Shape optimization of elasto-plastic bodies under plane strains: Sensitivity analysis and numerical implementation

J. Haslinger

KFK MFF UK, Ke Karlovu 5, 120 00 Praha, CSFR

Dept. of Mathematics, University of Jyväskylä, P.O. Box 35, SF-40351 Jyväskylä, Finland

R. Mäkinen

Dept. of Mathematics, University of Jyväskylä, P.O. Box 35, SF-40351 Jyväskylä, Finland

Abstract. Optimal shape design problem, for an elastic body made from physically nonlinear material is presented. Sensitivity analysis is done by differentiating the discrete equations of equilibrium. Numerical examples are included.

This is the final draft (post-print) of the paper:

J. Haslinger & R. Mäkinen,
Shape optimization of elasto-plastic bodies under plane strains: Sensitivity analysis and numerical implementation, *Structural Optimization*, **4**, 133—141, 1992.

Notations

U^{ad}	set of admissible continuous design parameters
U_h^{ad}	set of admissible discrete design parameters
α	function from U^{ad} defining shape of body
α_h	function from U_h^{ad} defining approximated shape of body
$\boldsymbol{\alpha}$	vector of nodal values of α_h
$\{\alpha_n\}$	sequence of functions tending to α
$\Omega(\alpha)$	domain defined by α
$\varkappa$	bulk modulus
μ	shear modulus
ε	penalty parameter for contact condition
$V(\alpha)$	space of virtual displacements in $\Omega(\alpha)$
$V_h(\alpha_h)$	finite element approximation of $V(\alpha)$
J	cost functional
J_h	discretized cost functional
$\mathcal{J}$	algebraic form of J_h
$\sigma(u)$	stress tensor
$e(u)$	strain tensor
$\mathbf{K}$	stiffness matrix
$\mathbf{f}$	force vector
$\mathbf{b}(\mathbf{q})$	term arising from nonlinear boundary conditions
$\mathbf{q}$	vector of nodal degrees of freedom
$\mathbf{p}$	vector of adjoint state variables
$\mathbf{J}$	Jacobian of isoparametric mapping
$\lvert\mathbf{J}\rvert$	determinant of $\mathbf{J}$
$\mathbf{N}$	vector of shape function values on parent element
$\mathbf{L}$	matrix of shape function derivatives on parent element
$\mathbf{G}$	matrix of cartesian derivatives of shape functions
$\mathbf{X}$	matrix of nodal coordinates of element
$\mathbf{D}$	matrix of elastic coefficients
$\mathbf{B}$	strain-displacement matrix
Γ_P	part of boundary where tractions are prescribed
Γ_u	part of boundary where dispacements are prescribed
Γ_α	variable part of boundary
$\mathit{\Gamma}$	strain invariant

1. Introduction

Shape optimization is a branch of the optimal control theory, in which the control (design) variable is connected with the geometry of the problem. The goal is to find a shape of a deformable body, assigning a minimum to an objective functional. Rigorous mathematical treatment of the subject can be found in Pironneau (1984) and Haslinger and Neittaanmäki (1988), for example. More application oriented treatment can be found in Haftka, Gürdal and Kamat (1990) and Banichuk (1990) and literature therein.

In this paper we consider optimization of the shape of an elastic body under plane strains using physically nonlinear model of elasto-plasticity introduced by Washizu

(1974) and mathematically analyzed by Nečas and Hlaváček (1981). The goal of the paper is to present the abstract formulation of the problem together with the sensitivity analysis and the numerical realization.

There are two ways, how these problems are numerically solved: 1) methods, based on the solution of necessary optimality conditions, satisfied by the solution 2) the application of nonlinear mathematical programming methods, generating a minimizing sequence, which under certain conditions converges to a minimum. Here we shall restrict to the second approach. The crucial point is to provide gradient informations, which are required by the most of the minimization methods. Such informations are obtained by combining the sensitivity analysis with the adjoint state technique. Design sensitivity analysis have been recently discussed from a mathematical point of view by Neittaanmäki and Salmenjoki (1989). The mechanical literature reviewed by Haftka and Adelman (1989) represents the state-of-the-art and should be consulted for more details. However, most of the literature devoted to the design sensitivity analysis of nonlinear structures deal with sizing problems.

