \in \text{span}_+ \left(\begin{array}{c} \{\pm\partial^\alpha \tilde{\Phi}_k(z - \cdot)|_D : z \in B_{R(n,k)} \setminus D, |\alpha| \leq 1\} \\ \cup\{-\tilde{\Phi}_k(z - \cdot)|_D : z \in B_{R(n,k)} \setminus D\} \end{array} \right)$$

with $w_j \to w \in L^1(D)$. From (7.10a) and (7.10b), we know that

$$(7.11) \qquad \int_D w_j(x)\,dx \geq \int_D w_j(x)\,d\mu(x) \quad \text{for all } j.$$

Since $\mu \in L^\infty(D)$ Hölder's inequality implies that

$$\left| \int_D (w_j(x) - w(x))\,d\mu(x) \right| \leq \|\mu\|_{L^\infty(D)} \|w - w_j\|_{L^1(D)}.$$

Taking the limit $j \to \infty$ in (7.11) we therefore arrive at

$$\int_D w(x)\,dx \geq \int_D w(x)\,d\mu(x).$$

This is the desired inequality (7.9), and thus completes the proof. $\qquad \square$

We are now ready to prove Theorem 7.1.

Proof of Theorem 7.1. **Step 1: Constructing the k-quadrature domain.** For $\epsilon < \delta < R(n,k)$ to be chosen set

$$(7.12) \qquad h_\delta = \frac{1}{c_{n,k,\delta}^{\text{MVT}}} \chi_{B_\delta}.$$

Then

$$(7.13) \qquad \mu * h_\delta(x) = \frac{\mu(B_\delta(x))}{c_{n,k,\delta}^{\text{MVT}}}$$

is non-negative and supported in $B_{\epsilon+\delta}$. Furthermore for all $x \in B_{\delta-\epsilon}$,

$$(7.14) \qquad \mu * h_\delta(x) = \frac{\mu(B_\delta(x))}{c_{n,k,\delta}^{\text{MVT}}} = \frac{\mu(\mathbb{R}^n)}{c_{n,k,\delta}^{\text{MVT}}}.$$

Let us for $\kappa \geq 1$ and $r > 0$ define

$$\mu_{\kappa,r} := \kappa \chi_{B_r} \mathsf{m}.$$

Set

$$\mu_1 := \mu_{\kappa,r} \quad \text{and} \quad \mu_2 := \mu * h_\delta - \mu_{\kappa,r}$$

with

$$(7.15) \qquad 1 < \kappa \leq \frac{\mu(\mathbb{R}^n)}{c_{n,k,\delta}^{\mathrm{MVT}}} \quad \text{and} \quad 0 < r \leq \delta - \epsilon$$

to be chosen. Note that these choices of r, κ imply that the measures μ_1, μ_2 are non-negative. Furthermore, both measures have bounded densities with respect to Lebesgue measure and

$$\mu_1(\mathbb{R}^n) = \kappa\mathsf{m}(B_r) \quad \text{and} \quad \mu_2(\mathbb{R}^n) = \frac{\mu(\mathbb{R}^n)\mathsf{m}(B_\delta)}{c_{n,k,\delta}^{\mathrm{MVT}}} - \kappa\mathsf{m}(B_r).$$

By (7.5) we know that $(\mu_1 + \mu_2)(\mathbb{R}^n) \leq (\mathrm{Bal}_k(\mu_1) + \mu_2)(\mathbb{R}^n)$. Let $r' > r$ be such that

$$(7.16) \qquad \mathrm{Bal}_k(\mu_{\kappa,r}) = \mu_{1,r'}.$$

Consequently, if

$$(7.17) \qquad 0 < k < c_n \min\left\{\frac{1}{r'}, \frac{1}{\delta+\epsilon}, \frac{1}{((c_{n,k,\delta}^{\mathrm{MVT}})^{-1}\mu(\mathbb{R}^n)\delta^n + (r')^n - \kappa r^n)^{1/n}}\right\},$$

then Proposition 6.1 implies that

$$(7.18) \qquad \mathrm{Bal}_k(\mu * h_\delta) = \mathrm{Bal}_k(\mu_1 + \mu_2) = \mathrm{Bal}_k(\mathrm{Bal}_k(\mu_1) + \mu_2) = \mathrm{Bal}_k(\mu_{1,r'} + \mu * h_\delta - \mu_{\kappa,r}).$$

By construction

$$(7.19) \qquad \mu_{1,r'} + \mu * h_\delta - \mu_{\kappa,r} = 0 \quad \text{outside } B_{r'} \cup B_{\epsilon+\delta}$$

and

$$(7.20) \qquad \mu * h_\delta - \mu_{\kappa,r} \geq 0.$$

Therefore,

$$(7.21) \qquad \mu_{1,r'} + \mu * h_\delta - \mu_{\kappa,r} \geq 1 \quad \text{in } B_{r'}.$$

Consequently, if we choose our parameters to satisfy

$$(7.22) \qquad r' > \epsilon + \delta,$$

then the fact that $\mu * h_\delta \in L^\infty(\mathbb{R}^n)$ combined with Proposition 5.1 implies

$$\mathrm{Bal}_k(\mu_{1,r'} + \mu * h_\delta - \mu_{\kappa,r}) = \chi_{D(\mu_{1,r'}+\mu*h_\delta-\mu_{\kappa,r})}\mathsf{m}.$$

Combining (7.18) and (7.23), we have

$$(7.23) \qquad \mathrm{Bal}_k(\mu * h_\delta) = \mathrm{Bal}_k(\mu_{1,r'} + \mu * h_\delta - \mu_{\kappa,r}) = \chi_{D(\mu_{1,r'}+\mu*h_\delta-\mu_{\kappa,r})}\mathsf{m},$$

which implies [2]

$$(7.24) \qquad \mathrm{Bal}_k(\mu * h_\delta) = \chi_{D(\mu*h_\delta)}\mathsf{m} \quad \text{and} \quad D(\mu * h_\delta) = D(\mu_{1,r'} + \mu * h_\delta - \mu_{\kappa,r}).$$

Under these assumptions, we can argue as in [Gus90, Corollary 2.3] to prove that $D(\mu_{1,r'} + \mu * h_\delta - \mu_{\kappa,r})$, and therefore $D(\mu * h_\delta)$, is connected.

Using again that we shall choose our parameters so that $r' > \epsilon+\delta$ and thus $\overline{B_{r'}} = \mathrm{supp}(\mu_{1,r'} + \mu * h_\delta - \mu_{\kappa,r}) \subset D(\mu * h_\delta)$, we find

$$\mathrm{supp}(\mu) \subset \mathrm{supp}(\mu * h_\delta) \subset B_{r'} \subset D(\mu * h_\delta).$$

---

[2]Note that $\mu * h_\delta$ does not necessarily have a density which is greater than 1 on its support and so the structure of its partial balayage in (7.23) does not follow from Proposition 5.1.

By Lemma 4.9,

$$(7.25\text{a}) \qquad U_k^{D(\mu*h_\delta)} = U_k^{\mu*h_\delta} = \tilde{\Phi}_k * \mu * h_\delta \quad \text{in } B_{R(n,k)} \setminus D(\mu*h_\delta), \text{ and}$$

$$(7.25\text{b}) \qquad U_k^{D(\mu*h_\delta)} \le U_k^{\mu*h_\delta} = \tilde{\Phi}_k * \mu * h_\delta \quad \text{in } B_{R(n,k)}.$$

Under our assumptions, Corollary 7.7 implies that $D(\mu*h_\delta)$ is a k-quadrature domain for $\mu*h$ and furthermore we have the quadrature inequality for sub-solutions.

The MVT (Proposition A.5) implies that $\tilde{\Phi}_k * h_\delta(y) \le \tilde{\Phi}_k(y)$ for all $y \in \mathbb{R}^n$ and equality holds if $|y| > \delta$. Therefore, by the non-negativity of μ,

$$U_k^{\mu*h_\delta}(x) = \tilde{\Phi}_k * \mu * h_\delta(x) = \int (\tilde{\Phi}_k * h_\delta)(x-y)d\mu(y) \le \tilde{\Phi}_k * \mu(x) = U_k^\mu(x)$$

with equality if $\text{dist}(x, \text{supp}(\mu)) > \delta$. In particular, since under our assumptions

$$\{x \in \mathbb{R}^n : \text{dist}(x, \text{supp}(\mu)) \le \delta\} \subset B_{\epsilon+\delta} \subset B_{r'} \subset D(\mu*h_\delta)$$

we have equality for $x \in B_{R(n,k)} \setminus D(\mu*h)$. We have thus arrived at [3]

$$(7.26\text{a}) \qquad U_k^{D(\mu*h)} = U_k^\mu \quad \text{in } B_{R(n,k)} \setminus D(\mu*h_\delta), \text{ and}$$

$$(7.26\text{b}) \qquad U_k^{D(\mu*h)} \le U_k^\mu \quad \text{in } B_{R(n,k)}.$$

We now show that we can choose the parameters r, δ, κ appropriately only depending on the measure μ, specifically we shall choose them depending on $\epsilon, \mu(\mathbb{R}^n)$. We choose $\delta = 2\epsilon$ and $r = \epsilon$, and let $\gamma = k\epsilon$.

