INVERSE PROBLEMS FOR ELLIPTIC PDE

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ABSTRACT. These notes for a minicourse in Beijing Analysis Autumn School 2023 give an exposition of Calderón type inverse problems for elliptic PDE. We begin with a fundamental uniqueness result in dimensions ≥ 3 based on complex geometrical optics solutions. The case of transversally anisotropic geometries is considered next. We then move to semilinear equations and show that nonlinearity may help in solving inverse problems. The final part considers p-Laplace type equations and discusses known results. Several open questions are stated along the way.

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PREFACE

Electrical Impedance Tomography (EIT) is an imaging method with potential applications in medical imaging and nondestructive testing. The method is based on the following important inverse problem.

> **Calderón problem:** Is it possible to determine the electrical conductivity of a medium by making voltage and current measurements on its boundary?

Let us discuss the mathematical model of EIT. The purpose is to determine the electrical conductivity $\gamma(x)$ at each point $x \in \Omega$, where $\Omega \subset \mathbb{R}^n$ represents the body which is imaged (in practice $n = 3$). We assume that Ω is a bounded open subset of $\mathbb{R}^n$ with C^∞ boundary, and that γ is a function in

$$L^\infty_+(\Omega) = \{\gamma \in L^\infty(\Omega) \,:\, \operatorname{ess\,inf} \gamma > 0\}.$$

Under the assumption of no sources or sinks of current in Ω, a voltage potential f at the boundary $\partial\Omega$ induces a voltage potential u in Ω, which solves the Dirichlet problem for the conductivity equation,

$$\left\{\begin{aligned} \mathrm{div}(\gamma \nabla u) &= 0 && \text{in } \Omega, \\ u &= f && \text{on } \partial\Omega. \end{aligned}\right.$$

Consider the Sobolev spaces

$$\begin{aligned} H^1(\Omega) &= \{u \in L^2(\Omega) \,:\, \partial_j u \in L^2(\Omega),\ 1 \leq j \leq n\}, \\ H^{1/2}(\partial\Omega) &= \{u|_{\partial\Omega} \,:\, u \in H^1(\Omega)\}. \end{aligned}$$

Since $\gamma \in L^\infty_+(\Omega)$, the conductivity equation is elliptic and there is a unique weak solution $u = u_f \in H^1(\Omega)$ for any boundary value $f \in H^{1/2}(\partial\Omega)$. One can define the Dirichlet-to-Neumann map (DN map) formally as

$$\Lambda_\gamma f = \gamma \partial_\nu u|_{\partial\Omega}.$$

Here ∂_ν is the normal derivative, and $\gamma\partial_\nu u$ is the current flowing through the boundary. More precisely, the DN map can be defined weakly by

$$(\Lambda_\gamma f, g)_{\partial\Omega} = \int_\Omega \gamma \nabla u_f \cdot \nabla \bar{v}\, dx, \quad f, g \in H^{1/2}(\partial\Omega),$$

where v is any function in $H^1(\Omega)$ with $v|_{\partial\Omega} = g$. One can show that Λ_γ is a bounded linear map from $H^{1/2}(\partial\Omega)$ into $H^{-1/2}(\partial\Omega) = (H^{1/2}(\partial\Omega))^*$.

The *Calderón problem*, also called the *inverse conductivity problem*, is to determine the conductivity function γ from the knowledge of the map Λ_γ. That is, if the measured current $\Lambda_\gamma f$ is known for all boundary voltages $f \in H^{1/2}(\partial\Omega)$, one would like to determine the conductivity γ. There are several aspects of this inverse problem which are interesting both for mathematical theory and practical applications.

1. **Uniqueness.** If $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, show that $\gamma_1 = \gamma_2$.
2. **Reconstruction.** Given the boundary measurements Λ_γ, find a procedure to reconstruct the conductivity γ.
3. **Stability.** If Λ_{γ_1} is close to Λ_{γ_2}, show that γ_1 and γ_2 are close (in a suitable sense).
4. **Partial data.** If Γ is a subset of $\partial\Omega$ and if $\Lambda_{\gamma_1} f|_\Gamma = \Lambda_{\gamma_2} f|_\Gamma$ for all boundary voltages f, show that $\gamma_1 = \gamma_2$.

The Calderón problem is a fundamental inverse problem for elliptic PDE and it has been studied intensively since the pioneering work of Calderón [Ca80]. In this minicourse we will focus on uniqueness results in Calderón type problems. We will begin with the basic uniqueness result for the Calderón problem from [SU87]. We will then move to more general geometries, semilinear equations and p-Laplace type equations and discuss results from [DKSU09, FO20, LL+21a, SZ12].

References. Readers are expected to be familiar with weak solutions of the Dirichlet problem, basic regularity results for linear elliptic PDE, and related functional analysis results roughly at the level of [Ev10]. The $H^{1/2}(\partial\Omega)$ space is discussed e.g. in [FSU]. For Section 1, more detailed accounts of the basic uniqueness result for the Calderón problem may be found in [Sa08] or [FSU]. A detailed exposition of the geometric uniqueness result discussed in Section 2 is given in [Sa12]. For a reference to basic Riemannian geometry we recommend [Le18]. In Section 3 we consider an inverse problem for semilinear PDE following [FO20, LL+21a]. Section 4 discusses an inverse problem for p-Laplace type equations and the boundary determination result from [SZ12].

1. Uniqueness in the Calderón problem

The Calderón problem is stated in terms of the conductivity equation $\mathrm{div}(\gamma\nabla u) = 0$. However, if $\gamma \in C^2(\overline{\Omega})$ is positive, there is a very useful reduction based on the *Liouville transform*: the substitution $u = \gamma^{-1/2}v$ and a short computation (exercise) yield

$$\mathrm{div}(\gamma\nabla(\gamma^{-1/2}v)) = \gamma^{1/2}(\Delta - q)v, \qquad q = \frac{\Delta(\gamma^{1/2})}{\gamma^{1/2}}. \tag{1.1}$$

In this way, the conductivity equation reduces to the *Schrödinger equation*

$$(-\Delta + q)u = 0 \text{ in } \Omega.$$

One advantage of this reduction is that the principal part of the equation is $-\Delta$ and the variable coefficient has moved to the zero order term. From now on we will exclusively work with the Schrödinger equation.

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $q \in L^\infty(\Omega)$ be real valued, and consider the Dirichlet problem

$$\left\{ \begin{aligned} (-\Delta + q)u &= 0 && \text{in } \Omega, \\ u &= f && \text{on } \partial\Omega. \end{aligned} \right. \tag{1.2}$$

We assume throughout that 0 is not a Dirichlet eigenvalue of $-\Delta + q$ in Ω (this is always true if q comes from a C^2 conductivity). Then for any $f \in H^{1/2}(\partial\Omega)$ the problem (1.2) has a unique solution $u \in H^1(\Omega)$. One can define the DN map

$$\Lambda_q : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega), \quad \Lambda_q f = \partial_\nu u|_{\partial\Omega},$$

where the normal derivative $\partial_\nu u|_{\partial\Omega}$ is interpreted in a natural weak sense. (If q and f are slightly more regular, then $u \in C^1(\overline{\Omega})$ and the normal derivative exists pointwise.)

The following result from [SU87], together with a boundary determination result, implies uniqueness for C^2 conductivities in the Calderón problem.

Theorem 1.1 (Uniqueness). *Let $n \geq 3$ and $q_1, q_2 \in L^\infty(\Omega)$. If $\Lambda_{q_1} = \Lambda_{q_2}$, then $q_1 = q_2$.*

For less regular conductivities the reduction to the Schrödinger equation becomes more difficult to use, and the low regularity case is not fully understood.

Question 1.1 (Low regularity). *Let $n \geq 3$. Is it true that Λ_γ determines $\gamma \in L^\infty_+(\Omega)$ uniquely, or can one find counterexamples for uniqueness?*

Uniqueness is known for Lipschitz conductivities if $n \geq 3$ [HT13, CR16], and for $\gamma \in W^{1,n} \cap L^\infty_+$ when $n = 3, 4$ [Ha15]. For a partial data problem there is a counterexample to uniqueness for C^α conductivities based on a counterexample to the unique continuation principle [DKN20]. When $n = 2$ the Calderón problem can be studied by using methods from complex analysis and quasiconformal mappings, and uniqueness is known for L^∞ conductivities [AP06].

Another open question is related to local data, where one only measures voltages and currents on a subset Γ of $\partial\Omega$.

Question 1.2 (Local data). *Let $n \geq 3$ and let Γ be a nonempty open subset of $\partial\Omega$. If $q_1, q_2 \in L^\infty(\Omega)$ and $\Lambda_{q_1} f|_\Gamma = \Lambda_{q_2} f|_\Gamma$ for all $f \in H^{1/2}(\partial\Omega)$ with $\mathrm{supp}(f) \subset \Gamma$, does it follow that $q_1 = q_2$?*

This result is known when $n = 2$ [IUY10]. When $n \geq 3$ there are only partial results for the case where $\partial\Omega \setminus \Gamma$ has a conformal symmetry, such as being part of a hyperplane or a surface of revolution [Is07, KS13].

The proof of Theorem 1.1 has three main components:

1. An integral identity showing that $q_1 - q_2$ is L^2-orthogonal to products $u_1 u_2$ where $(-\Delta + q_j) u_j = 0$.
2. Construction of special *complex geometrical optics* (CGO) solutions u_j based on a Carleman estimate and duality.
3. Inserting the CGO solutions in the integral identity and showing that $q_1 - q_2$ has vanishing Fourier transform.

1.1. **Integral identity.** The first step in the proof is the following integral identity, often called the *Alessandrini identity.*

Proposition 1.2 (Integral identity). *Let $q_1, q_2 \in L^\infty(\Omega)$. For any $f_1, f_2 \in H^{1/2}(\partial\Omega)$, one has*

$$((\Lambda_{q_1} - \Lambda_{q_2}) f_1, f_2)_{\partial\Omega} = \int_\Omega (q_1 - q_2) u_1 \bar{u}_2 \, dx$$

where $u_j \in H^1(\Omega)$ solve $(-\Delta + q_j) u_j = 0$ in Ω with $u_j|_{\partial\Omega} = f_j$.

Proof. We prove the result formally, i.e. assuming that all functions are sufficiently smooth (the proof in the general case can be done by using suitable weak formulations). We first show that the DN map Λ_q is formally self-adjoint, i.e.

$$(\Lambda_q g_1, g_2)_{\partial\Omega} = (g_1, \Lambda_q g_2)_{\partial\Omega}. \tag{1.3}$$

To prove this, we let v_j solve $(-\Delta + q)v_j = 0$ in Ω with $v_j|_{\partial\Omega} = g_j$ and integrate by parts using Green's formula:

$$\begin{aligned}(\Lambda_q g_1, g_2)_{\partial\Omega} &= \int_{\partial\Omega} (\partial_\nu v_1)\bar{v}_2 \, dS \\ &= \int_{\partial\Omega} (\nabla v_1 \cdot \nabla \bar{v}_2 + (\Delta v_1)\bar{v}_2) \, dx \\ &= \int_{\partial\Omega} (\nabla v_1 \cdot \nabla \bar{v}_2 + q v_1 \bar{v}_2) \, dx.\end{aligned} \tag{1.4}$$

We continue the computation:

$$\begin{aligned}&= \int_{\partial\Omega} (\nabla v_1 \cdot \nabla \bar{v}_2 + v_1(\Delta \bar{v}_2)) \, dx \\ &= \int_{\partial\Omega} v_1 \partial_\nu \bar{v}_2 \, dS \\ &= (g_1, \Lambda_q g_2)_{\partial\Omega}.\end{aligned} \tag{1.5}$$

This proves (1.3).

Now, if u_1 and u_2 are as in the statement, the computation (1.4) gives

$$(\Lambda_{q_1} f_1, f_2)_{L^2(\partial\Omega)} = \int_\Omega (\nabla u_1 \cdot \nabla \bar{u}_2 + q_1 u_1 \bar{u}_2) \, dx.$$

Similarly, (1.3) and the computation (1.5) give

$$\begin{aligned}(\Lambda_{q_2} f_1, f_2)_{L^2(\partial\Omega)} &= (f_1, \Lambda_{q_2} f_2)_{L^2(\partial\Omega)} \\ &= \int_\Omega (\nabla u_1 \cdot \nabla \bar{u}_2 + q_2 u_1 \bar{u}_2) \, dx\end{aligned}$$

The result follows by subtracting these two identities. □

Thus, if $\Lambda_{q_1} = \Lambda_{q_2}$, it follows from Proposition 1.2 that

$$\int_\Omega (q_1 - q_2) u_1 \bar{u}_2 \, dx = 0$$

for *any* $u_j \in H^1(\Omega)$ solving $(-\Delta + q_j)u_j = 0$ in Ω. The proof of Theorem 1.1 thus reduced to showing that

$$\{u_1 u_2 \,:\, u_j \in H^1(\Omega) \text{ solves } (-\Delta + q_j)u_j = 0\}$$

is a *complete set* in $L^1(\Omega)$ (i.e. its linear span is dense). In particular, it would be sufficient to construct a very rich family of solutions of the Schrödinger equation so that products of these solutions would form a complete set.