There are two general approaches for computing sensitivities: differentation of the continuum equations followed by discretization, and the reverse approach of discretization followed by differentation. The former one is widely used due to its simplicity, although it is not mathematically correct in general and can sometimes cause severe accuracy problems (Haug, Choi and Komkov (1986)). We shall use the latter approach which produces exact sensitivities. The crucial point is that we use the isoparametric element technique for the construction of element matrices and their sensitivities. This allows one to write general purpose computer code for the numerical solution of shape optimization problems.

The paper is organized as follows: Section 2 contains the definition of the state problem for a class of nonlinear Hooke's laws and the abstract setting of the optimal shape design problem. Section 3 deals with the approximation of the problem by finite elements. In Section 4, the sensitivity analysis and the adjoint state technique is used for finding the gradient of a cost functional. Section 5 deals with the implementation on computer. Finally, Section 6 presents numerical tests of model examples.

2. Setting of the problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, the boundary $\partial\Omega$ of which is decomposed into three disjoint open parts:

$$\partial\Omega = \bar{\Gamma}_u \cup \bar{\Gamma}_P \cup \bar{\Gamma}_\alpha.$$

We shall assume that $\Gamma_\alpha \neq \emptyset$. A deformable body will be represented by the inifinite tube Q,

$$Q = \{(x_1, x_2, x_3) \mid (x_1, x_2) \in \Omega,\ x_3 \in \mathbb{R}\} \equiv \Omega \times \mathbb{R}$$

with the boundary $\partial Q = \bar{\Theta}_u \cup \bar{\Theta}_P \cup \bar{\Theta}_\alpha$, where

$$Q_j = \Gamma_j \times \mathbb{R},\ \ j = u, P, \alpha.$$

The body will be subjected to a body force $F = (F_1, F_2, F_3)$ and surface tractions $P = (P_1, P_2, P_3)$ will be applied on Θ_P.

Next we shall suppose that:

(i) material properties of Q do not depend on x_3-coordinate;

(ii)

$$F_i = F_i(x_1, x_2),\ i = 1, 2; \quad F_3 \equiv 0;$$
$$P_i = P_i(x_1, x_2),\ i = 1, 2; \quad P_3 \equiv 0;$$

(iii) $u_3 \equiv 0$ on $\Theta_u \cup \Theta_\alpha$.

Under these hypothesis, $u_3 \equiv 0,\ u_i(x_1, x_2, x_3) = u_i(x_1, x_2),\ i = 1, 2$. As a consequence, we may analyze a plane problem for the cross section Ω, only.

Let us assume that Ω is made from a material, obeying the theory of small elasto-plastic deformations (see Washizu (1974), pp. 231–239).

The classical formulation consists of finding the displacement field $u = (u_1, u_2)$ satisfying:

the equilibrium equations: (Here and in the sequel, summation convention is used)

$$\frac{\partial \sigma_{ij}}{\partial x_j} + F_i = 0 \quad \text{in } \Omega,\ i = 1, 2; \tag{1}$$

the nonlinear Hooke's law:

$$\sigma_{ij} = \sigma_{ij}(u) = \varkappa e_{ll}\delta_{ij} + 2\mu(\Gamma)(e_{ij} - \frac{1}{3}\delta_{ij}e_{ll}), \tag{2}$$

$$e_{ij} = e_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial u_i}\right), \quad i, j = 1, 2 \tag{3}$$