Since $t \mapsto t^{-\frac{n}{2}} J_{\frac{n}{2}}(t)$ is a decreasing function on $[0, j_{\frac{n-2}{2},1}]$ satisfying that $\lim_{y \to 0_+} y^{-\frac{n}{2}} J_{\frac{n}{2}}(y) = \frac{2^{-\frac{n}{2}}}{\Gamma(1+\frac{n}{2})}$, we by using explicit form of $c_{n,k,r}^{\text{MVT}}$ find that

$$(7.27) \qquad (c_{n,k,2\epsilon}^{\text{MVT}})^{-1}(2\epsilon)^n = \frac{\gamma^{\frac{n}{2}}}{\pi^{\frac{n}{2}} J_{\frac{n}{2}}(2\gamma)} \ge \frac{\Gamma(1+\frac{n}{2})}{\pi^{\frac{n}{2}}}.$$

The required bound on k (7.17) is then valid if

$$(7.28) \qquad 0 < k < c_n \min\left\{\frac{1}{r'}, \frac{1}{\epsilon}, \frac{1}{(\pi^{-\frac{n}{2}}\Gamma(1+\frac{n}{2})\mu(\mathbb{R}^n) + (r')^n)^{1/n}}\right\}.$$

Assume that $c_n \le j_{\frac{n-2}{2},1}/4$ so that $0 \le \gamma \le j_{\frac{n-2}{2},1}/4$. Then, since $t \mapsto t^{\frac{n}{2}} J_{\frac{n}{2}}(t)$ is strictly increasing on $[0, j_{\frac{n-2}{2},1}]$ we can choose

$$\kappa = \frac{c_{n,k,4\gamma}^{\text{MVT}}}{c_{n,k,\gamma}^{\text{MVT}}} = \frac{(4\gamma)^{\frac{n}{2}} J_{\frac{n}{2}}(4\gamma)}{\gamma^{\frac{n}{2}} J_{\frac{n}{2}}(\gamma)} = \frac{2^n J_{\frac{n}{2}}(4\gamma)}{J_{\frac{n}{2}}(\gamma)} > 1.$$

Since $\gamma = k\epsilon = kr$, by Lemma 7.4 we know that (7.16) holds with

$$r' = 4r = 4\epsilon > 3\epsilon = \epsilon + \delta,$$

thus (7.22) is satisfied.

---

[3] If $\mu \in L^\infty(B_{R(n,k)})$ then another application of Lemma 4.9 combined with (7.26a) and (7.26b) implies that

$$\text{Bal}_k(\mu) = \text{Bal}_k(\mu*h_\delta) = \chi_{D(\mu*h_\delta)}\mathsf{m}.$$

We now want to find a sufficient condition so that (7.15) holds. By the choice of κ and the definition of γ what we need to verify is the inequality

$$(7.29) \qquad \kappa = \frac{2^n J_{\frac{n}{2}}(4\gamma)}{J_{\frac{n}{2}}(\gamma)} \leq \frac{\mu(\mathbb{R}^n)}{c_{n,k,2\epsilon}^{\mathrm{MVT}}} = \frac{\mu(\mathbb{R}^n)\gamma^{\frac{n}{2}}}{(2\epsilon)^n \pi^{\frac{n}{2}} J_{\frac{n}{2}}(2\gamma)},$$

or equivalently,

$$4^n \pi^{\frac{n}{2}} \frac{J_{\frac{n}{2}}(4\gamma) J_{\frac{n}{2}}(2\gamma)}{J_{\frac{n}{2}}(\gamma)\gamma^{\frac{n}{2}}} \epsilon^n \leq \mu(\mathbb{R}^n).$$

Since the function

$$\gamma \mapsto \frac{J_{\frac{n}{2}}(4\gamma) J_{\frac{n}{2}}(2\gamma)}{J_{\frac{n}{2}}(\gamma)\gamma^{\frac{n}{2}}}$$

is continuous it is bounded from above for all $\gamma \in [0, j_{\frac{n-2}{2},1}/4]$ by a constant depending only on n. Therefore, the require bound (7.15) holds if we assume that

$$(7.30) \qquad \epsilon \leq c_n \mu(\mathbb{R}^n)^{1/n}$$

provided c_n is sufficiently small.

Since we chose our parameters so that $r' = 4\epsilon$, the require bound on k (7.28) (and thus also (7.17)) is valid if

$$(7.31) \qquad 0 < k < \frac{c_n}{\mu(\mathbb{R}^n)^{1/n}}.$$

Consequently, all our requirements are met provided

$$(7.32) \qquad 0 < k < \frac{c_n}{\mu(\mathbb{R}^n)^{1/n}} \quad \text{and} \quad \epsilon \leq c_n \mu(\mathbb{R}^n)^{1/n}$$

for some constant c_n depending only on n.

From here and on we let h denote the function h_δ with the particular choice $\delta = 2\epsilon$ in the discussion above. By the construction we have that

$$(7.33) \qquad \mathrm{Bal}_k(\mu * h) = \chi_{D(\mu*h)}\mathsf{m}$$

and

$$(7.34) \qquad \mathrm{supp}(\mu) \subset \overline{B_\epsilon} \subset \mathrm{supp}(\mu * h) \subset B_{4\epsilon} \subset D(\mu * h).$$

We now show that $D(\mu * h) = \omega(\mu * h)$. Using (7.14) and (7.29), we have

$$\mu * h_{2\epsilon} \geq \frac{\mu(\mathbb{R}^n)}{c_{n,k,2\epsilon}^{\mathrm{MVT}}} \chi_{B_\epsilon}\mathsf{m} \geq \frac{c_{n,k,4\epsilon}^{\mathrm{MVT}}}{c_{n,k,\epsilon}^{\mathrm{MVT}}} \chi_{B_\epsilon}\mathsf{m}.$$

From (7.1), we verify that

$$0 < 10\epsilon < 10c_n^2 k^{-1} < \frac{1}{2}j_{\frac{n-2}{2},1}k^{-1} = R(n,k).$$

Therefore from Remark 6.2 and Lemma 7.4 we know that

$$\omega(\mu * h) \supset \omega\left(\frac{c_{n,k,4\epsilon}^{\mathrm{MVT}}}{c_{n,k,\epsilon}^{\mathrm{MVT}}} \chi_{B_\epsilon}\mathsf{m}\right) = B_{4\epsilon} \supset \mathrm{supp}(\mu * h)$$

By definition of $\omega(\mu * h)$, we see that (7.26a) and (7.26b) hold by replacing $D(\mu * h)$ with $\omega(\mu * h)$. Therefore applying Lemma 5.4 with $u = U_k^{\mu * h} - U_k^{\omega(\mu * h)}$ (note that $\mu * h$ is bounded) we know that

$$\text{Bal}_k(\mu * h) = \chi_{\omega(\mu * h)}\mathsf{m}.$$

Combining with (7.33) we conclude that $D(\mu * h) \equiv \omega(\mu * h)$.

Step 2: $D(\mu * h) \equiv \omega(\mu * h)$ **is a k-quadrature domain.** The equations (7.26a) and (7.26b) imply that $U_k^{\mu} - U_k^{D(\mu * h)} \geq 0$ with equality in $B_{R(n,k)} \setminus D(\mu * h)$. By elliptic regularity $U_k^{D(\mu * h)}$ is $C^1(B_{R(n,k)})$ and U_k^{μ} is smooth away from $\text{supp}(\mu)$. Thus $U_k^{\mu} - U_k^{D(\mu * h)}$ attains its minimum in $B_{R(n,k)} \setminus D(\mu * h)$, and consequently

$$(7.35) \qquad \nabla U_k^{D(\mu * h)} = \nabla U_k^{\mu} \quad \text{in } B_{R(n,k)} \setminus D(\mu * h).$$

Since $\text{supp}(\mu) \subset D(\mu * h) \subset B_{R(n,k)}$ the extension by zero of $U_k^{\mu} - U_k^{D(\mu * h)}$ to all of $\mathbb{R}^n$ satisfies the assumptions of Proposition 2.1 and hence $D(\mu * h)$ is a k-quadrature domain for μ.

As noted above, (7.25a), (7.25b) and Corollary 7.7 imply that

$$\int_{D(\mu * h)} w(x)\,dx \geq \int w(x)\,d(\mu * h)(x)$$

for all $w \in L^1(D(\mu * h))$ satisfying $(\Delta + k^2)w(x) \geq 0$ in $D(\mu * h)$. Assume further that $w \in L^1(\mu)$. By Fubini and since by construction $B_\delta(y) \in D(\mu * h)$ for all $y \in \text{supp}(\mu)$ the integral above can be rewritten as

$$\int_{D(\mu * h)} w(x)\,d(\mu * h)(x) = \int \left(\int_{D(\mu * h)} w(x) h_\delta(x - y)\,dx \right) d\mu(y)$$
$$= \int \left(\frac{1}{c_{n,k,\delta}^{\text{MVT}}} \int_{B_\delta(y)} w(x)\,dx \right) d\mu(y).$$

Since $B_\delta(y) \subset D(\mu * h)$ for all $y \in \text{supp}(\mu)$ the function $w(x)$ is a sub-solution of the Helmholtz equation on $B_\delta(y)$ for all $y \in \text{supp}(\mu)$. Therefore, the mean value inequality for sub-solutions of the Helmholtz equation implies that the expression within the parenthesis is greater than $w(y)$ for each $y \in \text{supp}(\mu)$. Since $w \in L^1(\mu)$ and μ is non-negative this proves (7.2).