1.2. **Products of harmonic functions are complete.** In his original work [Ca80], Calderón showed that products of solutions of $\Delta u = 0$ form a complete set (thus solving a linearized version of the inverse conductivity problem). In the proof he used special exponential solutions. This approach has been a model for many later developments.

Proposition 1.3 (Completeness)**.** *If $f \in L^\infty(\Omega)$ and*

$$\int_\Omega f u_1 u_2 \, dx = 0$$

for any $u_j \in H^1(\Omega)$ with $\Delta u_j = 0$ in Ω, then $f = 0$.

Proof. Let $\rho = \eta + i\xi \in \mathbb{C}^n$ where $\eta, \xi \in \mathbb{R}^n$, and let $u = e^{\rho \cdot x} = e^{\eta \cdot x + i\xi \cdot x}$. One has

$$\Delta u = \sum \partial_j^2 (e^{\rho \cdot x}) = \sum \partial_j (\rho_j e^{\rho \cdot x}) = (\sum \rho_j^2) e^{\rho \cdot x}$$

Thus $\Delta u = 0$ iff $\rho \cdot \rho = \sum \rho_j^2 = 0$. Moreover, since

$$\rho \cdot \rho = (\eta + i\xi) \cdot (\eta + i\xi) = |\eta|^2 - |\xi|^2 + 2i\eta \cdot \xi,$$

we have

$$\rho \cdot \rho = 0 \quad \Longleftrightarrow \quad |\eta| = |\xi| \text{ and } \eta \perp \xi.$$

Now fix any $\xi \in \mathbb{R}^n$, and choose some $\eta \in \mathbb{R}^n$ with $\eta \perp \xi$ and $|\eta| = |\xi|$. Then choose

$$u_1 = e^{(\eta + i\xi) \cdot x}, \qquad u_2 = e^{(-\eta + i\xi) \cdot x}.$$

If $\int f u_1 u_2 \, dx = 0$, then we have

$$\int_\Omega f e^{2ix \cdot \xi} \, dx = 0.$$

This is true for any $\xi \in \mathbb{R}^n$, which implies that the Fourier transform of f (extended by zero to $\mathbb{R}^n$) vanishes identically. By Fourier inversion, we obtain $f = 0$. $\square$

1.3. **Complex geometrical optics solutions.** We now move to solutions of $(-\Delta + q_j) u_j = 0$. Since these are variable coefficient equations, it is not reasonable to expect that there would be explicit exact solutions such as $e^{\rho \cdot x}$. A key insight in [SU87] was that when $\rho \cdot \rho = 0$ and $|\rho|$ is very large, it is possible to find solutions that look *approximately* like $e^{\rho \cdot x}$. These solutions are called *complex geometrical optics* (CGO) solutions.

Proposition 1.4 (CGO solutions)**.** *Let $q \in L^\infty(\Omega)$. There are $C, \tau_0 > 0$ such that whenever $\rho \in \mathbb{C}^n$ satisfies $\rho \cdot \rho = 0$ and $|\rho| \geq \tau_0$, the equation $(-\Delta + q)u = 0$ has a distributional solution*

$$u = e^{\mathrm{Re}(\rho)\cdot x}(e^{i\mathrm{Im}(\rho)\cdot x} + r)$$

where $r \in L^2(\Omega)$ satisfies

$$\|r\|_{L^2(\Omega)} \leq \frac{C}{|\rho|}.$$

Let us show how Theorem 1.1 follows from Proposition 1.4. The idea is that when $n \geq 3$, for any $\xi \in \mathbb{R}^n$ it is possible to find two vectors $\rho_j \in \mathbb{C}^n$ such that $\rho_j \cdot \rho_j = 0$, $|\rho_j| \to \infty$, and $e^{\rho_1 \cdot x} e^{\rho_2 \cdot x} = e^{2ix\cdot\xi}$.

Proof of Theorem 1.1. As discussed above, the assumption $\Lambda_{q_1} = \Lambda_{q_2}$ together with the integral identity in Proposition 1.2 imply that

$$\int_\Omega (q_1 - q_2) u_1 u_2 \, dx = 0 \tag{1.6}$$

whenever $u_j \in H^1(\Omega)$ solve $(-\Delta + q_j)u_j = 0$ in Ω.

Now fix a vector $\xi \in \mathbb{R}^n$, and use the assumption $n \geq 3$ to find some unit vectors $\alpha, \gamma \in \mathbb{R}^n$ such that

$$\{\alpha, \gamma, \xi\} \text{ is an orthogonal set.}$$

Next, for $\tau \geq |\xi|$ define two complex vectors

$$\rho_1 = \tau\alpha + i\left[\xi + \sqrt{\tau^2 - |\xi|^2}\gamma\right],$$
$$\rho_2 = -\tau\alpha + i\left[\xi - \sqrt{\tau^2 - |\xi|^2}\gamma\right].$$

Then $|\mathrm{Re}(\rho_j)| = |\mathrm{Im}(\rho_j)| = \tau$ and $\mathrm{Re}(\rho_j) \perp \mathrm{Im}(\rho_j)$, so $\rho_j \cdot \rho_j = 0$.

By Proposition 1.4, for $\tau > 0$ sufficiently large there exist solutions u_j of $(-\Delta + q_j)u_j = 0$ of the form

$$u_j = e^{\mathrm{Re}(\rho_j)\cdot x}(e^{i\mathrm{Im}(\rho_j)\cdot x} + r_j),$$

with $\|r_j\|_{L^2(\Omega)} \to 0$ as $\tau \to \infty$. (These solutions are in principle only in $L^2(\Omega)$, but doing the construction in a slightly larger set and using elliptic regularity in Ω yields solutions in $H^1(\Omega)$.) Inserting these solutions in (1.6) and using that $\rho_1 + \rho_2 = 2i\xi$, we obtain

$$\begin{aligned} \int_\Omega (q_1 - q_2) e^{2ix\cdot\xi} \, dx \\ = -\int_\Omega (q_1 - q_2)(e^{i\mathrm{Im}(\rho_1)\cdot x} r_2 + e^{i\mathrm{Im}(\rho_2)\cdot x} r_1 + r_1 r_2). \end{aligned} \tag{1.7}$$

Since $|e^{i\mathrm{Im}(\rho_j)\cdot x}| = 1$ and $\|r_j\|_{L^2(\Omega)} \to 0$ as $\tau \to \infty$, the right hand side of (1.7) converges to zero. Taking the limit of (1.7) as $\tau \to \infty$ yields

$$\int_\Omega (q_1 - q_2)e^{2ix\cdot\xi}\,dx = 0.$$

This is true for any $\xi \in \mathbb{R}^n$, which implies that the Fourier transform of $q_1 - q_2$ (extended by zero to $\mathbb{R}^n$) vanishes identically. Fourier inversion then ensures that $q_1 = q_2$. $\square$

1.4. **Carleman estimate.** It remains to prove Proposition 1.4. There are several proofs of this result. Some of these are based on Fourier analysis, but we will give a proof using integration by parts and a duality argument.

We choose coordinates so that $\alpha = e_1$, and then

$$\rho = \tau(e_1 + i\beta)$$

where $|\beta| = 1$ and $\beta \cdot e_1 = 0$. Then u has the form

$$u = e^{\tau x_1}(a(x') + r) \tag{1.8}$$

where $x = (x_1, x')$ and $a(x') = e^{i\tau\beta\cdot x}$.

Define the conjugated operator P_τ by

$$P_\tau v = e^{-\tau x_1}\Delta(e^{\tau x_1}v) = ((\partial_1 + \tau)^2 + \Delta_{x'})v = (\Delta + \tau^2 + 2\tau\partial_1)v.$$

Then u solves $(-\Delta + q)u = 0$ iff r solves

$$(P_\tau - q)r = f \text{ in } \Omega,$$

where $f = -(P_\tau - q)(a(x')) = -(\Delta + \tau^2 - q)(e^{i\tau\beta\cdot x}) = qe^{i\tau\beta\cdot x}$. Proposition 1.4 is therefore a consequence of the following solvability result for P_τ.

Proposition 1.5 (Solvability for P_τ)**.** *Let $q \in L^\infty(\Omega)$. There are $C, \tau_0 > 0$ such that whenever $\tau \in \mathbb{R}$ and $|\tau| \geq \tau_0$, for any $f \in L^2(\Omega)$ the equation*

$$(P_\tau - q)u = f \text{ in } \Omega$$

has a solution $u \in L^2(\Omega)$ satisfying

$$\|u\|_{L^2(\Omega)} \leq \frac{C}{|\tau|}.$$

This solvability result will in turn be a consequence of an *a priori* estimate for the adjoint operator $P_{-\tau}$.

Proposition 1.6 (Carleman estimate)**.** *Let $q \in L^\infty(\Omega)$. There are $C, \tau_0 > 0$ such that whenever $\tau \in \mathbb{R}$ and $|\tau| \geq \tau_0$, one has the estimate*

$$\|u\|_{L^2(\Omega)} \leq \frac{C}{|\tau|}\|(P_{-\tau} - q)u\|_{L^2(\Omega)}, \qquad u \in C_c^\infty(\Omega). \tag{1.9}$$

Remark 1.7. Writing $u = e^{\tau x_1} v$, the estimate (1.9) may be rewritten as

$$\|e^{\tau x_1} v\|_{L^2(\Omega)} \leq \frac{C}{|\tau|} \|e^{\tau x_1}(\Delta - q)v\|_{L^2(\Omega)}, \qquad v \in C_c^\infty(\Omega). \tag{1.10}$$

This is an exponentially weighted L^2 estimate involving a large parameter τ, i.e. a *Carleman estimate*. Estimates of this type appear frequently in unique continuation problems, control theory and inverse problems.

We now prove the Carleman estimate by a standard argument involving a nonnegative commutator and the Poincaré inequality.

Proof of Proposition 1.6. We note that $P_{-\tau} = e^{\tau x_1} \circ \Delta \circ e^{-\tau x_1}$ is not self-adjoint (its formal adjoint is P_τ). We may decompose $P_{-\tau}$ in a self-adjoint and skew-adjoint part as

$$P_{-\tau} = \Delta + \tau^2 - 2\tau \partial_1 = A + iB$$

where

$$A = \Delta + \tau^2, \qquad B = 2i\tau\partial_1.$$

Note that A and B are formally self-adjoint in the inner product $(u, v) = \int_\Omega uv\,dx$.

For any $u \in C^\infty(\Omega)$, we compute

$$\begin{aligned} \|P_{-\tau} u\|^2 &= (P_{-\tau} u, P_{-\tau} u) \\ &= ((A + iB)u, (A + iB)u) \\ &= \|Au\|^2 + \|Bu\|^2 + i(Bu, Au) - i(Au, Bu) \\ &= \|Au\|^2 + \|Bu\|^2 + (i[A, B]u, u). \end{aligned}$$

Here $[A, B] = AB - BA$ is the *commutator* of A and B. In particular, if $(i[A, B]u, u) \geq 0$ we obtain a lower bound for $\|P_{-\tau} u\|$. But in our case A and B are constant coefficient operators, so

$$[A, B] \equiv 0.$$

In particular, we have

$$\|P_{-\tau} u\| \geq \|Bu\| = 2|\tau| \|\partial_1 u\|. \tag{1.11}$$

We next invoke the Poincaré inequality[1]

$$\|u\|_{L^2(\Omega)} \leq C\|\partial_1 u\|_{L^2(\Omega)}, \qquad u \in C_c^\infty(\Omega).$$

Combining this with (1.11) gives

$$\|u\|_{L^2(\Omega)} \leq \frac{C}{|\tau|}\|P_{-\tau}u\|_{L^2(\Omega)} \leq \frac{C}{|\tau|}(\|(P_{-\tau} - q)u\|_{L^2(\Omega)} + \|qu\|_{L^2(\Omega)}).$$

If $|\tau| \geq \tau_0 := 2C\|q\|_{L^\infty(\Omega)}$, we can absorb the last term on the right to the left. This proves (1.9). □

Given the a priori estimate (1.9), the solvability result follows by a duality argument.