and *the boundary conditions:*

$$u_i = 0 \text{ on } \Gamma_u \cup \Gamma_\alpha,\ i = 1, 2; \tag{4}$$

$$\sigma_{ij}(u)n_j = P_i \text{ on } \Gamma_P,\ i = 1, 2. \tag{5}$$

The stress tensor $\sigma(u) = \{\sigma_{ij}(u)\}_{i,j=1}^2$ is related to the strain tensor $e(u) = \{e_{ij}(u)\}_{i,j=1}^2$ by means of the constitutive relation (2), justification and validity of which is discussed in Washizu (1974). Symbols $\varkappa$ and μ stand for the bulk and shear moduli, respectively. We assume that μ is a function of the invariant

$$\mathit{\Gamma} = \frac{1}{\sqrt{3}}\left[(e_{11} - e_{22})^2 + e_{11}^2 + e_{22}^2 + 6e_{12}^2\right]^{\frac{1}{2}}. \tag{6}$$

The symbol $n = (n_1, n_2)$ denotes the outward unit normal vector to $\partial\Omega$.

In what follows we shall assume that $\varkappa = \varkappa(x),\ \mu = \mu(t, x),\ t \geq 0,\ x \in \bar{\Omega}$ are continuous functions of their variables and μ is continuously differentiable with respect to t:

$$\varkappa \in C(\bar{\Omega}),\ \mu \in C(\mathbb{R}_+ \times \bar{\Omega}),\ \frac{\partial \mu}{\partial t} \in C(\mathbb{R}_+ \times \bar{\Omega}).$$

Moreover

$$0 < \varkappa_0 \leq \varkappa(x) \leq \varkappa_1 \quad \forall x \in \Omega, \tag{7}$$

$$\tag{8}$$

$$0 < \mu_0 \leq \mu(t, x) \leq \frac{3}{2}\varkappa(x) \quad \forall x \in \Omega,\ \forall t > 0, \tag{9}$$

$$0 < \vartheta_0 \leq \mu(t, x) + 2\frac{\partial \mu(t, x)}{\partial t}t \leq \vartheta_1 \quad \forall x \in \Omega,\ \forall t > 0, \tag{10}$$

where $\varkappa_0,\ \varkappa_1,\ \mu_0,\ \vartheta_0,\ \vartheta_1$ are given positive constants.

In order to present the variational formulation of the problem given by (1)–(6), we introduce the following functional spaces. By $H^1(\Omega)$ we denote the Sobolev space of functions, which are square integrable in Ω, together with their first derivatives, i.e. they are elements of $L^2(\Omega)$:

$$H^1(\Omega) = \{v \mid v,\ \frac{\partial v}{\partial x_1},\ \frac{\partial v}{\partial x_2} \in L^2(\Omega)\}.$$

It is well-known that $H^1(\Omega)$ is a Hilbert space with the scalar product $(u,v)_{1,\Omega}$, defined as follows:

$$(u,v)_{1,\Omega} \equiv (u,v)_{0,\Omega} + \sum_{i=1}^{2} \left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \right)_{0,\Omega},$$

where the symbol $(u,v)_{0,\Omega}$ denotes the usual $L^2(\Omega)$-scalar product $\int_\Omega uv\,dx,\ dx = dx_1 dx_2$.

Let $\mathcal{H}^1(\Omega) \equiv H^1(\Omega) \times H^1(\Omega)$ and let

$$V = \{v = (v_1, v_2) \in \mathcal{H}^1(\Omega) \mid v_i = 0 \text{ on } \Gamma_u \cup \Gamma_\alpha,\ i = 1,2\}$$

be the space of virtual displacements.

The weak form of the problem given by (1)–(6) reads as follows:

$$\begin{cases} \text{Find } u \in V \text{ such that} \\ (\sigma(u), e(v))_\Omega = \langle L, v\rangle_\Omega \quad \forall v \in V, \end{cases} \tag{$\mathcal{P}$}$$

where

$$(\sigma(u), e(v))_\Omega \equiv \int_\Omega \sigma_{ij}(u) e_{ij}(v)\,dx,$$

$$\langle L, v\rangle_\Omega \equiv \int_\Omega F_i v_i\,dx + \int_{\Gamma_P} P_i v_i\,ds,$$

$F \in (L^2(\Omega))^2,\ P \in (L^2(\Gamma_P))^2$ and the relation between $\sigma(u)$ and $e(u)$ is given by (2).