Step 3: Regularity of $\partial D(\mu * h) \equiv \partial\omega(\mu * h)$. We now show that $D(\mu * h)$ has real-analytic boundary $\partial D(\mu * h)$ by using the moving plane technique as in [Gus04, Theorem 5.4]. Set $u := U_k^{\mu * h} - U_k^{D(\mu * h)} \in \bigcap_{0 < \alpha < 1} C^{1,\alpha}(\overline{B_{R(n,k)}})$. By Proposition 4.3, we know that u is the smallest among all $w \in H_0^1(B_{R(n,k)})$ satisfying

$$(7.36) \qquad w \geq 0 \quad \text{and} \quad (\Delta + k^2)w \leq -\mu * h + 1 \quad \text{in } B_{R(n,k)}.$$

Moreover, Lemma 4.9 and the fact that $\text{Bal}_k(\mu * h) = \chi_{D(\mu * h)}\mathsf{m}$ implies

$$(7.37) \qquad u = 0 \quad \text{in } B_{R(n,k)} \setminus D(\mu * h).$$

Given $x_0 \in \partial D(\mu * h)$ there by (7.34) exists a hyperplane that separates $\text{supp}(\mu * h)$ and x_0. Since Laplacian is translation and rotation invariant, without loss of generality, we may assume that the hyperplane is

$$(7.38) \qquad \left\{ \, x \in \mathbb{R}^n \mid x_n = 0 \, \right\},$$

and that $\text{supp}(\mu * h) \subset \left\{ \, x \in \mathbb{R}^n \mid x_n < 0 \, \right\}$. We define

$$(D(\mu * h))_{\text{loc}} = D(\mu * h) \cap \left\{ \, x \in \mathbb{R}^n \mid x_n > 0 \, \right\}$$

and
$$(\partial D(\mu * h))_{\mathrm{loc}} = \partial D(\mu * h) \cap \left\{ \ x \in \mathbb{R}^n \ \middle| \ x_n > 0 \ \right\}.$$

Let u^* be the reflection of u with respect to the hyperplane (7.38), that is, $u^*(x', x_n) = u^*(x', -x_n)$. We now define
$$v = u - \inf\{u, u^*\} = (u - u^*)_+.$$

Since $(\Delta + k^2)u \leq -\mu * h + 1 \leq 1$ in $B_{R(n,k)}$, we have $(\Delta + k^2)u^* \leq 1$ in $B_{R(n,k)}$. Since there exists a unique $\phi \in H_0^1(B_{R(n,k)})$ such that $(\Delta + k^2)\phi = 1$ in $B_{R(n,k)}$, using Proposition A.3, we know that
$$(\Delta + k^2)\inf\{u, u^*\} \leq 1 \quad \text{in } B_{R(n,k)}.$$

From $(\Delta + k^2)u = 1$ in $(D(\mu * h))_{\mathrm{loc}}$, we have
$$(\Delta + k^2)v \geq 0 \quad \text{in } (D(\mu * h))_{\mathrm{loc}}.$$

The boundary condition $v = 0$ on $\partial((D(\mu * h))_{\mathrm{loc}})$ and using maximum principle in Proposition A.2 yield
$$v \leq 0 \quad \text{in } (D(\mu * h))_{\mathrm{loc}}.$$

Thus we have
$$\frac{\partial u}{\partial x_n} \leq 0 \quad \text{on } \left\{ \ x \in \mathbb{R}^n \ \middle| \ x_n = 0 \ \right\}.$$

From (7.37), we know that $\frac{\partial u}{\partial x_n} = 0$ on $(\partial D(\mu * h))_{\mathrm{loc}}$. On the other hand, we know that
$$(\Delta + k^2)u = 1 \quad \text{in } (D(\mu * h))_{\mathrm{loc}}.$$

Hence $\frac{\partial u}{\partial x_n} \in C^\infty((D(\mu * h))_{\mathrm{loc}}) \cap \bigcap_{0 < \alpha < 1} C^{0,\alpha}(\overline{(D(\mu * h))_{\mathrm{loc}}})$ and it satisfies
$$(\Delta + k^2)\frac{\partial u}{\partial x_n} = \frac{\partial}{\partial x_n}(\Delta + k^2)u = 0 \quad \text{in } (D(\mu))_{\mathrm{loc}}.$$

Applying the strong maximum principle in Proposition A.2 on $\frac{\partial u}{\partial x_n}$, we obtain $\frac{\partial u}{\partial x_n} < 0$ in $(D(\mu * h))_{\mathrm{loc}}$ (because $\frac{\partial u}{\partial x_n} \not\equiv 0$).

Since x_0 can be separated from $\mathrm{supp}(\mu * h)$ by hyperplanes whose normals form an open convex cone, this argument implies that in a neighbourhood of x_0 the function u is decreasing in a cone of directions. We deduce that in a neighbourhood of x_0 the free boundary $\partial D(\mu * h)$ is the graph of a Lipschitz function. Since the choice of $x_0 \in \partial D(\mu * h)$ was arbitrary, we conclude that the free boundary $\partial D(\mu * h)$ is locally a Lipschitz graph. Using [Caf77, Caf80, Caf98], we know that $\partial D(\mu * h)$ is C^1, and then from [KN77] we conclude that $\partial D(\mu * h)$ is real-analytic. We also refer to the monograph [Fri88] for the general regularity theory for free boundaries.

Step 4: Uniqueness. Let $\tilde{D}$ be any domain such that (7.2) holds with $\overline{\tilde{D}} \subset B_{R(n,k)}$. We define
$$u_1 := U_k^{\mu * h} - U_k^{\tilde{D}} \quad \text{in } B_{R(n,k)}.$$

We observe that $u_1 \in H_0^1(B_{R(n,k)})$ and it satisfies
$$(\Delta + k^2)u_1 = -\mu * h + \chi_{\tilde{D}} \leq -\mu * h + 1 \quad \text{in } B_{R(n,k)}.$$

By (7.36), $u_1 \geq u$ in $B_{R(n,k)}$. Consequently,
$$(7.39) \qquad u_2 := u_1 - u = U_k^{D(\mu * h)} - U_k^{\tilde{D}} \geq 0 \quad \text{in } B_{R(n,k)}.$$

Since $\tilde{D}$ is a k-quadrature domain corresponding to μ, we have $U_k^{\tilde{D}} = U_k^{\mu}$ in $B_{R(n,k)} \setminus \tilde{D}$. Since $U_k^{D(\mu*h)} \leq U_k^{\mu}$ in $B_{R(n,k)}$ have that

$$(7.40) \qquad u_2 = U_k^{D(\mu*h)} - U_k^{\mu} \leq 0 \quad \text{in } B_{R(n,k)} \setminus \tilde{D}.$$

So, in fact, $u_2 = 0$ in $B_{R(n,k)} \setminus \tilde{D}$. Furthermore,

$$(\Delta + k^2)u_2 = -\chi_{D(\mu*h)} + \chi_D \geq 0 \quad \text{in } \tilde{D}.$$

Thus the maximum principle in Proposition A.2 combined with that $u_2 = 0$ on $\partial\tilde{D}$ implies that

$$(7.41) \qquad u_2 \leq 0 \quad \text{in } \tilde{D}.$$

Therefore, u_2 must vanish also in $\tilde{D}$. This completes the proof with $D \equiv D(\mu * h) \equiv \omega(\mu * h)$. $\qquad \square$

Appendix A. Auxiliary propositions

A.1. **A real-valued fundamental solution.** In this section we give an exact expression for a real-valued radial fundamental solution to the Helmholtz equation. This solution is positive in a ball with suitable radius, which is crucial for our construction of k-quadrature domains.

Proposition A.1. *Fix $k > 0$ and $n \geq 2$. For any $R > 0$, there exists a distribution $\tilde{\Phi}_{k,R}$ in $\mathbb{R}^n$ which is radial, smooth outside the origin and satisfies*

$$(A.1) \qquad \begin{cases} (\Delta + k^2)\tilde{\Phi}_{k,R} = -\delta & \text{in } \mathcal{D}'(\mathbb{R}^n), \\ \tilde{\Phi}_{k,R}(x) = 0 & \text{for } x \in \partial B_R(0). \end{cases}$$

Furthermore, in the case when $0 < R < j_{\frac{n-2}{2}} k^{-1}$, the distribution $\tilde{\Phi}_{k,R}$ is positive in $B_R(0)$.

Proof. Let $\tilde{\Phi}_{k,R}$ be given by

$$(A.2) \quad \tilde{\Phi}_{k,R}(x) = \frac{k^{\frac{n-2}{2}}}{4(2\pi)^{\frac{n-2}{2}} J_{\frac{n-2}{2}}(kR)} |x|^{-\frac{n-2}{2}} \left(Y_{\frac{n-2}{2}}(kR) J_{\frac{n-2}{2}}(k|x|) - J_{\frac{n-2}{2}}(kR) Y_{\frac{n-2}{2}}(k|x|) \right).$$

Clearly $\tilde{\Phi}_{k,R}$ is smooth outside the origin and vanishes on $\partial B_R(0)$. A standard calculation verifies that in the distributional sense $(\Delta + k^2)\tilde{\Phi}_{k,R} = -\delta$.