Proof of Proposition 1.5. Suppose that $|\tau| \geq \tau_0$ and let $f \in L^2(\Omega)$. Define $X = (P_{-\tau} - q)C_c^\infty(\Omega)$ considered as a subspace of $L^2(\Omega)$, and define a linear functional

$$\ell : X \to \mathbb{R}, \quad \ell(P_{-\tau}\varphi) = \int f\varphi\,dx.$$

This functional is well-defined, since any $w \in X$ satisfies $w = (P_{-\tau} - q)\varphi$ for a unique $\varphi \in C_c^\infty(\Omega)$ by the Carleman estimate (1.9). Using (1.9) again we have

$$|\ell(P_{-\tau}\varphi)| \leq \|f\|_{L^2(\Omega)}\,\|\varphi\|_{L^2(\Omega)} \leq \frac{C}{|\tau|}\|f\|_{L^2(\Omega)}\,\|(P_{-\tau} - q)\varphi\|_{L^2(\Omega)}.$$

Thus ℓ is a continuous linear functional on X. By Hahn-Banach, it extends as a continuous linear functional $\bar{\ell}$ on $L^2(\Omega)$ with $\|\bar{\ell}\| \leq \frac{C}{|\tau|}\|f\|_{L^2(\Omega)}$. The Riesz representation theorem ensures that there is $u \in L^2(\Omega)$ such that

$$\bar{\ell}(w) = \int_\Omega uw\,dx, \qquad \|u\|_{L^2(\Omega)} = \|\bar{\ell}\| \leq \frac{C}{|\tau|}\|f\|_{L^2(\Omega)}.$$

Now if $\varphi \in C_c^\infty(\Omega)$, then we have in the distributional pairing $\langle\,\cdot\,,\,\cdot\,\rangle$ on Ω that

$$\begin{aligned}\langle (P_\tau - q)u, \varphi\rangle &= \langle u, (P_{-\tau} - q)\varphi\rangle = \bar{\ell}((P_{-\tau} - q)\varphi) = \ell((P_{-\tau} - q)\varphi) \\ &= \int_\Omega f\varphi\,dx.\end{aligned}$$

Thus $(P_\tau - q)u = f$ in the sense of distributions. □

[1]Since $\mathrm{supp}(u) \subset \Omega \subset \{a < x_1 < b\}$ for some $a < b$, for any $x \in \Omega$ we have

$$u(x_1, x') = u(x_1, x') - u(a, x') = \int_a^{x_1} \partial_1 u(t, x')\,dt$$

and therefore by Cauchy-Schwarz

$$|u(x_1, x')| \leq \int_a^b |\partial_1 u(t, x')|\,dt \leq (b - a)^{1/2}\|\partial_1 u\|_{L^2(\Omega)}.$$

Squaring this and integrating over Ω gives inequality $\|u\|_{L^2(\Omega)}^2 \leq (b - a)|\Omega|\|\partial_1 u\|_{L^2(\Omega)}^2$.

2. Uniqueness in transversally anisotropic geometry

In the preface we considered the conductivity equation,

$$\mathrm{div}(\gamma \nabla u) = 0,$$

in the *isotropic* case where the conductivity γ is a scalar function in $L^\infty_+(\Omega)$. There are many practical situations, such as imaging muscle tissue, where the conductivity is *anisotropic* and is given by a symmetric positive definite matrix $\gamma = (\gamma^{jk}(x))_{j,k=1}^n$. Then the conductivity equation takes the basic form

$$\sum_{j,k=1}^n \partial_j(\gamma^{jk}(x)\partial_k u) = 0.$$

There are only partial results for the anisotropic Calderón problem when $n \geq 3$, and many important questions remain open.

It is convenient to think of the conductivity matrix (γ^{jk}) geometrically in terms of a *Riemannian metric*. Let $\Omega \subset \mathbb{R}^n$ be a bounded C^∞ domain, and let $g = (g_{jk}(x))_{j,k=1}^n$ be a Riemannian metric on $\overline{\Omega}$. This means that (g_{jk}) is a symmetric positive definite matrix whose entries g_{jk} are functions in $C^\infty(\overline{\Omega})$. The pair $(\overline{\Omega}, g)$ becomes a compact Riemannian manifold with smooth boundary. There is a canonical associated elliptic operator, the *Laplace-Beltrami operator* Δ_g, defined by

$$\Delta_g u = \sum_{j,k=1}^n |g|^{-1/2}\partial_j(|g|^{1/2} g^{jk}\partial_k u).$$

Here $(g^{jk}(x))$ is the inverse matrix of $(g_{jk}(x))$, and $|g| = \det(g_{jk})$. Thus the conductivity equation $\mathrm{div}(\gamma\nabla u) = 0$ can be rewritten as $\Delta_g u = 0$ if g is chosen so that

$$|g|^{1/2} g^{jk} = \gamma^{jk}.$$

It is easy to see (exercise) that such a choice is always possible when $n \geq 3$. Rewriting the conductivity equation in this way gives us access to powerful geometric notions in the study of the anisotropic Calderón problem.

For the purposes of these notes we will consider an inverse problem for a Schrödinger equation as in Section 1. Let $(\overline{\Omega}, g)$ be as above and let $q \in C^\infty(\overline{\Omega})$ be real valued. We assume throughout that 0 is not a Dirichlet eigenvalue of the problem

$$\left\{ \begin{array}{rll} (-\Delta_g + q)u &= 0 & \text{in } \Omega, \\ u &= f & \text{on } \partial\Omega. \end{array} \right. \tag{2.1}$$

Then for any $f \in C^\infty(\partial\Omega)$ there is a unique solution $u \in C^\infty(\overline{\Omega})$ by elliptic regularity. We have a DN map

$$\Lambda_{g,q} : C^\infty(\partial\Omega) \to C^\infty(\partial\Omega), \quad \Lambda_{g,q} f = \partial_\nu u|_{\partial\Omega}.$$

Here ∂_ν denotes the normal derivative with respect to g.

One of the fundamental open questions is to determine a potential q from the knowledge of $\Lambda_{g,q}$, when the metric g is known.

Question 2.1. *Let g be a Riemannian metric on $\overline{\Omega}$, and let $q_1, q_2 \in C^\infty(\overline{\Omega})$. If $\Lambda_{g,q_1} = \Lambda_{g,q_2}$, is it true that $q_1 = q_2$?*

We saw in Theorem 1.1 that the answer is affirmative when g is the Euclidean metric, i.e. $g_{jk}(x) = \delta_{jk}$. The argument was based on complex geometrical optics (CGO) solutions that look approximately like $e^{\rho\cdot x}$ for suitable $\rho \in \mathbb{C}^n$. One challenge in the geometric case is that there is no obvious analogue of CGO solutions. However, there is a special class of metrics where certain CGO solutions have been constructed.

Definition. A metric g is *CTA (conformally transversally anisotropic)* if there are some coordinates $x = (x_1, x')$ in $\overline{\Omega}$ such that

$$g(x_1, x') = c(x) \begin{pmatrix} 1 & 0 \\ 0 & g_0(x') \end{pmatrix}$$

where $c \in C^\infty(\overline{\Omega})$ is positive, and $g_0(x')$ is a Riemannian metric in some bounded open set $\Omega_0 \subset \mathbb{R}^{n-1}$ with C^∞ boundary such that $\overline{\Omega} \subset \mathbb{R} \times \Omega_0$. If $c \equiv 1$, the metric is called *TA (transversally anisotropic)*.

The class of CTA metrics is related to Carleman estimates. Generalizing the linear weight $\varphi(x) = x_1$ appearing in (1.10), [KSU07] introduced the notion of a *limiting Carleman weight* and solved a partial data inverse problem in the presence of such a weight. It was proved in [DKSU09] that locally a metric g admits a limiting Carleman weight iff g is CTA. Moreover, the Laplace-Beltrami operator of a TA metric takes the form

$$\Delta_g = \partial_1^2 + \Delta_{g_0}$$

where Δ_{g_0} only acts in the x' variables. This is formally similar to the wave operator

$$\Box = \partial_t^2 - \Delta_{g_0}.$$

Even though the Laplace and wave equations have a very different character, there are a number of analogies between inverse problems for the wave equation and Laplace-Beltrami equation on CTA manifolds.

Specializing to CTA manifolds, also the following case of Question 2.1 is open.

Question 2.2. *Let g be a CTA metric on $\overline{\Omega}$, and let $q_1, q_2 \in C^\infty(\overline{\Omega})$. If $\Lambda_{g,q_1} = \Lambda_{g,q_2}$, is it true that $q_1 = q_2$?*

We will discuss a result from [DKSU09] showing that this is true under an extra geometric condition on the metric g_0 (this condition was later weakened in [DKLS16]).

Theorem 2.1 (Uniqueness)**.** *Let g be a CTA metric and $q_1, q_2 \in C^\infty(\overline{\Omega})$. Assume that the metric g_0 on $\overline{\Omega}_0$ is* simple. *If $\Lambda_{g,q_1} = \Lambda_{g,q_2}$, then $q_1 = q_2$.*

The simplicity condition is a restriction on the behaviour of geodesic curves of the manifold $(\overline{\Omega}_0, g_0)$. We will give a precise definition later. The simplicity condition is satisfied e.g. for strictly convex domains in $\mathbb{R}^n$ or in a space of nonpositive sectional curvature, and small metric perturbations of these.

The proof of Theorem 2.1 follows the same pattern as the proof of Theorem 1.1 and consists of three steps:

1. An integral identity showing that $q_1 - q_2$ is L^2-orthogonal to products $u_1 u_2$ where $(-\Delta_g + q_j)u_j = 0$.
2. Construction of CGO solutions u_j based on a Carleman estimate and duality.
3. Inserting the CGO solutions in the integral identity and showing that a certain transform of $q_1 - q_2$ vanishes.

The integral identity and Carleman estimate will be almost the same as in the Euclidean case. However, the construction of CGO solutions will be more geometric, and the transform used in Step 3 will be different.

2.1. **Integral identity.** Before stating the integral identity we will introduce without proofs some facts from Riemannian geometry. The Riemannian metric g induces a natural inner product

$$(u, v)_{L^2(\Omega)} = \int_\Omega u\bar{v}\,dV_g, \qquad dV_g(x) = |g(x)|^{1/2}\,dx.$$

There is an induced inner product on $L^2(\partial\Omega)$ based on a measure dS_g. One has a Riemannian normal derivative $\partial_\nu u$ and the integration by parts (or Green's) identity

$$\int_{\partial\Omega} (\partial_\nu u)v\,dS_g = \int_\Omega ((\Delta_g u)v + \langle \nabla u, \nabla v\rangle_g)\,dV_g$$

where $\langle \nabla u, \nabla v\rangle_g$ denotes the Riemannian inner product

$$\langle \nabla u, \nabla v\rangle_g = \sum_{j,k=1}^n g^{jk}\partial_j u \partial_k v.$$

We will also use the Riemannian length

$$|\nabla u|_g = \langle \nabla u, \nabla u\rangle_g^{1/2}.$$

With these facts, the following integral identity in the Riemannian case can be proved in exactly the same way as its Euclidean counterpart in Proposition 2.2.

Proposition 2.2 (Integral identity)**.** *Let g be a Riemannian metric on $\overline{\Omega}$ and let $q_1, q_2 \in C^\infty(\overline{\Omega})$. For any $f_1, f_2 \in C^\infty(\partial\Omega)$, one has*

$$\int_{\partial M} ((\Lambda_{g,q_1} - \Lambda_{g,q_2})f_1)\bar{f}_2 \, dS_g = \int_\Omega (q_1 - q_2) u_1 \bar{u}_2 \, dV_g$$

where $u_j \in H^1(\Omega)$ solve $(-\Delta_g + q_j)u_j = 0$ in Ω with $u_j|_{\partial\Omega} = f_j$.

2.2. **Carleman estimate and solvability.** Recall from (1.8) that in the Euclidean case we constructed CGO solutions of the form

$$u = e^{\tau x_1}(a(x') + r) \tag{2.2}$$

where $a(x') = e^{i\tau\beta\cdot x}$ with β being a unit vector satisfying $\beta \cdot e_1 = 0$, and $\|r\|_{L^2(\Omega)} \to 0$ as $\tau \to \infty$. We wish to make a similar construction in the CTA manifold case based on a Carleman estimate.