Remark (2.1)**.** *($\mathcal{P}$) is nothing else than the principle of virtual work. The potential energy of the problem is given by*

$$\Phi_\Omega(v) = \int_\Omega \left[\frac{1}{2} \varkappa e_{ll}^2(v) + \frac{1}{2} \int_0^{2\Gamma^2(u)} \mu(t)\,dt \right] dx - \langle L, v\rangle_\Omega,$$

with $\Gamma(u)$ define by (6). Using (8)–(10) it can be shown (see Nečas and Hlaváček (1981)) that Φ_Ω is differentiable, it is strictly convex and coercive in V. As a result, there exists a unique minimizer of Φ_Ω over V which is nothing else than the solution of ($\mathcal{P}$).

Up to now we assumed that the shape of Ω is given. In the shape optimization problems, the boundary $\partial\Omega$ (or at least a part of it), plays the role of the control (design) variable, variations of which change properties of the structure. Our aim will be to find such a shape of Ω from an à priori given class of domains, assigning a minimum to a cost functional J, associated with the problem. Such shape will be called to be *optimal* with respect to the choice of J. In the sequel we shall assume that only the part Γ_α of $\partial\Omega$ is variable.

Remark (2.2)**.** *If $\Gamma_u = \Gamma_P = \emptyset$, then $\partial\Omega = \bar{\Gamma}_\alpha$, i.e. the shape of the whole Ω is the design variable. From reasons, which are related purely to mathematical aspects, we assume that zero displacements are prescribed on Γ_α (in this case, several mathematical results are already available and they will be presented below). Nevertheless, the extension to other boundary conditions on Γ_α, including contact conditions, is possible.*

Let the variable part Γ_α of the boundary $\partial\Omega$ be defined by means of the design variable α, which belongs to an admissible set U^{ad}.

Remark (2.3)**.** The nature of α may be different. It can be done by a single function of one variable (this will be the case of model examples presented below) or it can be done by a parametric expression of a curve, the graph of which is Γ_α. In order to emphasize the dependence of Ω on α, we shall write $\Omega(\alpha)$ in what follows.

Let $\mathcal{O} = \{\Omega(\alpha) \mid \alpha \in U^{ad}\}$ be the class of admissible variations of domains. On any $\Omega(\alpha) \in \mathcal{O}$ we shall solve the problem $(\mathcal{P}(\alpha))$:

$$\begin{cases} \text{Find } u(\alpha) \in V(\alpha) \text{ such that} \\ (\sigma(u(\alpha)), e(v))_{\Omega(\alpha)} = \langle L, v\rangle_{\Omega(\alpha)} \quad \forall v \in V(\alpha). \end{cases} \tag{$\mathcal{P}(\alpha)$}$$

The problem $(\mathcal{P}(\alpha))$ is nothing else than the problem $(\mathcal{P})$ introduced before and solved on $\Omega := \Omega(\alpha) \in \mathcal{O}$. The meaning of symbols appearing in the definition of $(\mathcal{P}(\alpha))$ is the same as before: all integrals are evaluated on $\Omega := \Omega(\alpha)$ and $V(\alpha)$ is the space V of functions defined on $\Omega := \Omega(\alpha)$.

Let $\hat{\Omega} \subset \mathbb{R}^2$ be a fixed domain, containing all $\Omega(\alpha) \in \mathcal{O}$. We shall suppose that material functions $\varkappa$, μ as well as a body force vector F are defined on $\hat{\Omega}$ and conditions (8)–(10) hold on $\hat{\Omega}$. Then there exists a unique solution $u(\alpha)$ of $(\mathcal{P}(\alpha))$ for any $\alpha \in U^{ad}$.