It remains to prove that $\tilde{\Phi}_{k,R}$ is positive in $B_R(0)$. By the asymptotic behaviour of Bessel functions at 0 one readily checks that $\lim_{|x|\to 0} \tilde{\Phi}_{k,R}(x) = +\infty$. Similarly, one computes

$$(A.3) \qquad \begin{aligned} &\frac{x}{|x|} \cdot \nabla \tilde{\Phi}_{k,R}(x) \\ &= -\frac{k^{\frac{n}{2}}}{4(2\pi)^{\frac{n-2}{2}} J_{\frac{n-2}{2}}(kR))} |x|^{-\frac{n-2}{2}} \left(Y_{\frac{n-2}{2}}(kR) J_{\frac{n}{2}}(k|x|) - J_{\frac{n-2}{2}}(kR) Y_{\frac{n}{2}}(k|x|) \right). \end{aligned}$$

When $|x| = R$ the right-hand side simplifies to

$$(A.4) \qquad -\frac{k^{\frac{n-2}{2}}}{(2\pi)^{\frac{n-2}{2}} J_{\frac{n-2}{2}}(kR)} R^{-\frac{n}{2}},$$

which is negative since $J_{\frac{n-2}{2}}$ is positive up to its first zero. Thus in order to show positivity of $\tilde{\Phi}_{k,R}$ in $B_R(0)$, it is enough to show that there is at most one $r \in (0, R)$ so that the radial derivative of $\tilde{\Phi}_{k,R}$ vanishes when $|x| = r$.

Finally, if $r \in (0, R)$ is such that the radial derivative vanishes when $|x| = r$ the expression for the derivative implies that

$$(A.5) \qquad \frac{Y_{\frac{n}{2}}(kr)}{J_{\frac{n}{2}}(kr)} = \frac{Y_{\frac{n-2}{2}}(kR)}{J_{\frac{n-2}{2}}(kR)}.$$

Since

$$(A.6) \qquad \frac{d}{dr}\frac{Y_{\frac{n}{2}}(kr)}{J_{\frac{n}{2}}(kr)} = \frac{2}{\pi r J_{\frac{n}{2}}(kr)^2}$$

is positive on $(0, R)$ we conclude that there is at most one solution of (A.5) in $(0, R)$. This concludes the proof. $\qquad\square$

A.2. **Maximum principle.** We will need the following (generalized) maximum principle and properties of sub/super-solutions in small domains. These are well known, but for completeness we give a proof based on using a positive solution as in [PW84] (variational proofs would also be possible, see e.g. [GT01]).

Proposition A.2. *Fix $k > 0$, $n \geq 2$, and $0 < R < j_{\frac{n-2}{2},1}k^{-1}$. Let U be any open set in $\mathbb{R}^n$ with $U \subset B_R$. If $w \in H^1(U)$ satisfies $w \leq 0$ on ∂U (i.e. $w_+ := \max\{w, 0\} \in H_0^1(U)$) and $(\Delta + k^2)w \geq 0$ in the sense of $H^{-1}(U)$, then $w \leq 0$ in U. If we further assume that $w \in C^2(U) \cap C^0(\overline{U})$ and $w = 0$ on ∂U, then either w vanishes identically or $w < 0$ in U.*

Proposition A.3. *Fix $k > 0$, $n \geq 2$, and $0 < R < j_{\frac{n-2}{2},1}k^{-1}$. Let U be any open set in $\mathbb{R}^n$ with $U \subset B_R$. If $w_1, w_2 \in H^1(U)$ satisfy $(\Delta + k^2)w_j \leq 0$ in the sense of $H^{-1}(U)$ for $j = 1$ and 2, then the same is true for $w = \min\{w_1, w_2\}$.*

Remark A.4. The statements of the propositions are valid in an open set $U \subset \mathbb{R}^n$ as soon as there exists a function $u_0 \in C^\infty(\overline{U})$ which is positive in $\overline{U}$ and satisfies $(\Delta + k^2)u_0 = 0$.

Proof of Propositions A.2 & A.3. Define

$$(A.7) \qquad u_0(x) = |x|^{-\frac{n-2}{2}} J_{\frac{n-2}{2}}(k|x|)$$

so that $u_0 \in C^\infty(\mathbb{R}^n)$ and $(\Delta + k^2)u_0(x) = 0$ in $\mathbb{R}^n$. Moreover, since $R < j_{\frac{n-2}{2},1}k^{-1}$ we find that $u > 0$ in $\overline{B_R}$.

Since u_0 is smooth and bounded away from zero $v = u_0^{-1}w$ belongs to $H^1(U)$ if and only if $v \in H^1(U)$.

Moreover, a direct computation yields that $(\Delta + k^2)w \geq 0$ if and only if $v = u_0^{-1}w$ satisfies

$$(A.8) \qquad u_0^{-1}\nabla \cdot u_0^2\nabla v \geq 0 \quad \text{in the sense of } H^{-1}(U).$$

By the smoothness and positivity of u_0 this is equivalent to

$$(A.9) \qquad \nabla \cdot u_0^2\nabla v \geq 0 \quad \text{in the sense of } H^{-1}(U).$$

In particular, the maximum principle (as formulated in [GT01, Theorem 8.19]) implies that $v \leq 0$ in U. Therefore $w = vu_0 \leq 0$.

We next prove that with the additional assumption that $w \in C^2(U) \cap C^0(\overline{U})$, then either w vanishes identically or is $w < 0$ in U. If $w < 0$ on ∂U, the statement immediately follows by combining the negativity in $H^{-1}(U)$ with the assumed regularity of w. It remains to consider the case when w is equals to zero at some point on ∂U. Assume there exists $x_* \in U$ such that $w(x_*) = 0$. Since $u_0 > 0$ in U this is the case if and only if $v(x_*) = 0$. By the strong maximum principle (as formulated in [GT01, Theorem 8.19]) we conclude that v and thus w vanishes identically. Hence we conclude that either w vanishes identically or $w < 0$ in U, which concludes the proof of Proposition A.2.

The same computation yields that if we set $v_j = u_0^{-1} w_j$ for w_j as in Proposition A.3, then

$$(A.10) \qquad \nabla \cdot u_0^2 \nabla v_j \leq 0 \quad \text{for } j = 1, 2 \text{ in the sense of } H^{-1}(U).$$

By [KS00, Theorem II.6.6] the same property holds for $v = \min\{v_1, v_2\}$. Since $u_0 > 0$ we find $w = \min\{w_1, w_2\} = u_0 \min\{v_1, v_2\} = u_0 v$. By the same argument as before (but in the opposite direction) one deduces that $(\Delta + k^2)w \leq 0$. This concludes the proof of Proposition A.3. $\quad \square$

Alternative proof of the first statement in Proposition A.2. We observe $\langle \nabla w_-, \nabla w_+ \rangle = 0$ and test the equation $(\Delta + k^2)w \geq 0$ in $H^{-1}(U)$ by $w_+ \in H_0^1(U)$. From this we have

$$\|\nabla w_+\|_{L^2(U)}^2 = \langle \nabla w, \nabla w_+ \rangle = -\langle \Delta w, w_+ \rangle \leq k^2 \|w_+\|_{L^2(U)}^2.$$

Using the Poincaré inequality, we have

$$\|w_+\|_{L^2(U)}^2 \leq R^2 j_{\frac{n-2}{2},1}^{-2} \|\nabla w_+\|_{L^2(U)}^2 \leq k^2 R^2 j_{\frac{n-2}{2},1}^{-2} \|w_+\|_{L^2(U)}^2.$$

Since $k^2 R^2 j_{\frac{n-2}{2},1}^{-2} < 1$, we conclude that $w_+ \equiv 0$. $\qquad \square$

A.3. The mean value theorem.

Proposition A.5. *Let $n \geq 2$ be an integer, and let $R > 0$ be any constant. If $u \in L^1(B_R(x_0))$ is a solution to*

$$(\Delta + k^2)u = 0 \quad \text{in } B_R(x_0),$$

then

$$(A.11) \qquad \int_{B_R(x_0)} u(x)\, dx = c_{n,k,R}^{\mathrm{MVT}} u(x_0) \quad \text{with} \quad c_{n,k,R}^{\mathrm{MVT}} = (2\pi)^{n/2} \frac{R^{\frac{n}{2}} J_{\frac{n}{2}}(kR)}{k^{\frac{n}{2}}}.$$

In addition, if we assume that $0 < R < j_{\frac{n-2}{2},1} k^{-1}$ and $u \in L^1(B_R(x_0))$ is a sub-solution of the Helmholtz equation,

$$(\Delta + k^2)u \geq 0 \quad \text{in } B_R(x_0),$$

then

$$(A.12) \qquad \int_{B_R(x_0)} u(x)\, dx \geq c_{n,k,R}^{\mathrm{MVT}} u(x_0),$$

and the mapping

$$r \mapsto \frac{1}{c_{n,k,r}^{\mathrm{MVT}}} \int_{B_r(x_0)} u(x)\, dx$$

is monotone non-decreasing on the interval $(0, R)$.