First it is convenient to make a Liouville reduction as in (1.1): if c is a positive scalar function, a short computation (exercise) gives the identity

$$(-\Delta_{c\bar{g}} + q)(c^{-\frac{n-2}{4}} v) = c^{-\frac{n+2}{4}}(-\Delta_{\bar{g}} + c(q - q_c))v,$$

where $q_c = c^{\frac{n-2}{4}} \Delta_{c\bar{g}}(c^{-\frac{n+2}{4}})$. With this argument, at the expense of changing the potential q in a controlled way, we can reduce the case of a CTA metric to a TA metric of the form

$$g(x_1, x') = \begin{pmatrix} 1 & 0 \\ 0 & g_0(x') \end{pmatrix} \tag{2.3}$$

From now on we will assume that g is a TA metric of this form.

If g has the form (2.3), the Laplace-Beltrami operator is

$$\Delta_g = \partial_1^2 + \Delta_{g_0}$$

where Δ_{g_0} only acts in the x' variables. We consider the ansatz (2.2) and note that the conjugated Laplace-Beltrami operator $P_\tau := e^{-\tau x_1} \circ \Delta_g \circ e^{\tau x_1}$ satisfies

$$\begin{aligned} P_\tau v &= e^{-\tau x_1}(\partial_1^2 + \Delta_{g_0})(e^{\tau x_1} v) = ((\partial_1 + \tau)^2 + \Delta_{g_0})v \\ &= (\partial_1^2 + \Delta_{g_0} + \tau^2 + 2\tau\partial_1)v. \end{aligned} \tag{2.4}$$

In particular, the function u in (2.2) solves $(-\Delta_g + q)u = 0$ iff r solves

$$(P_\tau - q)r = f$$

where

$$f = -(P_\tau - q)(a(x')) = (-\Delta_{g_0} - \tau^2 + q(x))a(x'). \tag{2.5}$$

Iin the Euclidean case we chose explicit functions $a(x') = e^{i\tau\beta\cdot x}$ which satisfy $(-\Delta_{x'} - \tau^2)(a(x')) = 0$. These functions are called *plane waves* propagating in direction β. In the Riemannian case, we will instead construct functions

$$a(x') = a_\tau(x')$$

that are *approximate* solutions of $(-\Delta_{g_0} - \tau^2)a = 0$ in Ω_0 in the sense that

$$\|(-\Delta_{g_0} - \tau^2)a_\tau\|_{L^2(\Omega_0)} \leq C, \tag{2.6}$$

$$\|a_\tau\|_{L^2(\Omega_0)} \leq C, \tag{2.7}$$

uniformly over $\tau \geq 1$. Such functions are called *approximate eigenfunctions* or *quasimodes*. (An eigenfunction is a solution of $(-\Delta_{g_0} - \tau^2)a = 0$.)

Assuming that we have found a satisfying (2.6)–(2.7), the function f in (2.5) satisfies $\|f\|_{L^2(\Omega)} \leq C$. The following solvability result will then produce a correction term r satisfying $\|r\|_{L^2(\Omega)} \leq \frac{C}{|\tau|}$.

Proposition 2.3 (Solvability for P_τ)**.** *Let g be a TA metric and let $q \in L^\infty(\Omega)$. There are $C, \tau_0 > 0$ such that whenever $\tau \in \mathbb{R}$ and $|\tau| \geq \tau_0$, for any $f \in L^2(\Omega)$ the equation*

$$(P_\tau - q)r = f \text{ in } \Omega$$

has a solution $r \in L^2(\Omega)$ satisfying

$$\|r\|_{L^2(\Omega)} \leq \frac{C}{|\tau|}\|f\|_{L^2(\Omega)}.$$

Proof. The proof is almost identical to the Euclidean case and is based on the Carleman estimate

$$\|u\|_{L^2(\Omega)} \leq \frac{C}{|\tau|}\|(P_{-\tau} - q)u\|_{L^2(\Omega)}, \qquad u \in C_c^\infty(\Omega), \tag{2.8}$$

which is the Riemannian counterpart of (1.9). The proof in the Euclidean case was based on the decomposition

$$P_{-\tau} = A + iB$$

where A and B are self-adjoint. Since g is a TA metric, from (2.4) we see that in our case

$$A = \partial_1^2 + \Delta_{g_0} + \tau^2, \qquad B = 2i\tau\partial_1.$$

As in the proof of Proposition 1.6, for $u \in C_c^\infty(\Omega)$ we have

$$\begin{aligned} \|P_{-\tau}u\|^2 &= ((A + iB)u, (A + iB)u) \\ &= \|Au\|^2 + \|Bu\|^2 + (i[A, B]u, u). \end{aligned}$$

But since Δ_{g_0} only acts on the x' variables, also in this case we have

$$[A, B] \equiv 0.$$

We can then invoke a Poincaré inequality using the $\|Bu\|^2$ term as in the Euclidean case to prove the Carleman estimate (2.8). The solvability result, Proposition 2.3, follows from (2.8) by duality just as in Proposition 1.5. $\square$

2.3. **Quasimodes and geodesics.** By the discussion above, in order to construct CGO solutions of the form

$$u = e^{\tau x_1}(a(x') + r),$$

it is enough to find suitable quasimodes $a(x') = a_\tau(x') \in C^\infty(\overline{\Omega}_0)$ satisfying (2.6)–(2.7). In the Euclidean case we chose explicit plane waves $a(x') = e^{i\tau\beta\cdot x}$ that satisfy $(-\Delta_{x'} - \tau^2)a = 0$. In the Riemannian case one can instead construct quasimodes of geometric nature that are associated with *geodesic curves*.

Proposition 2.4 (Quasimodes)**.** *Let Ω_0 be a bounded C^∞ domain, let g_0 be a Riemannian metric, and let $\gamma : [0,T] \to \overline{\Omega}_0$ be a maximal unit speed geodesic curve in $(\overline{\Omega}_0, g_0)$. There is $C > 0$ such that for any $\tau \geq 1$ there is $a = a_\tau \in C^\infty(\overline{\Omega}_0)$ supported in a small neighborhood of $\gamma([0,T])$, satisfying*

$$\begin{aligned} \|(-\Delta_{g_0} - \tau^2)a_\tau\|_{L^2(\Omega_0)} &\leq C, \\ \|a_\tau\|_{L^2(\Omega_0)} &\leq C, \end{aligned}$$

and satisfying for any $\varphi \in C_c^\infty(\Omega_0)$

$$\lim_{\tau\to\infty} \int_{\Omega_0} \varphi |a_\tau|^2 \, dV_{g_0} = \int_0^T \varphi(\gamma(t))\, dt. \tag{2.9}$$

Above, a geodesic curve $\gamma : [0,T] \to \overline{\Omega}_0$ is a smooth curve that minimizes the length functional

$$L_{g_0}(\eta) = \int_0^T |\dot{\eta}(t)|_{g_0} \, dt$$

among all (piecewise) smooth curves $\eta : [0,T] \to \overline{\Omega}_0$ with $\eta(0) = \gamma(0)$ and $\eta(T) = \gamma(T)$. As a minimizer of L_{g_0}, any geodesic curve satisfies an Euler-Lagrange equation, called *geodesic equation*, which is given by (exercise)

$$\ddot{\gamma}^l(t) + \Gamma_{jk}^l(\gamma(t))\dot{\gamma}^j(t)\dot{\gamma}^k(t) = 0$$

where $\Gamma_{jk}^l = \frac{1}{2} g_0^{lm}(\partial_j g_{0,km} + \partial_k g_{0,jm} - \partial_m g_{0,jk})$ are the Christoffel symbols of the metric g_0.

It follows from the geodesic equation that for any $x_0 \in \overline{\Omega}_0$ and $v_0 \in \mathbb{R}^n$, there is a unique geodesic γ_{x_0,v_0} with

$$\gamma(0) = x_0, \qquad \dot{\gamma}(0) = v_0.$$

Moreover, any geodesic satisfies

$$|\dot{\gamma}(t)|_{g_0} \equiv \text{const.}$$

By doing a constant reparametrization we may assume that all geodesics have unit speed, i.e. that

$$|\dot{\gamma}(t)|_{g_0} \equiv 1.$$

If g_0 is the Euclidean metric (i.e. $g_{0,jk} = \delta_{jk}$), then all Christoffel symbols vanish and geodesics are precisely straight line segments. A geodesic curve $\gamma : [0, T] \to \overline{\Omega}_0$ is *maximal* if it cannot be extended to a larger interval. Finally, the identity (2.9) means that

$$|a|^2 \, dV_{g_0} \to \delta_\gamma$$

in the sense of distributions. Here δ_γ denotes the delta function of the geodesic γ.

2.4. **Simple manifolds and proof of Theorem 2.1.** We will sketch a proof of Proposition 2.4 below in the special case where the Riemannian manifold $(\overline{\Omega}_0, g_0)$ is *simple*. Given $x_0 \in \overline{\Omega}_0$, we define the g_0-unit sphere

$$S_{x_0} = \{v \in \mathbb{R}^n \,:\, |v|_{g_0} = 1\}.$$

The geodesics $\gamma_{x_0,v}$ with $v \in S_{x_0}$ are called *radial geodesics* starting at x_0.

Definition. The Riemannian manifold $(\overline{\Omega}_0, g_0)$ is *simple* is $\overline{\Omega}_0$ has strictly convex boundary (with respect to g_0-geodesics), and if for any $x_0 \in \overline{\Omega}_0$ the radial geodesics starting at x_0 parametrize $\overline{\Omega}_0$ bijectively.

Simplicity in particular implies that for any x_0, radial geodesics γ_{x_0,v_1} and γ_{x_0,v_2} for $v_1 \neq v_2$ never intersect after $t = 0$. This is a restriction on the behaviour of geodesics that is always satisfied e.g. for metrics g_0 with nonpositive curvature.

Simple manifolds have the very important property that functions are uniquely determined by their integrals over maximal geodesics, i.e. the *geodesic X-ray transform* is injective. This result due to [Mu77] is fundamental in geometric inverse problems (see [PSU23] for a detailed account). For the Euclidean metric, i.e. integration over straight lines, this result was proved by J. Radon already in 1917.

Theorem 2.5 (Geodesic X-ray transform)**.** *Let $(\overline{\Omega}_0, g_0)$ be simple. If $f \in C(\overline{\Omega}_0)$ and if*

$$\int_\gamma f(\gamma(t)) \, dt = 0$$

for any maximal geodesic γ on $\overline{\Omega}_0$, then $f = 0$.

After all these prerequisites we can give the proof of Theorem 2.1. The transform used in recovering the coefficients will be a mixed Fourier/geodesic X-ray transform.

Proof of Theorem 2.1. As discussed above, we may reduce to the case where g_0 is a TA metric of the form (2.3). In order to remove some technicalities in the presentation, we will also assume that

$$\partial^\alpha q_1(z) = \partial^\alpha q_1(z) \tag{2.10}$$

for any $z \in \partial\Omega$ and any multi-index α. This can be proved by a boundary determination result, see e.g. [DKSU09, Section 8].

The assumption $\Lambda_{g,q_1} = \Lambda_{g,q_2}$ together with the integral identity in Proposition 2.2 imply that

$$\int_\Omega (q_1 - q_2) u_1 \bar{u}_2 \, dV_g = 0 \tag{2.11}$$

for any $u_j \in H^1(\Omega)$ solving $(-\Delta_g + q_j)u_j = 0$. Let $\gamma : [0,T] \to \overline{\Omega}_0$ be a maximal geodesic in $(\overline{\Omega}_0, g_0)$. For $\tau > 0$ sufficiently large we construct CGO solutions

$$u_1 = e^{\tau x_1}(a_\tau(x') + r_1), \tag{2.12}$$

$$u_2 = e^{-\tau x_1}(a_\tau(x') + r_2), \tag{2.13}$$

where $a_\tau(x')$ is a quasimode associated with the geodesic γ constructed in Proposition 2.4 (note that a_τ does not depend on the potentials q_j). The correction terms r_j are obtained from Proposition 2.3 and they satisfy $\|r_j\|_{L^2(\Omega)} \leq \frac{C}{\tau}$.