Let $J : [\alpha, y] \to \mathbb{R},\ \alpha \in U^{ad},\ y \in V(\alpha)$ be a cost functional which has to be minimized. *The optimal shape design problem* in an abstract setting can be formulated as follows

$$\begin{cases} \text{Find } \alpha^* \in U^{ad} \text{ such that} \\ J(\alpha^*, u(\alpha^*)) \le J(\alpha, u(\alpha)) \quad \forall \alpha \in U^{ad}, \end{cases} \tag{$\mathbb{P}$}$$

with $u(\alpha) \in V(\alpha)$ being the solution of $(\mathcal{P}(\alpha))$.

If the system $\mathcal{O}$ (or equivalently the set of design parameters U^{ad}) possesses certain compactness property and the cost functional is continuous in the appropriate sense, then $(\mathbb{P})$ has at least one solution $\alpha^* \in U^{ad}$ (i.e. $\Omega(\alpha^*) \in \mathcal{O}$) (see Haslinger and Neittaanmäki (1988)).

Below we present a case, when domains belonging to $\mathcal{O}$ can be parametrized by one function of one variable. Their simple geometry makes easier the rigorous mathematical analysis, which is done in Haslinger and Dimitrovova (1991) as well as the numerical realization itself of model examples, presented here. Let

$$\Omega(\alpha) = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_1 < \alpha(x_2),\ x_2 \in (0, 1)\}$$

be a "curved rectangle" with Γ_α being the graph of a nonnegative function α:

$$\Gamma_\alpha = \{(x_1, x_2) \mid x_1 = \alpha(x_2),\ x_2 \in (0, 1)\}.$$

Next we shall suppose that α belongs to U^{ad}, where

$$U^{ad} = \{\alpha \in C^{0,1}([0,1]) \mid 0 < c_0 \le \alpha(x_2) \le c_1,\ x_2 \in (0,1),$$
$$|\alpha(x_2) - \alpha(\bar{x}_2)| \le c_2|x_2 - \bar{x}_2|,\ x_2, \bar{x}_2 \in (0,1),\ \text{meas}\,\Omega(\alpha) = c_3\}. \quad (11)$$

Positive constants $c_0,\ c_1,\ c_2$ and c_3 are chosen in such a way that U^{ad} is non-empty. U^{ad} contains all functions which are uniformly bounded, uniformly Lipschitz-continuous and preserve the area of $\Omega(\alpha)$. Thus $\mathcal{O} = \{\Omega(\alpha) \mid \alpha \in U^{ad}\}$.

Let a cost functional J be continuous in the following sense:

$$\left.\begin{array}{l} \alpha_n, \alpha \in U^{ad},\ \alpha_n \to \alpha \text{ (uniformly) in } C([0,1]) \\ y_n, y \in H^1(\hat{\Omega}),\ y_n \rightharpoonup y \text{ (weakly) in } H^1(\hat{\Omega}) \end{array}\right\}$$
$$\Longrightarrow \liminf_{n\to\infty} J(\alpha_n, y_n|_{\Omega(\alpha_n)}) \ge J(\alpha, y|_{\Omega(\alpha)}). \quad (12)$$

If U^{ad} is given by (11) and the condition (12) is satisfied, the problem $(\mathbb{P})$ has at least one solution.

3. Approximation of $(\mathbb{P})$

As the exact solution of optimal shape design problems is hardly available, the approximation of $(\mathbb{P})$ is necessary.

Instead of the system $\mathcal{O}$, which may contain complicated shapes we assume its approximation $\mathcal{O}_h$, containing domains, boundaries of which have simpler shape and are determined by a finite number of parameters (splines,...). Approximation $\mathcal{O}_h$ is usually done through the approximation of U^{ad} denoted by U_h^{ad}, i.e. $\mathcal{O}_h = \{\Omega(\alpha_h) \mid \alpha_h \in U_h^{ad}\}$. $\mathcal{O}_h$ is chosen in such a way that domains belonging there can be generated by standard finite elements.