Remark A.6. In particular,

$$\text{when } n = 2, \quad c_{2,k,R}^{\text{MVT}} = \frac{2\pi R J_1(kR)}{k},$$

$$\text{when } n = 3, \quad c_{3,k,R}^{\text{MVT}} = \frac{4\pi(\sin(kR) - kR\cos(kR))}{k^3}.$$

Unlike the mean value theorem for harmonic functions, there are radii for which $c_{n,k,R}^{\text{MVT}}$ is zero or even negative.

Proof of Proposition A.5. Without loss of generality, we may assume that $x_0 = 0$. For each $0 < r < R$, using Proposition A.1 as well as (A.3) and (A.4), we have

$$
\begin{aligned}
u(0) &= -\int_{B_r} (\Delta + k^2)\tilde{\Phi}_{k,r}(x)u(x)\,dx \\
&= -\int_{B_r} \left[\Delta\tilde{\Phi}_{k,r}(x)u(x) - \tilde{\Phi}_{k,r}(x)\Delta u(x)\right] dx - \int_{B_r} \tilde{\Phi}_{k,r}(x)(\Delta + k^2)u(x)\,dx \\
&= -\int_{\partial B_r} \left[\partial_{|x|}\tilde{\Phi}_{k,r}(x)u(x) - \overbrace{\tilde{\Phi}_{k,r}(x)}^{=0}\partial_{|x|}u(x)\right] dS - \int_{B_r} \tilde{\Phi}_{k,r}(x)(\Delta + k^2)u(x)\,dx \\
&= \frac{k^{\frac{n-2}{2}}}{(2\pi)^{\frac{n}{2}}J_{\frac{n-2}{2}}(kr)}r^{-\frac{n}{2}}\int_{\partial B_r} u(x)\,dS - \int_{B_r} \tilde{\Phi}_{k,r}(x)(\Delta + k^2)u(x)\,dx,
\end{aligned}
$$

(A.13)

that is,

$$\frac{k^{\frac{n-2}{2}}}{(2\pi)^{\frac{n}{2}}}\int_{\partial B_r} u(x)\,dS = J_{\frac{n-2}{2}}(kr)r^{\frac{n}{2}}u(0) + J_{\frac{n-2}{2}}(kr)r^{\frac{n}{2}}\int_{B_r}\tilde{\Phi}_{k,r}(x)(\Delta + k^2)u(x)\,dx.$$

If $(\Delta + k^2)u = 0$ in B_R, then

$$\frac{k^{\frac{n-2}{2}}}{(2\pi)^{\frac{n}{2}}}\int_{B_R} u(x)\,dx = \left(\int_0^R J_{\frac{n-2}{2}}(kr)r^{\frac{n}{2}}\,dr\right)u(0) = \frac{R^{\frac{n}{2}}J_{\frac{n}{2}}(kR)}{k}u(0),$$

which proves (A.11). By an almost identical argument (A.12) can be deduced from the positivity of $\tilde{\Phi}_{k,R}$ in $B_R(0)$.

Since for each $0 < t < j_{\frac{n-2}{2},1}k^{-1}$ and $0 < |x| < j_{\frac{n-2}{2},1}k^{-1}$ we have

$$
\begin{aligned}
\frac{\partial}{\partial t}\tilde{\Phi}_{k,t}(x) &= \frac{k^{\frac{n-2}{2}}}{4(2\pi)^{\frac{n-2}{2}}}|x|^{-\frac{n-2}{2}}J_{\frac{n-2}{2}}(k|x|)\frac{\partial}{\partial t}\left(\frac{Y_{\frac{n-2}{2}}(kt)}{J_{\frac{n-2}{2}}(kt)}\right) \\
&= \frac{k^{\frac{n-2}{2}}}{2(2\pi)^{\frac{n-2}{2}}\pi}|x|^{-\frac{n-2}{2}}\frac{J_{\frac{n-2}{2}}(k|x|)}{tJ_{\frac{n-2}{2}}(kt)^2} > 0,
\end{aligned}
$$

then for each $0 < s < r < R < j_{\frac{n-2}{2},1}k^{-1}$ we have

(A.14) $$0 \leq \tilde{\Phi}_{k,s} \leq \tilde{\Phi}_{k,r} \text{ in } B_s \quad \text{and} \quad \tilde{\Phi}_r \geq 0 \text{ in } B_r.$$

Since $(\Delta + k^2)u \geq 0$ in B_R, then from (A.13) and (A.14) we have

$$\int_{B_r}\tilde{\Phi}_{k,r}(x)(\Delta + k^2)u(x)\,dx \geq \int_{B_s}\tilde{\Phi}_{k,r}(x)(\Delta + k^2)u(x)\,dx \geq \int_{B_s}\tilde{\Phi}_{k,s}(x)(\Delta + k^2)u(x)\,dx,$$

which implies

(A.15) $$\frac{1}{r^{\frac{n}{2}} J_{\frac{n-2}{2}}(kr)} \int_{\partial B_r} u(x)\, dS \geq \frac{1}{s^{\frac{n}{2}} J_{\frac{n-2}{2}}(ks)} \int_{\partial B_s} u(x)\, dS$$

for all $0 < s < r < R < j_{\frac{n-2}{2},1} k^{-1}$. Note that

(A.16)
$$\frac{\partial}{\partial r}\left(\frac{1}{c_{n,k,r}^{\mathrm{MVT}}} \int_{B_r} u(x)\, dx\right)$$
$$= (2\pi)^{n/2} k^{\frac{n}{2}} \left[\frac{\partial}{\partial r}\left(\frac{1}{r^{\frac{n}{2}} J_{\frac{n}{2}}(kr)}\right) \int_{B_r} u(x)\, dx + \frac{1}{r^{\frac{n}{2}} J_{\frac{n}{2}}(kr)} \int_{\partial B_r} u(x)\, dS \right]$$
$$= (2\pi)^{n/2} k^{\frac{n}{2}} \left[-\frac{k J_{\frac{n-2}{2}}(kr)}{r^{\frac{n}{2}} J_{\frac{n}{2}}(kr)^2} \int_0^r s^{\frac{n}{2}} J_{\frac{n-2}{2}}(ks) \left(\frac{1}{s^{\frac{n}{2}} J_{\frac{n-2}{2}}(ks)} \int_{\partial B_s} u(x)\, dS\right) ds \right.$$
$$\left. + \frac{1}{r^{\frac{n}{2}} J_{\frac{n}{2}}(kr)} \int_{\partial B_r} u(x)\, dS \right].$$

From (A.15) and (A.16) we have

$$\frac{\partial}{\partial r}\left(\frac{1}{c_{n,k,r}^{\mathrm{MVT}}} \int_{B_r} u(x)\, dx\right)$$
$$\geq \left((2\pi)^{n/2} k^{\frac{n}{2}} r^{-\frac{n}{2}} \int_{\partial B_r} u(x)\, dS\right)\left(-\frac{1}{r^{\frac{n}{2}} J_{\frac{n}{2}}(kr)^2} \int_0^r k s^{\frac{n}{2}} J_{\frac{n-2}{2}}(ks)\, ds + \frac{1}{J_{\frac{n}{2}}(kr)}\right)$$
$$\geq \left((2\pi)^{n/2} k^{\frac{n}{2}} r^{-\frac{n}{2}} \int_{\partial B_r} u(x)\, dS\right)\left(-\frac{1}{r^{\frac{n}{2}} J_{\frac{n}{2}}(kr)^2} r^{\frac{n}{2}} J_{\frac{n}{2}}(kr) + \frac{1}{J_{\frac{n}{2}}(kr)}\right)$$
$$= 0$$

for all $0 < r < R < j_{\frac{n-2}{2},1} k^{-1}$, which conclude Proposition A.5. $\square$

A.4. **Laplacian under analytic maps.** The following classical fact was used for seeing how the Helmholtz equation changes under an an analytic change of coordinates.

Lemma A.7. *Let φ be analytic in an open set $U \subset \mathbb{C}$ and let u be C^∞ in $\varphi(U)$. Then*
$$\Delta(u \circ \varphi)(z) = \Delta u(\varphi(z))|\varphi'(z)|^2.$$

Proof. We introduce the complex (Wirtinger) derivatives
$$\partial = \frac{1}{2}(\partial_1 - i\partial_2), \quad \overline{\partial} = \frac{1}{2}(\partial_1 + i\partial_2).$$

Define $\tilde{u}(z) := u(\varphi(z))$ for $z \in U$. Note that φ satisfies the Cauchy-Riemann equation $\overline{\partial}\varphi = 0$, and write $\varphi' = \partial\varphi$. Using the chain rule (see e.g. [AIM09, equation (2.48)]), we see that

(A.17) $$\overline{\partial}\tilde{u}(z) = \partial u\Big|_{\varphi(z)} \overline{\partial}\varphi(z) + \overline{\partial}u\Big|_{\varphi(z)} \overline{\partial\varphi(z)} = \overline{\partial}u\Big|_{\varphi(z)} \overline{\partial\varphi(z)}$$

and

$$\frac{1}{4}\Delta \tilde{u}(z) = \partial\overline{\partial}\tilde{u}(z) = \partial\overline{\partial}u\Big|_{\varphi(z)} \partial\varphi(z)\overline{\partial\varphi(z)} + \overline{\partial}^2 u\Big|_{\varphi(z)} \partial\overline{\varphi}(z)\overline{\partial\varphi(z)}$$

$$= \partial\overline{\partial}u\Big|_{\varphi(z)} |\varphi'(z)|^2 = \frac{1}{4}\Delta u\Big|_{\varphi(z)} |\varphi'(z)|^2. \qquad \square$$

A.5. **On the boundary of conformal images of the disk.** In this appendix we prove a number of results concerning the structure of the boundary of the two-dimensional k-quadrature domains constructed in Theorem 1.5. The results we shall prove are certainly well-known to experts in the field but since we have been unsuccessful in finding the precise statements in the literature we choose to include the proof.