Let f be the zero extension of $q_1 - q_2$ outside Ω. By (2.10) we have $f \in C_c^\infty(\mathbb{R}^n)$. Since $\overline{\Omega} \subset \mathbb{R} \times \Omega_0$, the identity (2.11) implies

$$\int_{-\infty}^{\infty} \int_{\Omega_0} f(x_1, x') |a_\tau(x')|^2 \, dV_{g_0} \, dx_1 = O(\tau^{-1})$$

as $\tau \to \infty$. We rewrite this in terms of the partial Fourier transform

$$\tilde{f}(\lambda, x') = \int_{-\infty}^{\infty} e^{-i\lambda x_1} f(x_1, x') \, dx_1$$

as

$$\int_{\Omega_0} \tilde{f}(0, \,\cdot\,) |a_\tau|^2 \, dV_{g_0} = O(\tau^{-1}).$$

Since $\tilde{f}(0, \,\cdot\,) \in C_c^\infty(\Omega_0)$, taking the limit as $\tau \to \infty$ and using (2.9) gives

$$\int_\gamma \tilde{f}(0, \gamma(t)) \, dt = 0.$$

This is true for any maximal geodesic γ in $(\overline{\Omega}_0, g_0)$, so the geodesic X-ray transform of $\tilde{f}(0, \,\cdot\,)$ vanishes. We invoke Theorem 2.5 to conclude that

$$\tilde{f}(0, x') = 0 \text{ for all } x' \in \overline{\Omega}_0. \tag{2.14}$$

The conclusion (2.14) is not quite enough to show that $f = 0$. However, one can employ the trick of replacing the positive number τ by a *slightly* *complex* parameter $\tau + i\mu$ where $\mu \in \mathbb{R}$ is fixed and $\tau \to \infty$. After doing this and taking some derivatives in μ, one can prove that

$$\partial_\lambda^k \tilde{f}(0, x') = 0 \text{ for all } k \geq 0 \text{ and } x' \in \overline{\Omega}_0.$$

But now, since $f(\,\cdot\,, x')$ is compactly supported, $\tilde{f}(\,\cdot\,, x')$ is real-analytic by the Paley-Wiener theorem for the Fourier transform. This implies that $\tilde{f} \equiv 0$ and consequently $q_1 = q_2$. $\square$

2.5. **Quasimodes in the simple case.** It remains to sketch a proof of Proposition 2.4 when $(\overline{\Omega}_0, g_0)$ is simple. A detailed proof may be found in e.g. [Sa12]. We use a *geometrical optics* (or WKB) ansatz and look for a in the form

$$a(x') = e^{i\tau\psi(x')} b(x') \tag{2.15}$$

where $\psi \in C^\infty(\overline{\Omega}_0)$ is a real *phase function* and $b \in C^\infty(\overline{\Omega}_0)$ is an *amplitude*. (In the Euclidean case we have $\psi(x') = \beta \cdot x$ and $b(x') = 1$.) The geometrical optics ansatz is a classical method in the construction of asymptotic solutions depending on a large parameter τ (see e.g. [Ev10, Section 4.5.3]). If g_0 is not simple, one can use a more general construction of *Gaussian beam quasimodes* to prove Proposition 2.4 in full (see [DKLS16]).

We wish to apply the operator $\Delta_{g_0} + \tau^2$ to the ansatz (2.15). To do this, we observe that for any Riemannian metric g one has

$$\begin{aligned}\Delta_g v &= \sum_{j,k} |g|^{-1/2} \partial_j (|g|^{1/2} g^{jk} \partial_k v) \\ &= \mathrm{div}_g(\nabla_g v)\end{aligned}$$

where the Riemannian gradient and divergence are defined by

$$\begin{aligned}\nabla_g v &= (\sum_k g^{1k} \partial_k v, \ldots, \sum_k g^{nk} \partial_k v), \\ \mathrm{div}_g(X) &= \sum_j |g|^{-1/2} \partial_j (|g|^{1/2} X^j).\end{aligned}$$

We thus compute (exercise)

$$\begin{aligned}(\Delta_{g_0} + \tau^2) a &= \mathrm{div}_{g_0}(\nabla_{g_0}(e^{i\tau\psi} b)) + \tau^2 b \\ &= \mathrm{div}_{g_0}[e^{i\tau\psi}(\nabla_{g_0} b + i\tau(\nabla_{g_0}\psi) b)] + \tau^2 b \\ &= e^{i\tau\psi}\big(\tau^2[-\langle \nabla_{g_0}\psi, \nabla_{g_0}\psi\rangle + 1] b \\ &\qquad + i\tau[2\langle \nabla_{g_0}\psi, \nabla_{g_0} b\rangle + (\Delta_{g_0}\psi) b] \\ &\qquad + \Delta_{g_0} b\big).\end{aligned} \tag{2.16}$$

By looking at the τ^2 and $i\tau$ terms, in order to have $\|(-\Delta_{g_0}-\tau^2)a\|_{L^2(\Omega_0)} \leq C$ for $|\tau|$ large it would be enough to find a phase function $\psi \in C^\infty(\overline{\Omega}_0)$ and an amplitude $b \in C^\infty(\overline{\Omega}_0)$ satisfying the equations

$$\begin{cases} \langle \nabla_{g_0}\psi, \nabla_{g_0}\psi\rangle = 1 \quad \text{(eikonal equation)}, \\ 2\langle \nabla_{g_0}\psi, \nabla_{g_0} b\rangle + (\Delta_{g_0}\psi)b = 0 \quad \text{(transport equation)}. \end{cases}$$

The *eikonal equation* is an important first order nonlinear PDE. It can always be solved locally (e.g. by the method of characteristics), but C^∞ global solutions do not exist in general. However, the assumption that $(\overline{\Omega}_0, g_0)$ is simple will ensure that there are many global solutions. Given a solution ψ of the eikonal equation, the transport equation is a *linear* first order PDE and it is easy to find smooth solutions.

We start the construction by choosing a slightly larger bounded C^∞ domain Ω_1 with $\overline{\Omega}_0 \subset \Omega_1$, and extend g_0 to $\overline{\Omega}_1$ so that also $(\overline{\Omega}_1, g_0)$ is simple. Let $\gamma : [0, T] \to \overline{\Omega}_0$ be a maximal geodesic, extend γ to Ω_1, and fix some point $p \in \Omega_1 \setminus \overline{\Omega}_0$ on γ. We also identify the unit sphere S_p with S^{n-1}. The radial geodesics starting at p are given by $\gamma_{p,\omega}(r)$ for $r \geq 0$ and our original geodesic γ is part of γ_{p,ω_0}. By the definition of a simple manifold, the map

$$(r, \omega) \mapsto \gamma_{p,\omega}(r)$$

is bijective onto $\overline{\Omega}_1$. We consider (r, ω) as *Riemannian polar coordinates.* The metric g_0 in these coordinates takes the form

$$g_0(r, \omega) = \begin{pmatrix} 1 & 0 \\ 0 & h(r, \omega) \end{pmatrix}. \tag{2.17}$$

In fact, the 1 in the upper left corner comes from the fact that geodesics are unit speed, and the 0 entries in the first row and column are due to the Gauss lemma in Riemannian geometry. Note that since p is outside of $\overline{\Omega}$, the coordinates (r, ω) are smooth in $\overline{\Omega}_0$.

We may now choose

$$\psi(r, \omega) = r.$$

One can alternatively express ψ as the distance function

$$\psi = \mathrm{dist}_{g_0}(\,\cdot\,, p).$$

Then $\psi \in C^\infty(\overline{\Omega}_0)$ and by (2.17) one has

$$\langle \nabla_{g_0}\psi, \nabla_{g_0}\psi\rangle = \langle \nabla_{g_0} r, \nabla_{g_0} r\rangle = 1.$$

This gives the required solution of the eikonal equation.

We proceed to solving the transport equation. We will need to make the amplitude b slightly τ-dependent and write

$$b = b_0 + \tau^{-1} b_1.$$

Since ψ solves the eikonal equation, we obtain from (2.16) after grouping like powers of τ that

$$(\Delta_{g_0} + \tau^2)a = e^{i\tau\psi}\big(i\tau L b_0 + (iLb_1 + \Delta_{g_0} b_0) + \tau^{-1}\Delta_{g_0} b_1\big).$$

Here we denoted by L transport operator

$$Lw = 2\langle \nabla_{g_0}\psi, \nabla_{g_0} w\rangle + (\Delta_{g_0}\psi) w.$$

We now wish to find b_0 and b_1 solving

$$\begin{aligned} Lb_0 &= 0, \\ Lb_1 &= i\Delta_{g_0} b_0. \end{aligned}$$

Using (2.17), the transport equation for b_0 becomes

$$2\partial_r b_0 + h b_0 = 0$$

where $h = \Delta_{g_0} r = |g_0|^{-1/2}\partial_r(|g_0|^{1/2})$ is a smooth function in $\overline{\Omega}_0$. This ODE has the solution

$$b_0(r,\omega) = |g_0|^{-1/4}\chi(\omega)$$

where $\chi \in C^\infty(S^{n-1})$.

The transport equation for b_1 becomes

$$2\partial_r b_1 + h b_1 = f$$

with $f = i\Delta_{g_0} b_0$. This ODE can be solved by integrating in r and it has a smooth solution b_1 satisfying

$$\|b_1\|_{H^2(\overline{\Omega}_0)} \leq C\|b_0\|_{H^4(\overline{\Omega}_0} \leq C\|\chi\|_{H^4(\overline{\Omega}_0}.$$

Recall that we need a to satisfy the estimates in Proposition 2.4. With the above choices, we have

$$\|a\|_{L^2(\Omega_0)} = \|e^{i\tau\psi} b\|_{L^2(\Omega_0)} \leq C\|\chi\|_{L^2(\Omega_0)}$$

and

$$\|(\Delta_{g_0} + \tau^2)a\|_{L^2(\Omega_0)} = \|e^{i\tau\psi}\tau^{-1}\Delta_{g_0} b_1\|_{L^2(\Omega_0)} \leq C\tau^{-1}\|\chi\|_{H^4(\overline{\Omega}_0}.$$

Moreover, if $\varphi \in C_c^\infty(\Omega_0)$ we have the integral

$$\int_{\Omega_0} \varphi |a|^2 \, dV_{g_0} = \int_{\Omega_0} \varphi |g_0|^{-1/2}|\chi|^2 |g_0|^{1/2} \, dx' = \int \varphi(r,\omega)|\chi(\omega)|^2 \, dr \, d\omega.$$

We would like the last integral to converge to $\int_0^T \varphi(r,0)\,dr = \int_0^T \varphi(\gamma(t))\,dt$ as $\tau \to \infty$. It is enough to choose χ as the mollifier

$$\chi(\omega) = \chi_\tau(\omega) = \varepsilon^{-\frac{n-1}{2}}\chi_0(\omega/\varepsilon)$$

where $\chi_0 \in C_c^\infty(S^{n-1})$ is supported near ω_0, satisfies $0 \leq \chi_0 \leq 1$ and $\int \chi_0^2 = 1$. Here $\varepsilon = \varepsilon(\tau) = \tau^\alpha$ for a suitable $\alpha > 0$. With this choice one has the required convergence

$$\lim_{\tau \to \infty} \int_{\Omega_0} \varphi |a|^2 \, dV_{g_0} = \int_0^T \varphi(\gamma(t)) \, dt.$$

We also have $\|\chi\|_{L^2(\Omega_0)} \leq C$, so $\|a\|_{L^2(\Omega_0)} \leq C$. Finally,

$$\|(\Delta_{g_0} + \tau^2) a\|_{L^2(\Omega_0)} \leq C\tau^{-1} \|\chi\|_{H^4(\overline{\Omega}_0} \leq C\tau^{-1}\varepsilon^{-4} = C\tau^{4\alpha - 1}.$$

If we choose $\alpha \leq 1/4$ then we have $\|(\Delta_{g_0} + \tau^2) a\|_{L^2(\Omega_0)} \leq C$ uniformly over $\tau \geq 1$. This concludes the proof of Proposition 2.4.

3. Inverse problems for semilinear PDE

In the previous sections we considered the linear equation

$$(-\Delta_g + q)u = 0 \text{ in } \Omega.$$

We proved uniqueness in inverse problems when g is Euclidean or g has a special form (it is CTA and the transversal metric g_0 is simple). Now we move to *nonlinear* equations. It is well known that the existence theory for solutions of nonlinear elliptic PDE is often more involved than in the linear case [Ev10, Part III]. However, starting with the works [FO20, LL+21a] it was observed that inverse problems for certain nonlinear elliptic PDE may be easier to solve than their counterparts for linear equations. The idea is that nonlinear interactions may create new phenomena that can be helpful in solving inverse problems. In other words, *nonlinearity helps*!