Our first aim is to prove the following proposition which is essentially a restatement of Remark 3.2.

Proposition A.8. *Let φ be an analytic function in a neighbourhood of $\overline{\mathbb{D}}$ which is injective in $\mathbb{D}$ and set $D = \varphi(\mathbb{D})$. Then each $z \in \partial D$ falls into one of the following three categories.*

- (I) *(Smooth points) There is an $r > 0$ so that $\partial D \cap B_r(z)$ is an analytic curve.*
- (II) *(Inward cusp points) There is an $r > 0$ so that $D \cap B_r(z)$ contains a semidisk and $\partial D \cap B_r(z)$ consists of two analytic curves ending and tangent to each other at z and having no other common points.*
- (III) *(Double points) There is an $r > 0$ so that $\partial D \cap B_r(z)$ is the union of two analytic curves which are tangent to each other at z and with no other common points. The set $D \cap B_r(z)$ consists of the two components into which the common normals to the analytic curves at z point.*

Furthermore, there are only finitely many non-smooth points.

The characterization of the different points in Proposition A.8 is classical. The sketch of proof that we provide focuses on proving the statement that the set of all non-smooth points is finite.

Sketch of the proof of Proposition A.8. Recall that we in the proof of Theorem 1.5 proved that $\varphi(\partial\mathbb{D}) = \partial D$. We also note that since φ is analytic in a neighbourhood of $\overline{\mathbb{D}}$ the set D is bounded.

Let $z \in \partial D$ be the image of $\zeta \in \partial\mathbb{D}$. If $\varphi'(\zeta) \neq 0$ then the inverse function theorem for analytic mappings implies that z satisfies the conclusion of (I). Since φ' is analytic in a neighbourhood of $\overline{\mathbb{D}}$ and $\partial\mathbb{D}$ is compact we find that φ' either has finitely many zeroes on $\partial\mathbb{D}$ or $\varphi' \equiv 0$ in $\mathbb{D}$. Since φ is injective in $\mathbb{D}$ we know that $\varphi' \not\equiv 0$. The injectivity of φ in the interior implies that each zero of φ' on $\partial\mathbb{D}$ is simple, see (ii) in Remark 3.2. It is well-known that the zeroes of φ' on $\partial\mathbb{D}$ correspond to ∂D having inward cusps.

Let $\{\zeta_1, \ldots, \zeta_k\}$ be the (finitely many) zeroes of φ' on $\partial\mathbb{D}$ counted in counter-clockwise manner starting from the positive real axis. Let for $j = 1, \ldots, k-1$ define γ_j to be the open sub-arc of $\partial\mathbb{D}$ between the consecutive zeroes ζ_j and ζ_{j+1}, and γ_k be the open sub-arc from ζ_k to ζ_1.

Finally, we want to show there are at most finitely many double points. We argue by contradiction. Suppose the contrary, that there are infinitely many double points. We define $\mathscr{D} := \{(\zeta_\alpha^-, \zeta_\alpha^+)\}_{\alpha \in \Lambda}$ with $(\zeta_\alpha^-, \zeta_\alpha^+) \in \partial\mathbb{D} \times \partial\mathbb{D}$ such that $\zeta_\alpha^- \neq \zeta_\alpha^+$ and $\varphi(\zeta_\alpha^+) = \varphi(\zeta_\alpha^-)$. By

compactness of $\partial \mathbb{D} \times \partial \mathbb{D}$, we can find a countable sequence of distinct pairs $\{(\zeta_j^-, \zeta_j^+)\}_{j=1}^\infty$ in $\mathscr{D}$ such that

$$\lim_{j \to \infty} \zeta_j^\pm = \zeta^\pm$$

for some $(\zeta^-, \zeta^+) \in \mathscr{D}$. We now divide our discussions into three cases:

(1) neither ζ^- nor ζ^+ are zeros of φ';
(2) one of the points ζ^-, ζ^+ is a zero of φ' and the other is not;
(3) both ζ^- and ζ^+ are zeros of φ'.

Case 1: neither ζ^- nor ζ^+ are zeros of φ'. In this case, since φ' is analytic and non-zero near both points $\zeta^\pm \in \partial \mathbb{D}$, then there exists two (sufficiently short) open arcs $\gamma^\pm \subset \partial \mathbb{D}$, passing through $\zeta^\pm$, with $\gamma^- \subset \gamma_j$ and $\gamma^+ \subset \gamma_{j'}$, as well as φ is analytic and invertible near $\gamma^\pm$. Without loss of generality, we may assume that $\zeta_j^\pm \in \gamma^\pm$ for $j = 1, 2, \cdots$. In particular, the arcs $\varphi(\gamma^-)$ and $\varphi(\gamma^+)$ intersect at infinitely many points. Let $\varphi_{\mathrm{loc}}^{-1}$ denotes the local inverse of φ near γ^+, and let $M_\pm$ be Möbius transforms with $M_\pm((0,1)) = \gamma^\pm$ such that

$$G := M_+^{-1} \circ \varphi_{\mathrm{loc}}^{-1} \circ \varphi \circ M_-$$

is analytic and bijective in a neighborhood of $[0, 1]$. By assuming γ^- is sufficiently short, we can further assuming $\Re G((0,1)) \subset (0,1)$, where $\Re G$ is the real part of G.

For each $x_j := M_-^{-1}(\zeta_j^-) \in (0,1)$, we see that

$$\varphi_{\mathrm{loc}}^{-1} \circ \varphi \circ M_-(x_j) = \varphi_{\mathrm{loc}}^{-1} \circ \varphi(\zeta_j^-) = \zeta_j^+ \in \gamma^+,$$

and hence $G(x_j) \in (0,1)$. This implies $\Im G(x_j) = 0$, where $\Im G$ is the imaginary part of G. Since $\zeta_j^\pm$ accumulate at $\zeta^\pm \in \gamma^\pm$, then $\Im G$ has an infinite number of zeros in $(0,1)$ and they accumulate at $x_0 := M_-^{-1}(\zeta^-) \in (0,1)$. Since $\Im G$ is real-analytic, then $\Im G \equiv 0$. Hence we know that $G((0,1)) \subset (0,1)$, and consequently

$$\varphi(\gamma^-) \subset \varphi(\gamma^+),$$

that is, $\varphi(\gamma^-)$ is a sub-arc of $\varphi(\gamma^+)$, and hence all points in $\varphi(\gamma^-)$ are double points.

Let p be the endpoint of $\varphi(\gamma^-)$. If p is not an endpoint of $\varphi(\gamma_j)$ or $\varphi(\gamma_{j'})$ then by the arguments above there exists a neighborhood of p in $\varphi(\gamma_j)$ which is an sub-arc of $\varphi(\gamma_{j'})$. By continuity we conclude that $\varphi(\gamma_j) \cap \varphi(\gamma_{j'})$ contains an arc the endpoints of which are also endpoints of either $\varphi(\gamma_j)$ or $\varphi(\gamma_{j'})$. Since the endpoints of $\gamma_j, \gamma_{j'}$ are zeros of φ' we have found an accumulation point of double-points which is the image of a zero of φ', that is, we are in the setting of Cases 2 and 3. After analysing Cases 2 and 3 we shall conclude that p is actually an endpoint of both $\varphi(\gamma_j)$ and $\varphi(\gamma_{j'})$, and hence $\varphi(\gamma_j) = \varphi(\gamma_{j'})$.

Case 2: one of the points ζ^-, ζ^+ is a zero of φ' and the other is not. Without loss of generality, we may assume that $\varphi'(\zeta^-) = 0$ and $\varphi'(\zeta^+) \neq 0$. The image of $\partial \mathbb{D}$ contained in a small neighbourhood of ζ^+ is an analytic curve passing through $\varphi(\zeta^+) = \varphi(\zeta^-)$ while the image of a small neighbourhood of ζ^- is an inward cusp to D with singular point at $\varphi(\zeta^+) = \varphi(\zeta^-)$. It is easy to deduce that the sets $\varphi(B_r(\zeta^+) \cap \mathbb{D})$ and $\varphi(B_r(\zeta^-) \cap \mathbb{D})$ must intersect for every $r > 0$, which contradicts the bijectivity of φ.

Case 3: both ζ^- and ζ^+ are zeros of φ'. Note that both $\varphi(\zeta^-)$ and $\varphi(\zeta^+)$ are vertices of cusps. If $\zeta^- \neq \zeta^+$, then there are two different inward cusps sharing the same vertex which can be shown to violate that φ maps $\mathbb{D}$ bijectively and continuously to D. Thus we can assume that $\zeta^- = \zeta^+$. By rotation we can further assume that $\zeta^- = \zeta^+ = 1$.

We define $\phi(\theta) = \varphi(e^{i\theta})$, and we are interested the behavior near $\theta = 0$. There exists $\epsilon > 0$ such that

$$\phi(\theta) = \sum_{k=0}^{\infty} a_k \theta^k \quad \text{for all } \theta \in (-\epsilon, \epsilon).$$

We now want to show that ϕ is even near 0, that is, $a_k = 0$ for all odd integer k. If this is the case then $\phi((-\epsilon, 0)) = \phi((0, \epsilon))$, that is two of the arcs $\{\varphi(\gamma_j)\}_j$ coincide along a sub-arc and in particular we have accumulation of double points in the interior of the arcs, that is we are in the setting of Case 1.

If ϕ is not even then there exists a smallest odd integer k_0 such that $a_{k_0} \neq 0$. By assumption $\varphi'(1) = 0$ which implies that $\phi'(0) = 0$. This shows that $k_0 \geq 3$. Using that

$$\phi(\theta) = \sum_{k=0}^{k_0} a_k \theta^k + O(\theta^{k_0+1}),$$

it is not difficult to see that there exists a sufficiently small $0 < \epsilon' < \epsilon$ such that there are only finitely many pairs of points $(\theta^-, \theta^+) \in (-\epsilon', \epsilon') \times (-\epsilon', \epsilon')$ with

$$\varphi(e^{i\theta^-}) = \phi(\theta^-) = \phi(\theta^+) = \varphi(e^{i\theta^-}) \quad \text{and} \quad \theta^- < \theta^+.$$

This contradicts that $\zeta^- = \zeta^+$ was an accumulation point the set of double points.

Conclusion. By combining these three cases we conclude that if there exist an infinite number of double points then there exists $j \neq j'$ so that $\varphi(\gamma_j) = \varphi(\gamma_{j'})$. From the injectivity of φ on $\mathbb{D}$ and the construction of the γ_j one deduces that D is the complement of the arc $\varphi(\gamma_j)$, but this is impossible under the assumptions on φ since they imply that D is bounded. $\qquad\square$

For the readers' convenience we next consider the case of inward cusp points in more detail. Suppose that $z_0 = e^{it_0} \in \partial B_1$ and $\varphi'(z_0) = 0$. We let $\gamma(t) = e^{i(t+t_0)}$ be the boundary curve for ∂B_1 with $\gamma(0) = z_0$. Then

$$\gamma^{(k)}(t) = i^k \gamma(t).$$

Let $\eta(t) = \varphi(\gamma(t))$ be the corresponding boundary curve for ∂D with $\eta(0) = \varphi(z_0)$. Then η is a smooth curve. We compute the derivatives of η:

$$\eta' = \varphi'(\gamma)\gamma' = i\varphi'(\gamma)\gamma$$
$$\eta'' = \varphi''(\gamma)(\gamma')^2 + \varphi'(\gamma)\gamma'' = i^2(\varphi''(\gamma)\gamma^2 + \varphi'(\gamma)\gamma),$$
$$\eta''' = i^3(\varphi'''(\gamma)\gamma^3 + 3\varphi''(\gamma)\gamma^2 + \varphi'(\gamma)\gamma).$$

Evaluating at $t = 0$ and using $\varphi'(z_0) = 0$ we obtain

$$\eta(0) = \varphi(z_0),$$
$$\eta'(0) = 0,$$
$$\eta''(0) = i^2 \varphi''(z_0) z_0^2,$$
$$\eta'''(0) = i\eta''(0) \left(\frac{\varphi'''(z_0) z_0}{\varphi''(z_0)} + 3 \right).$$

We now choose orthonormal coordinates (x_1, x_2) so that the origin corresponds to $\varphi(z_0)$ and the vector $\eta''(0)$ corresponds to λe_1 for some $\lambda > 0$. It follows that

$$\eta_1(t) = t^2 g(t)$$

where g is real-analytic near 0 with $g(0) > 0$. We choose a new time variable $s(t)$ near $t = 0$ as

$$s(t) = t\sqrt{g(t)}.$$

This is a valid change of coordinates since s is real-analytic near 0 and satisfies $s'(0) > 0$. Writing $\tilde{\eta}(s) = \eta(t(s))$, we have $\tilde{\eta}_1(s) = s^2$. Moreover, since $\eta'(0) = 0$, we have

$$\tilde{\eta}'''(0) = (t'(0))^3 \eta'''(0) + 3t'(0)t''(0)\eta''(0) = \eta''(0)\left(t'(0)^3 i\left(\frac{\varphi'''(z_0)z_0}{\varphi''(z_0)} + 3\right) + 3t'(0)t''(0)\right).$$

Here $t'(0) > 0$, $t''(0) \in \mathbb{R}$ and $\eta''(0) = \lambda > 0$. If we assume additionally that

$$(A.18) \qquad\qquad \frac{\varphi'''(z_0)z_0}{\varphi''(z_0)} + 3 \notin i\mathbb{R},$$

then we have $\tilde{\eta}_2'''(0) \neq 0$.

We have proved that assuming (A.18), the boundary curve of ∂D near $\varphi(z_0)$ takes the form

$$\tilde{\eta}_1(s) = s^2,$$
$$\tilde{\eta}_2(s) = h(s)$$

where $h(0) = h'(0) = h''(0) = 0$ but $h'''(0) \neq 0$. This is the simplest possible (ordinary) cusp. In particular, D has C^0 boundary near such a point.

If (A.18) does not hold whenever $z_0 \in \partial B_1$ and $\varphi'(z_0) = 0$, then $h(s)$ vanishes to order ≥ 4 at 0. If it vanishes to infinite order, then $h \equiv 0$ and the domain would look like the slit disk, but this is excluded by Proposition A.8. If the vanishing order of h is $k + 1$ for some $k \geq 2$, then we have a singularity of type A_k. If k is even, the boundary is C^0 near such a point, but if k is odd then a curved cusp may occur, see Example 3.5 as well as Figure 3.3.

References

[AH96]   D. R. Adams and L. I. Hedberg. *Function spaces and potential theory*, volume 314 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1996. MR1411441, doi:10.1007/978-3-662-03282-4.

[Ahl64]   L. V. Ahlfors. Finitely generated kleinian groups. *Amer. J. Math.*, 86:413–429, 1964. MR0167618, doi:10.2307/2373173.

[AIM09]   K. Astala, T. Iwaniec, and G Martin. *Elliptic partial differential equations and quasiconformal mappings in the plane*, volume 48 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 2009. MR2472875, doi:10.1515/9781400830114.

[Ber65]   L. Bers. An approximation theorem. *J. Analyse Math.*, 14:1–4, 1965. MR0178287, doi:10.1007/BF02806376.

[Blå18]   E. Blåsten. Nonradiating sources and transmission eigenfunctions vanish at corners and edges. *SIAM J. Math. Anal.*, 50(6):6255–6270, 2018. MR3885754, doi:10.1137/18M1182048, https://arxiv.org/abs/1803.10917.

[BL17]   E. Blåsten and H. Liu. On vanishing near corners of transmission eigenfunctions. *J. Funct. Anal.*, 273(11):3616–3632, 2017. MR3706612, doi:10.1016/j.jfa.2017.08.023, arXiv:1701.07957 [Addendum: arXiv:1710.08089].

[BL21]   E. Blåsten and H. Liu. On corners scattering stably and stable shape determination by a single far-field pattern. *Indiana Univ. Math. J.*, 70(3):907–947, 2021. MR4284101, doi:10.1512/IUMJ.2021.70.8411, arXiv:1611.03647.

[BPS14]   E. Blåsten, L. Päivärinta, and J. Sylvester. Corners always scatter. *Comm. Math. Phys.*, 331(2):725–753, 2014. MR3238529, doi:10.1007/s00220-014-2030-0.

[Bre11]   H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011. MR2759829, doi:10.1007/978-0-387-70914-7.

[BK82]   L. Brown and J.-P. Kahane. A note on the Pompeiu problem for convex domains. *Math. Ann.*, 259(1):107–110, 1982. MR0656655, doi:10.1007/BF01456832.

[BST73] L. Brown, B. M. Schreiber, and B. A. Taylor. Spectral synthesis and the Pompeiu problem. *Ann. Inst. Fourier (Grenoble)*, 23(3):125–154, 1973. MR0352492, doi:10.5802/aif.474.

[Caf77] L. A. Caffarelli. The regularity of free boundaries in higher dimensions. *Acta Math.*, 139(3–4):155–184, 1977. MR0454350, doi:10.1007/BF02392236.

[Caf80] L. A. Caffarelli. Compactness methods in free boundary problems. *Comm. Partial Differential Equations*, 5(4):427–448, 1980. MR0567780, doi:10.1080/0360530800882144.

[Caf98] L. A. Caffarelli. The obstacle problem revisited. *J. Fourier Anal. Appl.*, 4(4–5):383–402, 1998. MR1658612, doi:10.1007/BF02498216.

[CCH16] F. Cakoni, D. Colton, and H. Haddar. *Inverse scattering theory and transmission eigenvalues*, volume 88 of *Regional Conference Series in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2016. MR3601119, doi:10.1137/1.9781611974461.

[CV21]   F. Cakoni and M. S. Vogelius. Singularities almost always scatter: Regularity results for non-scattering inhomogeneities. *arXiv preprint*, 2021. arXiv:2104.05058.