In this section we will consider semilinear equations of the form

$$-\Delta_g u + a(x,u) = 0.$$

To simplify the presentation we will only discuss a model equation with cubic nonlinearity involving a potential $q \in C^\infty(\overline{\Omega})$. The equation is given by

$$\left\{ \begin{array}{rll} -\Delta_g u + qu^3 &= 0 & \text{in } \Omega, \\ u &= f & \text{on } \partial\Omega. \end{array} \right. \tag{3.1}$$

If $q \geq 0$, this equation has a maximum principle and for any $f \in C^\infty(\partial\Omega)$ there is a unique solution $u \in C^\infty(\overline{\Omega})$ [Ta96, Section 14.1]. For a general potential q we see that at least $u \equiv 0$ is a solution. Moreover, if u is small (say of size ε) then the nonlinearity qu^3 is negligible (of size ε^3) compared to the linear part $\Delta_g u$. Thus for *small* Dirichlet data f one can use existence theory for linear equations together with the Banach fixed point theorem to show that there is a unique *small* solution u [FO20, LL+21a].

One can then define the *nonlinear DN map for small data* by

$$\Lambda_{g,q}^{\mathrm{NL}} : \{f \in C^\infty(\partial\Omega) \,;\, \|f\|_{C^{2,\alpha}(\partial\Omega)} < \delta\} \to C^\infty(\partial\Omega), \quad f \mapsto \partial_\nu u|_{\partial\Omega}.$$

We will prove:

Theorem 3.1. *Let g be a CTA metric on $\overline{\Omega}$, and let $q_1, q_2 \in C^\infty(\overline{\Omega})$. If*

$$\Lambda_{g,q_1}^{\mathrm{NL}} = \Lambda_{g,q_2}^{\mathrm{NL}},$$

then $q_1 = q_2$.

Recall that we proved a similar result for linear equations in Theorem 2.1, but there we had to assume an extra condition on the transversal metric g_0. This is an example of a case where nonlinearity helps. Another example is the local data problem, which is open for linear equations (see Question 1.2) but the corresponding result for nonlinear PDE is known [KU20, LL+21b].

A standard method for dealing with inverse problems for nonlinear equations is *linearization*. Namely, if one knows the nonlinear DN map $\Lambda_{g,q}^{\mathrm{NL}}(f)$ for small f, then one also knows its linearization, or Fréchet derivative,

$$(D\Lambda_{g,q}^{\mathrm{NL}})_0(h) = \partial_\varepsilon \Lambda_{g,q}^{\mathrm{NL}}(\varepsilon h)|_{\varepsilon=0}, \qquad h \in C^\infty(\partial\Omega).$$

Let u_ε be the small solution of (3.1) with boundary value $f = \varepsilon h$, i.e.

$$\left\{ \begin{aligned} -\Delta_g u_\varepsilon + q u_\varepsilon^3 &= 0 && \text{in } \Omega, \\ u_\varepsilon &= \varepsilon h && \text{on } \partial\Omega. \end{aligned} \right. \tag{3.2}$$

Note that $u_0 = 0$, since $u = 0$ is the unique small solution with boundary value 0. Formally differentiating (3.2) in ε gives that

$$-\Delta_g(\partial_\varepsilon u_\varepsilon) + 3q u_\varepsilon^2 \partial_\varepsilon u_\varepsilon = 0.$$

Setting $\varepsilon = 0$ and using that $u_0 = 0$, we see that

$$v_h := \partial_\varepsilon u_\varepsilon|_{\varepsilon=0}$$

solves the linear equation

$$\left\{ \begin{aligned} -\Delta_g v_h &= 0 && \text{in } \Omega, \\ v_h &= h && \text{on } \partial\Omega. \end{aligned} \right. \tag{3.3}$$

Thus the linearized solution v_h is just the harmonic function in $(\overline{\Omega}, g)$ with boundary value h. This formal computation can be justified. Since

$$(D\Lambda_{g,q}^{\mathrm{NL}})_0(h) = \partial_\varepsilon \Lambda_{g,q}^{\mathrm{NL}}(\varepsilon h)|_{\varepsilon=0} = \partial_\varepsilon \partial_\nu u_\varepsilon|_{\varepsilon=0} = \partial_\nu v_h$$

this leads to the following:

Lemma 3.2 (Linearization of nonlinear DN map)**.**

$$(D\Lambda_{g,q}^{\mathrm{NL}})_0(h) = \Lambda_{g,0} h$$

where $\Lambda_{g,0}$ is the DN map for the Laplace equation (3.3) with no potential.

This shows that from the knowledge of $\Lambda_{g,q}^{\mathrm{NL}}$, we can recover its linearization $(D\Lambda_{g,q}^{\mathrm{NL}})_0 = \Lambda_{g,0}$. However, this *first linearization* does not contain any information about the unknown potential q. It turns out that for the nonlinearity qu^3, the right thing to do is to look at the *third linearization*, i.e. the *third order Fréchet derivative* $(D^3\Lambda_{g,q}^{\mathrm{NL}})_0$.

The third linearization can be computed by considering Dirichlet data of the form $f = \varepsilon_1 h_1 + \varepsilon_2 h_2 + \varepsilon_3 h_3$ where $h_j \in C^\infty(\partial\Omega)$. Writing $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$, let u_ε be the solution of

$$\left\{ \begin{aligned} -\Delta_g u_\varepsilon + q u_\varepsilon^3 &= 0 && \text{in } \Omega, \\ u_\varepsilon &= \varepsilon_1 h_1 + \varepsilon_2 h_2 + \varepsilon_3 h_3 && \text{on } \partial\Omega. \end{aligned} \right. \tag{3.4}$$

We formally apply the derivative $\partial_{\varepsilon_1\varepsilon_2\varepsilon_3}$ to this equation to obtain

$$\begin{aligned} 0 &= -\Delta_g(\partial_{\varepsilon_1\varepsilon_2\varepsilon_3} u_\varepsilon) + q\partial_{\varepsilon_1\varepsilon_2\varepsilon_3}(u_\varepsilon^3) \\ &= -\Delta_g(\partial_{\varepsilon_1\varepsilon_2\varepsilon_3} u_\varepsilon) + q\partial_{\varepsilon_1\varepsilon_2}(3u_\varepsilon^2 \partial_{\varepsilon_3} u_\varepsilon) \\ &= -\Delta_g(\partial_{\varepsilon_1\varepsilon_2\varepsilon_3} u_\varepsilon) + q\partial_{\varepsilon_1}(6u_\varepsilon \partial_{\varepsilon_2} u_\varepsilon \partial_{\varepsilon_3} u_\varepsilon + 3u_\varepsilon^2 \partial_{\varepsilon_2\varepsilon_3} u_\varepsilon) \\ &= -\Delta_g(\partial_{\varepsilon_1\varepsilon_2\varepsilon_3} u_\varepsilon) + 6q\partial_{\varepsilon_1} u_\varepsilon \partial_{\varepsilon_2} u_\varepsilon \partial_{\varepsilon_3} u_\varepsilon + \ldots \end{aligned}$$

where $\ldots$ consists of terms that contain a power of u_ε. Since $u_0 = 0$, when we set $\varepsilon = 0$ all the terms in $\ldots$ will vanish. Thus

$$w := \partial_{\varepsilon_1\varepsilon_2\varepsilon_3} u_\varepsilon|_{\varepsilon=0}$$

will solve the equation (recall that $v_{h_j} = \partial_{\varepsilon_j} u_\varepsilon|_{\varepsilon=0}$)

$$\left\{ \begin{aligned} -\Delta_g w &= -6q v_{h_1} v_{h_2} v_{h_3} && \text{in } \Omega, \\ w &= 0 && \text{on } \partial\Omega. \end{aligned} \right. \tag{3.5}$$

Now if the know the nonlinear DN map $\Lambda_{g,q}^{\mathrm{NL}}(\varepsilon_1 h_1 + \varepsilon_2 h_2 + \varepsilon_3 h_3) = \partial_\nu u_\varepsilon$, then we also know $\partial_\nu w = \partial_\nu \partial_{\varepsilon_1\varepsilon_2\varepsilon_3} u_\varepsilon|_\varepsilon$. Thus for any $h_4 \in C^\infty(\partial\Omega)$, we also know

$$\int_{\partial\Omega} (\partial_\nu w) h_4 \, dS_g = \int_\Omega ((\Delta_g w) v_{h_4} + \langle \nabla w, \nabla v_{h_4} \rangle_g) \, dV_g.$$

Integrating by parts in the last term, and using that $w|_{\partial\Omega} = 0$ and $\Delta_g v_{h_4} = 0$, we obtain that

$$\int_{\partial\Omega} (\partial_\nu w) h_4 \, dS_g = 6 \int_\Omega q v_{h_1} v_{h_2} v_{h_3} v_{h_4} \, dV_g.$$

Since $\partial_\nu w$ is determined by $\Lambda_{g,q}^{\mathrm{NL}}$, also the right hand side is determined by $\Lambda_{g,q}^{\mathrm{NL}}$. (One can check that the left hand side is equal to

$$((D^3\Lambda_{g,q}^{\mathrm{NL}})_0(h_1, h_2, h_3), h_4)_{L^2(\partial\Omega)},$$

where $(D^3\Lambda_{g,q}^{\mathrm{NL}})_0$ is the third Fréchet derivative of $\Lambda_{g,q}^{\mathrm{NL}}$ considered as a trilinear form.) This formal argument can be justified and it leads to the following identity:

Lemma 3.3 (Integral identity in nonlinear case)**.** *If $\Lambda^{\mathrm{NL}}_{g,q_1} = \Lambda^{\mathrm{NL}}_{g,q_2}$, then*

$$\int_\Omega (q_1 - q_2) v_1 v_2 v_3 v_4 \, dV_g = 0$$

for all $v_j \in C^\infty(\overline{\Omega})$ satisfying $\Delta_g v_j = 0$ in Ω.

This integral identity related to the nonlinear equation $-\Delta_g u + qu^3 = 0$ has two benefits over the identity for the linear equation $-\Delta_g u + qu = 0$:

- $q_1 - q_2$ is L^2-orthogonal to products of *four* solutions, instead of products of two solutions;
- the solutions v_j are solutions of the Laplace equation $\Delta_g v_j = 0$, which does *not* contain the potential q.

Let us finally sketch how one proves Theorem 3.1 based on the integral identity in Lemma 3.3 and the construction of special solutions in the proof of Theorem 2.1. The main point is that instead of considering a fixed geodesic γ in $(\overline{\Omega}_0, g_0)$, one can consider *two intersecting geodesics.*

Suppose that γ_1 and γ_2 are two maximal geodesics in $(\overline{\Omega}_0, g_0)$ that intersect only at one point $x_0' \in \Omega_0$. As in (2.12)–(2.13), we first construct two harmonic functions

$$\begin{aligned} v_1 &= e^{\tau x_1}(a_\tau(x') + r_1), \\ v_2 &= e^{-\tau x_1}(\bar{a}_\tau(x') + r_2), \end{aligned}$$

where a_τ is a quasimode supported near γ_1. Similarly, we construct harmonic functions

$$\begin{aligned} v_3 &= e^{\tau x_1}(b_\tau(x') + r_3), \\ v_4 &= e^{-\tau x_1}(\bar{b}_\tau(x') + r_4), \end{aligned}$$

where b_τ is a quasimode supported near γ_2. The correction terms satisfy $\|r_j\|_{L^2(\Omega)} = O(\tau^{-1})$ as $\tau \to \infty$ and they will go away in the limit. Then the product

$$v_1 v_2 v_3 v_4 = |a_\tau(x')|^2 |b_\tau(x')|^2 (1 + O(\tau^{-1}))$$

concentrates near the *one-dimensional manifold* $\mathbb{R} \times \{x_0'\}$ as $\tau \to \infty$. Using the properties of the quasimodes a_τ and b_τ (after multiplying by suitable normalizing constants), one has

$$0 = \lim_{\tau \to \infty} \int_\Omega (q_1 - q_2) v_1 v_2 v_3 v_4 \, dV_g = \int_{-\infty}^{\infty} (q_1 - q_2)(x_1, x_0') \, dx_1.$$

The point is that one has concentration at a *single point* x_0' in Ω_0, instead of concentration near a fixed geodesic in $\overline{\Omega}_0$. It follows that the Fourier transform of $(q_1 - q_2)(\,\cdot\,, x_0')$ vanishes at 0 for every $x_0' \in \Omega_0$. By the same trick of replacing τ by a slightly complex parameter $\tau + i\mu$ as in the proof of Theorem 2.1, we see that the full Fourier transform of $(q_1 - q_2)(\,\cdot\,, x_0')$ vanishes for every $x_0' \in \Omega_0$. This implies that $q_1 = q_2$.

In general, given $x_0' \in \Omega_0$ it may not be possible to find two finite length geodesics that only intersect at x_0'. The possibility of multiple intersection points can be handled by introducing another extra parameter in the solutions. See [LL+21a] for details. This proves Theorem 3.1 in general.