[CX21]   F. Cakoni and J. Xiao. On corner scattering for operators of divergence form and applications to inverse scattering. *Comm. Partial Differential Equations*, 46(3):413–441, 2021. MR4232500, doi:10.1080/03605302.2020.1843489, arXiv:https://arxiv.org/abs/1905.02558.

[CK19]   D. Colton and R. Kress. *Inverse acoustic and electromagnetic scattering theory*, volume 93 of *Applied Mathematical Sciences*. Springer, Cham, fourth edition, 2019. MR3971246, doi:10.1007/978-3-030-30351-8.

[CH89]   R. Courant and D. Hilbert. *Methods of mathematical physics (vol II)*. Wiley Classics Library. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1989. Reprint of the 1962 original, MR1013360, doi:10.1002/9783527617234.

[Dav74] P. J. Davis. *The Schwarz function and its applications*, volume 17 of *The Carus Mathematical Monographs*. The Mathematical Association of America, Buffalo, N. Y., 1974. MR0407252, doi:10.5948/9781614440178.

[Den12] J. Deng. Some results on the Schiffer conjecture in $\mathbb{R}^2$. *J. Differential Equations*, 253(8):2515–2526, 2012. MR2950461, doi:10.1016/j.jde.2012.06.002, arXiv:1112.0207.

[EH15]   J. Elschner and G. Hu. Corners and edges always scatter. *Inverse Problems*, 31(1):015003, 17 pp., 2015. MR3302364, doi:10.1088/0266-5611/31/1/015003.

[Fri57] A. Friedman. On n-metaharmonic functions and harmonic functions of infinite order. *Proc. Amer. Math. Soc.*, 8:223–229, 1957. MR0085430, doi:10.2307/2033716.

[Fri88] A. Friedman. *Variational principles and free-boundary problems*. Robert E. Krieger Publishing Co., Inc., Malabar, FL, second edition, 1988. MR1009785.

[GS09]   S. Gardiner and T. Sjödin. Partial balayage and the exterior inverse problem of potential theory. In *Potential theory and stochastics in Albac*, volume 11, pages 111–123, Theta, Bucharest, 2009. Theta Ser. Adv. Math. MR2681841.

[GT01]   D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order (reprint of the 1998 edition)*, volume 224 of *Classics in Mathematics*. Springer-Verlag Berlin Heidelberg, 2001. MR1814364, doi:10.1007/978-3-642-61798-0.

[Gus90] B. Gustafsson. On quadrature domains and an inverse problem in potential theory. *J. Analyse Math.*, 55:172–216, 1990. MR1094715, doi:10.1007/BF02789201.

[Gus04] B. Gustafsson. Lectures on balayage. In *Clifford algebras and potential theory*, volume 7 of *Univ. Joensuu Dept. Math. Rep. Ser.*, pages 17–63. Univ. Joensuu, Joensuu, 2004. MR2103705, diva2:492834.

[GR18]   B. Gustafsson and J. Roos. Partial balayage on Riemannian manifolds. *J. Math. Pures Appl. (9)*, 118:82–127, 2018. MR3852470,doi:10.1016/j.matpur.2017.07.013, arXiv:1605.03102.

[GS05]   B. Gustafsson and H. S. Shapiro. What is a quadrature domain? In *Quadrature domains and their applications*, volume 156 of *Oper. Theory Adv. Appl.*, pages 1–25. Birkhäuser, Basel, 2005. MR2129734, doi:10.1007/3-7643-7316-4_1.

[Hed73] L. I. Hedberg. Approximation in the mean by solutions of elliptic equations. *Duke Math. J.*, 40:9–16, 1973. MR0312071, doi:10.1215/S0012-7094-73-04002-7.

[Isa90] V. Isakov. *Inverse source problems*, volume 34 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1990. MR1071181, doi:10.1090/surv/034.

[KM96]   L. Karp and A. Margulis. Newtonian potential theory for unbounded sources and applications to free boundary problems. *J. Anal. Math.*, 70:1–63, 1996. MR1444257, doi:10.1007/BF02820440.

[KN77]   D. Kinderlehrer and L. Nirenberg. Regularity in free boundary problems. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 4(2):373–391, 1977. MR0440187, NUMDAM.

[KS00]   D. Kinderlehrer and G. Stampacchia. *An introduction to variational inequalities and their applications*, volume 31 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2000. Reprint of the 1980 original. MR1786735, doi:10.1137/1.9780898719451.

[Kuz19] N. Kuznetsov. Mean value properties of harmonic functions and related topics (a survey). *J. Math. Sci. (N.Y.)*, 242:177–199, 2019. MR4002408, doi:10.1007/s10958-019-04473-w, arXiv:1904.08312.

[Kuz21] N. Kuznetsov. Metaharmonic functions: mean flux theorem, its converse and related properties. *Algebra i Analiz*, 33(2):82–98, 2021. MR4240827, Mi aa1749, arXiv:2004.03433.

[LM16]   S.-Y. Lee and N. G. Makarov. Topology of quadrature domains. *J. Amer. Math. Soc.*, 29(2):333–369, 2016. MR3454377, doi:10.1090/jams828, arXiv:1307.0487.

[PSV17] L. Päivärinta, M. Salo, and E. V. Vesalainen. Strictly convex corners scatter. *Rev. Mat. Iberoam.*, 33(4):1369–1396, 2017. MR3729603, doi:10.4171/RMI/975, arXiv:1404.2513.

[PSU12] A. Petrosyan, H. Shahgholian, and N. Uraltseva. *Regularity of free boundaries in obstacle-type problems*, volume 136 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2012. MR2962060, doi:10.1090/gsm/136.

[Pom29] P. Pompeiu. Sur certains systèmes d'équations linéaires et sur une propriété intégrale des fonctions de plusieurs variables. *Comptes Rendus de l'Académie des Sciences. Série I. Mathématique*, 188:1138–1139, 1929.

[PW84]   M. Protter and H. Weinberger. *Maximum principles in differential equations*. Springer-Verlag, New York, 1984. Corrected reprint of the 1967 original. MR0762825, doi:10.1007/978-1-4612-5282-5.

[Rud87] W. Rudin. *Real and complex analysis*. McGraw-Hill Book Co., New York, third edition, 1987. MR0924157.

[Sak83] M. Sakai. Applications of variational inequalities to the existence theorem on quadrature domains. *Trans. Amer. Math. Soc.*, 276(1):267–279, 1983. MR0684507, doi:10.1090/S0002-9947-1983-0684507-2.

[Sak84] M. Sakai. Solutions to the obstacle problem as Green potentials. *J. Analyse Math.*, 44:97–116, 1984. MR0801289, doi:10.1007/BF02790192.

[Sak91] M. Sakai. Regularity of a boundary having a schwarz function. *Acta Math.*, 166(3–4):263–297, 1991. MR1097025, doi:10.1007/BF02398888.

[SS21]   M. Salo and H. Shahgholian. Free boundary methods and non-scattering phenomena. *Res. Math. Sci.*, 8(4):Paper No. 58, 2021. MR4323345, doi:10.1007/s40687-021-00294-z, arXiv:2106.15154.

[Sjö07] T. Sjödin. On the structure of partial balayage. *Nonlinear Anal.*, 67(1):94–102, 2007. MR2313881, doi:10.1016/j.na.2006.05.001.

[Trè75] F. Trèves. *Basic linear partial differential equations*, volume 62 of *Pure and Applied Mathematics*. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1975. MR0447753.

[Web68] H. Weber. Ueber einige bestimmte Integrale. *J. Reine Angew. Math.*, 69:222–237, 1868. MR1579416, doi:10.1515/crll.1868.69.222.

[Web69] H. Weber. Ueber die integration der partiellen differentialgleichung: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = 0$. *Math. Ann.*, 1(1):1–36, 1869. MR1509609, doi:10.1007/BF01447384.

[Wil76] S. A. Williams. A partial solution of the Pompeiu problem. *Math. Ann.*, 223(2):183–190, 1976. MR0414904, doi:10.1007/BF01360881.

[Wil81] S. A. Williams. Analyticity of the boundary for Lipschitz domains without the Pompeiu property. *Indiana Univ. Math. J.*, 30(3):357–369, 1981. MR0611225.

[Yaf10] D. R. Yafaev. *Mathematical scattering theory. Analytic theory*, volume 158 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2010. MR2598115, doi:10.1090/surv/158.

[Zal92] L. Zalcman. A bibliographic survey of the Pompeiu problem. In *Approximation by solutions of partial differential equations (Hanstholm, 1991)*, volume 365 of *NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci.*, pages 185–194. Kluwer Acad. Publ., Dordrecht, 1992. MR1168719.

Department of Mathematics and Statistics, P.O. Box 35 (MaD), FI-40014 University of Jyväskylä, Finland.
Email address: pu-zhao.pz.kow@jyu.fi

Mathematical Sciences, Chalmers University of Technology and the University of Gothenburg, SE-412 96 Göteborg, Sweden.
Email address: larsons@chalmers.se

Department of Mathematics and Statistics, P.O. Box 35 (MaD), FI-40014 University of Jyväskylä, Finland.
Email address: mikko.j.salo@jyu.fi

Department of Mathematics, KTH Royal Institute of Technology, SE-100 44 Stockholm, Sweden.
Email address: henriksh@kth.se