4. Inverse problems for p-Laplace type equations

The Calderón problem was originally stated for the conductivity equation $\mathrm{div}(\gamma \nabla u) = 0$, where $\gamma \in L^\infty_+(\Omega)$ and $\Omega \subset \mathbb{R}^n$ is a bounded C^∞ domain. However, there are special materials such as nonlinear dielectrics and electrorheological fluids whose electrical properties are instead governed by a power law. For such materials the conductivity equation may be replaced by the p-conductivity equation

$$\begin{cases} \mathrm{div}(\gamma |\nabla u|^{p-2} \nabla u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

Here $1 < p < \infty$, and $p = 2$ is the standard conductivity equation. If $\gamma \equiv 1$ this reduces to the p-Laplace equation [Li06].

Solutions of the p-conductivity equation are minimizers of the p-Dirichlet energy

$$E_p(u) = \int_\Omega \gamma |\nabla u|^p \,dx.$$

Consider the Sobolev spaces

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) \,:\, \nabla u \in L^p(\Omega)\},$$
$$W^{1-\frac{1}{p},p}(\partial\Omega) = \{u|_{\partial\Omega} \,:\, u \in W^{1,p}(\Omega)\}.$$

A standard variational argument [SZ12] shows that for any $f \in W^{1-\frac{1}{p},p}(\partial\Omega)$, the problem (4.1) has a unique solution $u \in W^{1,p}(\Omega)$.

One can define a nonlinear DN map formally by

$$\Lambda_\gamma^p : \ f \mapsto \gamma |\nabla u|^{p-2} \partial_\nu u|_{\partial\Omega}.$$

Using a suitable weak definition, the DN map is well defined as a map $\Lambda_\gamma^p : W^{1-\frac{1}{p},p}(\partial\Omega) \to (W^{1-\frac{1}{p},p}(\partial\Omega))^*$. We wish to study the inverse problem of determining a conductivity γ from the knowledge of the nonlinear DN map Λ_γ^p.

The p-conductivity equation is quasilinear degenerate and the unknown coefficient appears in its principal part. Consequently, linearization methods do not appear to be so helpful as for the PDE studied in Section 3. We will discuss the following result from [SZ12].

Theorem 4.1 (Boundary determination)**.** *Let $\gamma_1, \gamma_2 \in C(\overline{\Omega})$. If $\Lambda^p_{\gamma_1} = \Lambda^p_{\gamma_2}$, then $\gamma_1|_{\partial\Omega} = \gamma_2|_{\partial\Omega}$.*

If $\gamma_j \in C^{1,\alpha}(\overline{\Omega})$, it was proved in [Br16] that also $\partial_\nu \gamma_1|_{\partial\Omega} = \partial_\nu \gamma_2|_{\partial\Omega}$. It is reasonable to ask the following:

Question 4.1. *Let $\gamma_1, \gamma_2 \in C^\infty(\overline{\Omega})$. If $\Lambda^p_{\gamma_1} = \Lambda^p_{\gamma_2}$, is it true that $\partial^\alpha \gamma_1|_{\partial\Omega} = \partial^\alpha \gamma_2|_{\partial\Omega}$ for any multi-index α?*

Of course one would like to determine γ in the interior of Ω, but this inverse problem remains open. When $n = 2$ one can show that if $\Lambda^p_{\gamma_1} = \Lambda^p_{\gamma_2}$ and $\gamma_1 \leq \gamma_2$, then $\gamma_1 = \gamma_2$ [GKS16]. There is also a relation between p-Laplace type equations and quasiregular mappings in two dimensions, and one could ask if the method of [AP06] would extend to $p \neq 2$.

Question 4.2. *Let $\gamma_1, \gamma_2 \in L^\infty_+(\Omega)$. If $\Lambda^p_{\gamma_1} = \Lambda^p_{\gamma_2}$, is it true that $\gamma_1 = \gamma_2$?*

In the remainder of this section we will give a proof of Theorem 4.1 following [SZ12]. Below we will be working with complex valued solutions (the case of real valued solution is discussed in [SZ12]).

4.1. **Exponential p-harmonic functions.** Recall that in the previous sections we used CGO solutions modelled after the harmonic functions $e^{\rho\cdot x}$ where $\rho \in \mathbb{C}^n$ satisfies $\rho \cdot \rho = 0$. The proof of Theorem 4.1 is based on the fact that also the p-Laplace equation admits such solutions [Wo07].

Lemma 4.2 (Exponential solutions)**.** *Let $h(x) = e^{\rho\cdot x}$ where $\rho = \alpha + i\beta$ with $\alpha, \beta \in \mathbb{R}^n$. Then $\mathrm{div}(|\nabla h|^{p-2}\nabla h) = 0$ iff $(p-1)|\alpha|^2 = |\beta|^2$ and $\alpha \cdot \beta = 0$.*

Proof. Since $\nabla h = \rho e^{\rho\cdot x}$, we have

$$\begin{aligned}
\mathrm{div}(|\nabla h|^{p-2}\nabla h) &= \mathrm{div}(|\rho|^{p-2} e^{(p-2)\alpha\cdot x} \rho e^{\rho\cdot x}) \\
&= \mathrm{div}(|\rho|^{p-2} \rho e^{(p-1)\alpha\cdot x + i\beta\cdot x}) \\
&= |\rho|^{p-2} \rho \cdot ((p-1)\alpha + i\beta) e^{(p-1)\alpha\cdot x + i\beta\cdot x}.
\end{aligned}$$

Here $\rho \cdot ((p-1)\alpha + i\beta) = (p-1)|\alpha|^2 - |\beta|^2 + ip\alpha\cdot\beta$, which proves the result. ☐

4.2. **Solutions concentrating at a boundary point.** For simplicity, we will consider the case where x_0 is a point in $\partial\Omega$ such that $\partial\Omega$ is flat near x_0. By a translation and rotation, we may assume that $x_0 = 0$ and $\Omega \cap B(0, r) = \{x \in B(0, r)\,;\, x_n > 0\}$ for some small $r > 0$.

We wish to convert the p-harmonic function $e^{\rho\cdot x}$ in Lemma 4.2 into an exact solution of $\mathrm{div}(\gamma|\nabla u|^{p-2}\nabla u) = 0$ in Ω which concentrates near the boundary point 0. To this end, define the function

$$u_0(x) = \eta_M(x) h_N(x) \tag{4.2}$$

where $\eta_M(x) = \eta(Mx)$, $h_N(x) = h(Nx)$ where M and N are large positive numbers, $\eta \in C_c^\infty(\mathbb{R}^n)$ is a nonnegative cutoff function with $\eta = 1$ for $|x| \leq 1/2$ and $\eta = 0$ for $|x| \geq 1$, and

$$h(x) = e^{(i\beta - e_n)\cdot x}$$

with $\beta \in \mathbb{R}^n$ satisfying $|\beta|^2 = p-1$ and $\beta \cdot e_n = 0$. We will choose $N = N(M)$ so that $M/N \to 0$ as $M \to \infty$. The idea is that with these choices, since h_N solves the equation with γ frozen at 0 and since u_0 is supported in the ball $B(0, 1/M)$, u_0 becomes an approximate solution to the nonlinear equation in Ω when M is large. Lemma 4.5 below gives a precise meaning to this statement.

We obtain an exact solution u by solving the Dirichlet problem with boundary values u_0,

$$\left\{ \begin{array}{rl} \mathrm{div}(\gamma(x)|\nabla u|^{p-2}\nabla u) = 0 & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega. \end{array} \right.$$

Let $f = u_0|_{\partial\Omega}$. Then we have

$$\int_{\partial\Omega} \Lambda_\gamma(f)\bar{f}\,dS = \int_\Omega \gamma|\nabla u|^{p-2}\nabla u \cdot \nabla \bar{u}_0\,dx.$$

We write this as

$$\begin{aligned} \int_{\partial\Omega} \Lambda_\gamma(f)\bar{f}\,dS = &\int_\Omega \gamma|\nabla u_0|^p\,dx \\ &+ \int_\Omega \gamma(|\nabla u|^{p-2}\nabla u - |\nabla u_0|^{p-2}\nabla u_0)\cdot\nabla\bar{u}_0\,dx. \end{aligned} \tag{4.3}$$

Note that since f is an explicit function, the left hand side is determined by the nonlinear DN map. We will recover the value of γ at 0 by taking the limit of this identity as $M \to \infty$. To analyze the limit, we need a simple lemma.

Lemma 4.3. *Let $\zeta \in C_c^\infty(B(0,1))$ and let $a \geq 0$. Then as $M \to \infty$*

$$M^{n-1}N\int_\Omega \zeta(Mx)e^{-pNx_n}\,dx \to \frac{1}{p}\int_{\mathbb{R}^{n-1}} \zeta(x',0)\,dx',$$

$$\int_\Omega x_n^a \zeta(Mx)e^{-pNx_n}\,dx = O(M^{1-n}N^{-1-a}).$$

Proof. Follows from a direct computation. $\square$

We will also employ the following inequalities: if $z, w \in \mathbb{C}^n$ and $1 < p < \infty$ we have

$$||z|^p - |w|^p| \leq p(|z|^{p-1} + |w|^{p-1})|z - w|, \tag{4.4}$$

$$||z|^{p-2}z - |w|^{p-2}w| \lesssim (|z| + |w|)^{p-2}|z - w|. \tag{4.5}$$

$$(|z| + |w|)^{p-2}|z - w|^2 \sim \mathrm{Re}\left[(|z|^{p-2}z - |w|^{p-2}w) \cdot (\bar{z} - \bar{w})\right]. \tag{4.6}$$

We compute the limit of the first term on the right hand side of (4.3).

Lemma 4.4. *We have as $M \to \infty$*

$$M^{n-1}N^{1-p}\int_\Omega \gamma|\nabla u_0|^p \, dx \to c_p\gamma(0)$$

where $c_p = p^{\frac{p-2}{2}} \int_{\mathbb{R}^{n-1}} \eta(x', 0)^p \, dx'$. We also have

$$\int_\Omega |\nabla u_0|^p \, dx = O(M^{1-n}N^{p-1}),$$
$$\int_\Omega |\eta_M \nabla h_N|^p \, dx = O(M^{1-n}N^{p-1}),$$
$$\int_\Omega |\nabla u_0 - \eta_M \nabla h_N|^p \, dx = O(M^{1-n}N^{p-1}(M/N)^p).$$

Proof. Since $u_0 = \eta_M h_N$, we compute

$$\nabla u_0 = M\nabla\eta(M\,\cdot\,)h_N + \eta_M \nabla h_N.$$

Since $\nabla h_N = N(i\beta - e_n)e^{N(i\beta - e_n)\cdot x} = N(i\beta - e_n)h_N$, we have by Lemma 4.3

$$\|M\nabla\eta(M\,\cdot\,)h_N\|^p_{L^p(\Omega)} = O(M^{1-n}N^{-1}M^p),$$
$$\|\eta_M\nabla h_N\|^p_{L^p(\Omega)} = O(M^{1-n}N^{-1}N^p).$$

This shows the last three estimates since $M/N = o(1)$ as $M \to \infty$.

For the first statement, we use the inequality (4.4) to conclude that

$$\begin{aligned}
&\left|\int_\Omega (|\nabla u_0|^p - |N(i\beta - e_n)\eta_M h_N|^p) \, dx\right| \\
&\leq p\int_\Omega |M\nabla\eta(M\,\cdot\,)h_N|(|\nabla u_0|^{p-1} + |N(i\beta - e_n)\eta_M h_N|^{p-1}) \, dx \\
&\leq p\|M\nabla\eta(M\,\cdot\,)h_N\|_{L^p(\Omega)}(\|\nabla u_0\|^{p-1}_{L^p(\Omega)} + \|N(i\beta - e_n)\eta_M h_N\|^{p-1}_{L^p(\Omega)}) \\
&= O(M^{1-n}N^{p-1}(M/N)).
\end{aligned}$$

Using that $|i\beta - e_n|^2 = p$, we have by Lemma 4.3

$$\begin{aligned} \lim_{M\to\infty} M^{n-1}N^{1-p}\int_\Omega |\nabla u_0|^p\,dx &= \lim_{M\to\infty} M^{n-1}Np^{p/2}\int_\Omega |\eta_M h_N|^p\,dx \\ &= p^{\frac{p-2}{2}}\int_{\mathbb{R}^{n-1}} \eta(x',0)^p\,dx'. \end{aligned}$$

The result follows by writing $\gamma = \gamma(0)+(\gamma-\gamma(0))$ and by using the continuity of γ. □

We now move to the analysis of the second term on the right hand side of (4.3). Writing $u = u_0 + u_1$, the next result shows that $\|\nabla u_1\|_{L^p(\Omega)}$ is asymptotically smaller than $\|\nabla u_0\|_{L^p(\Omega)}$. This may be interpreted so that u_1 is a small correction term which corrects the approximate solution u_0 into an exact solution u. The important facts for the proof are that $\Delta_p h = 0$ and that u_0 is supported near the boundary which makes it possible to use Hardy's inequality [Ku85]: if $\delta(x) = \mathrm{dist}(x, \partial\Omega)$ then

$$\|v/\delta\|_{L^p(\Omega)} \leq C\|\nabla v\|_{L^p(\Omega)}, \quad v \in W_0^{1,p}(\Omega).$$

Lemma 4.5. *As $M \to \infty$*

$$\int_\Omega |\nabla u_1|^p\,dx = o(M^{1-n}N^{p-1}).$$

Proof. We will prove that

$$\begin{aligned} I &= \int_\Omega (|\nabla u| + |\nabla u_0|)^{p-2}|\nabla u_1|^2\,dx \\ &\leq o(M^{1-n}N^{p-1}) + o(1)\int_\Omega |\nabla u_1|^p\,dx. \end{aligned} \tag{4.7}$$

To prove (4.7), we start with

$$I \lesssim \int_\Omega \gamma(|\nabla u| + |\nabla u_0|)^{p-2}|\nabla u_1|^2\,dx,$$

since γ is positive on $\overline{\Omega}$. Then we invoke the inequality (4.6). Since $u_1 = u - u_0 \in W_0^{1,p}(\Omega)$ and since u is a solution, we obtain that

$$\begin{aligned} I &\lesssim \mathrm{Re}\left[\int_\Omega \gamma(|\nabla u|^{p-2}\nabla u - |\nabla u_0|^{p-2}\nabla u_0)\cdot(\nabla\bar{u} - \nabla\bar{u}_0)\,dx\right] \\ &= -\mathrm{Re}\left[\int_\Omega \gamma|\nabla u_0|^{p-2}\nabla u_0 \cdot \nabla\bar{u}_1\,dx\right]. \end{aligned}$$

The function u_0 is supported in the ball $B(0,1/M)$. Consequently, writing $\gamma = \gamma(0) + (\gamma - \gamma(0))$, we have

$$\begin{aligned} I \lesssim & \left| \int_\Omega |\nabla u_0|^{p-2} \nabla u_0 \cdot \nabla \bar{u}_1 \, dx \right| \\ & + \int_{B(0,1/M)\cap\Omega} |\gamma - \gamma(0)| |\nabla u_0|^{p-1} |\nabla u_1| \, dx \\ =& I_1 + I_2. \end{aligned}$$

Integral I_2 is bounded by $\|\gamma - \gamma(0)\|_{L^\infty(B(0,1/M)\cap\Omega)} \|\nabla u_0\|_{L^p}^{p-1} \|\nabla u_1\|_{L^p}$, which implies by Lemma 4.4, the continuity of γ and Young's inequality that

$$I_2 \le o(M^{1-n} N^{p-1}) + o(1) \int_\Omega |\nabla u_1|^p \, dx.$$

Then we estimate integral I_1 as follows. At this point it is convenient to replace ∇u_0 with $\eta_M \nabla h_N$ by writing

$$\begin{aligned} \int_\Omega |\nabla u_0|^{p-2} \nabla u_0 \cdot \nabla \bar{u}_1 \, dx = & \int_\Omega |\eta_M \nabla h_N|^{p-2} \eta_M \nabla h_N \cdot \nabla \bar{u}_1 \, dx \\ & + \int_\Omega (|\nabla u_0|^{p-2} \nabla u_0 - |\eta_M \nabla h_N|^{p-2} \eta_M \nabla h_N) \cdot \nabla \bar{u}_1 \, dx. \end{aligned}$$

Integrating by parts, we obtain that

$$\begin{aligned} I_1 \lesssim & \left| \int_\Omega \operatorname{div}(|\eta_M \nabla h_N|^{p-2} \eta_M \nabla h_N) \bar{u}_1 \, dx \right| \\ & + \left| \int_\Omega (|\nabla u_0|^{p-2} \nabla u_0 - |\eta_M \nabla h_N|^{p-2} \eta_M \nabla h_N) \cdot \nabla \bar{u}_1 \, dx \right|. \end{aligned}$$

In the first term on the right, we multiply and divide by δ (the distance to the boundary) and use the Hölder and Hardy inequalities so that

$$\begin{aligned} & \left| \int_\Omega \operatorname{div}(|\eta_M \nabla h_N|^{p-2} \eta_M \nabla h_N) \bar{u}_1 \, dx \right| \\ & \qquad \lesssim \|\delta \operatorname{div}(|\eta_M \nabla h_N|^{p-2} \eta_M \nabla h_N)\|_{L^{p'}} \|\nabla u_1\|_{L^p}. \end{aligned}$$

The second term on the right can be estimated by (4.5), and we have

$$\begin{aligned} & \left| \int_\Omega (|\nabla u_0|^{p-2} \nabla u_0 - |\eta_M \nabla h_N|^{p-2} \eta_M \nabla h_N) \cdot \nabla \bar{u}_1 \, dx \right| \\ & \qquad \lesssim \int_\Omega (|\nabla u_0| + |\eta_M \nabla h_N|)^{p-2} |\nabla u_0 - \eta_M \nabla h_N| |\nabla u_1| \, dx, \end{aligned}$$

which, by the Hölder inequality, is bounded by

$$\begin{aligned} & (\|\nabla u_0\|_{L^p} + \|\eta_M \nabla h_N\|_{L^p})^{p-2} \|\nabla u_0 - \eta_M \nabla h_N\|_{L^p} \|\nabla u_1\|_{L^p} \\ & \qquad = O((M^{1-n} N^{p-1})^{\frac{p-1}{p}} M/N) \|\nabla u_1\|_{L^p}, \end{aligned}$$

when $p \geq 2$, and by

$$\|\nabla u_0 - \eta_M \nabla h_N\|_{L^p}^{p-1} \|\nabla u_1\|_{L^p} = O((M^{1-n} N^{p-1})^{\frac{p-1}{p}} (M/N)^{p-1}) \|\nabla u_1\|_{L^p},$$

when $1 < p < 2$. In both cases, we used Lemma 4.4. Since $M/N = o(1)$, we obtain that

$$\left| \int_\Omega (|\nabla u_0|^{p-2} \nabla u_0 - |\eta_M \nabla h_N|^{p-2} \eta_M \nabla h_N) \cdot \nabla \bar{u}_1 \right| \lesssim o(M^{1-n} N^{p-1}) + o(1) \int_\Omega |\nabla u_1|^p \, dx.$$

Collecting these estimates together, we have proved that

$$I \lesssim \|\delta \operatorname{div}(|\eta_M \nabla h_N|^{p-2} \eta_M \nabla h_N)\|_{L^{p'}} \|\nabla u_1\|_{L^p} + o(M^{1-n} N^{p-1}) + o(1) \int_\Omega |\nabla u_1|^p \, dx.$$

We claim that as $M \to \infty$,

$$\|\delta \operatorname{div}(|\eta_M \nabla h_N|^{p-2} \eta_M \nabla h_N)\|_{L^{p'}}^{p'} = o(M^{1-n} N^{p-1}), \tag{4.8}$$

from which estimate (4.7) follows by Young's inequality. So, it remains to prove (4.8). Since η_M and h_N are explicit functions, this follows from a direct computation. Noting that $\operatorname{div}(|\nabla h_N|^{p-2} \nabla h_N) = \Delta_p h_N = N^p \Delta_p h = 0$, we have

$$\begin{aligned} \operatorname{div}(|\eta_M \nabla h_N|^{p-2} \eta_M \nabla h_N) &= \nabla(\eta_M^{p-1}) \cdot |\nabla h_N|^{p-2} \nabla h_N \\ &= (p-1) \eta_M^{p-2} M \nabla \eta(M\,\cdot) N^{p-1} (|\nabla h|^{p-2} \nabla h)(N\,\cdot). \end{aligned}$$

Consequently, since $\delta(x) = x_n$,

$$\begin{aligned} &\|\delta \operatorname{div}(|\eta_M \nabla h_N|^{p-2} \eta_M \nabla h_N)\|_{L^{p'}}^{p'} \\ &\quad \lesssim M^{\frac{p}{p-1}} N^p \int_{B(0,1/M) \cap \Omega} x_n^{\frac{p}{p-1}} |\nabla h(Nx)|^p \, dx \\ &\quad \leq M^{\frac{p}{p-1}} N^{p-1-\frac{p}{p-1}} \int_0^\infty \int_{|x'| \leq 1/M} x_n^{\frac{p}{p-1}} |\nabla h(Nx', x_n)|^p \, dx \\ &\quad \lesssim M^{\frac{p}{p-1}} N^{p-1-\frac{p}{p-1}} \int_0^\infty \int_{|x'| \leq 1/M} x_n^{\frac{p}{p-1}} e^{-p x_n} \, dx' \, dx_n \\ &= O(M^{\frac{p}{p-1}-n+1} N^{p-1-\frac{p}{p-1}}). \end{aligned}$$

This is $O(M^{1-n} N^{p-1} (M/N)^{\frac{p}{p-1}}) = o(M^{1-n} N^{p-1})$ as required. This finishes the proof of (4.8), and hence that of (4.7). Now the lemma follows easily from (4.7). When $p \geq 2$, we have

$$I = \int_{\Omega} (|\nabla u| + |\nabla u_0|)^{p-2} |\nabla u_1|^2 \, dx \geq \int_{\Omega} |\nabla u_1|^p \, dx,$$

which, together with (4.7), implies the desired estimate in the lemma. When $1 < p < 2$, we have by Hölder's inequality

$$\int_{\Omega} |\nabla u_1|^p \, dx \leq I^{\frac{p}{2}} \left(\int_{\Omega} (|\nabla u| + |\nabla u_0|)^p \, dx \right)^{\frac{2-p}{2}},$$

which implies the lemma. □

4.3. **Proof of Theorem 4.1.** We now prove the following result, which immediately implies Theorem 4.1 in the case where the boundary is flat near the point of interest.

Proposition 4.6. *If Ω is as above, there exists a sequence of explicit functions $(v_M) \subseteq C_c^\infty(\mathbb{R}^n)$ such that their boundary values $f_M = v_M|_{\partial\Omega}$ satisfy* $\mathrm{supp}(f_M) \subseteq B(0, 1/M) \cap \partial\Omega$ *and*

$$\lim_{M \to \infty} \int_{\partial\Omega} \Lambda_\gamma(f_M) \bar{f}_M \, dS = \gamma(0).$$

Proof. If $f = u_0|_{\partial\Omega}$ where u_0 is as in (4.2), then (4.3) holds true. By Lemma 4.4, we have

$$M^{n-1} N^{1-p} \int_{\Omega} \gamma |\nabla u_0|^p \, dx \to c_p \gamma(0)$$

where $c_p = p^{\frac{p-2}{2}} \int_{\mathbb{R}^{n-1}} \eta(x', 0)^p \, dx'$. By (4.5),

$$\left| \int_{\Omega} \gamma (|\nabla u|^{p-2} \nabla u - |\nabla u_0|^{p-2} \nabla u_0) \cdot \nabla \bar{u}_0 \, dx \right| \lesssim \int_{\Omega} (|\nabla u| + |\nabla u_0|)^{p-2} |\nabla u - \nabla u_0| |\nabla u_0| \, dx.$$

If $p \geq 2$ then the Hölder inequality and Lemmas 4.4 and 4.5 imply that the last expression is bounded by

$$\begin{aligned} &\lesssim (\|\nabla u\|_{L^p} + \|\nabla u_0\|_{L^p})^{p-2} \|\nabla u_1\|_{L^p} \|\nabla u_0\|_{L^p} \\ &= o(M^{1-n} N^{p-1}). \end{aligned}$$

If $1 < p < 2$ we obtain the same estimate from

$$\begin{aligned}
&\int_\Omega (|\nabla u| + |\nabla u_0|)^{p-2} |\nabla u - \nabla u_0| |\nabla u_0| \, dx \\
&\leq \int_\Omega |\nabla u - \nabla u_0|^{p-1} |\nabla u_0| \, dx \\
&\lesssim \|\nabla u_1\|_{L^p}^{p-1} \|\nabla u_0\|_{L^p} \\
&= o(M^{1-n} N^{p-1}).
\end{aligned}$$

Thus, if we define

$$v_M = \left(\frac{M^{n-1} N^{1-p}}{c_p} \right)^{1/2} u_0$$

then the result follows. □

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