Let $\Omega_h = \Omega(\alpha_h)$ denote the domain $\Omega(\alpha_h)$, $\alpha_h \in U_h^{ad}$, with a given partition. By $V_h(\Omega_h)$ we denote the finite dimensional subspace of the space $V(\Omega(\alpha_h))$ constructed by means of finite elements. Instead of the problem $(\mathcal{P}(\alpha))$, we assume its finite element approximation

$$\begin{cases} \text{Find } u_h \equiv u_h(\alpha_h) \in V_h(\Omega_h) \text{ such that} \\ (\sigma(u_h), e(v_h))_{\Omega_h} = \langle L, v_h\rangle_{\Omega_h} \quad \forall v_h \in V_h(\Omega_h). \end{cases} \quad ((\mathcal{P}(\alpha_h))_h)$$

Let us note that $(\mathcal{P}(\alpha_h))_h$ leads to a nonlinear system of algebraic equations.

Finally, let $J_h : [\alpha_h, y_h] \to \mathbb{R},\ \alpha \in U_h^{ad},\ y_h \in V_h(\Omega_h)$ be an approximation of the cost functional J (received for example by applying the numerical integration if J is an integral cost functional).

The *approximation* of the optimal control problem $(\mathbb{P})$ can be stated as follows:

$$\begin{cases} \text{Find } \alpha_h^* \in U_h^{ad} \text{ such that} \\ J_h(\alpha_h^*, u_h(\alpha_h^*)) \le J_h(\alpha_h, u_h(\alpha_h)) \quad \forall \alpha_h \in U_h^{ad}. \end{cases} \quad ((\mathbb{P}(\alpha_h))_h)$$

The problem $(\mathbb{P})_h$ can be practically realized by using mathematical programming methods.

To be sure that $(\mathbb{P})_h$ really represents the approximation of $(\mathbb{P})$, some kind of convergence, concerning shapes as well as states, has to be established. These questions are discussed in general framework in Haslinger and Neittaanmäki (1988).

If U^{ad} is given by (11), the simplest way of its approximation is to use piecewise linear functions, i.e.

$$U_h^{ad} = \{\alpha_h \in C([0,1]) \mid \alpha_h \text{ piecewise linear in } [0,1]\} \cap U^{ad}.$$

In that case $\Omega(\alpha_h)$ is a polygonal domain for any $\alpha_h \in U_h^{ad}$, therefore three-noded triangular or four-noded quadrilateral elements can be used. In Haslinger and Dimitrovova (1991), the finite-dimensional space $V_h(\Omega_h)$ of piecewise linear functions over a given triangulation is used for the realization of $(\mathcal{P}(\alpha_h))_h$ and the convergence analysis between $(\mathbb{P})$ and $(\mathbb{P})_h$ is done.

4. Design sensitivity analysis

The matrix form of the problem $(\mathbb{P})_h$, for $h > 0$ fixed leads to a nonlinear optimization problem. Methods, used for its numerical solution usually require gradient information on the basis of which a minimizing sequence is generated. In order to get these derivatives, the sensitivity analysis of the problem is necessary. This will be the goal of this part.

Performing a finite element discretization of $(\mathcal{P}(\alpha_h))_h$, the matrix form of the problem is given by a system of nonlinear algebraic equations

$$\mathbf{K}(\mathbf{q})\,\mathbf{q} + \mathbf{b}(\mathbf{q}) = \mathbf{f}, \tag{13}$$

where $\mathbf{K}(\mathbf{q})$ is the stiffness matrix and $\mathbf{f}$ is the force vector, respectively. The vector $\mathbf{q}$ of unknowns contains the nodal displacements.

Remark (4.1)**.** *In the case of boundary conditions given by* (4) *and* (5)*, the term* $\mathbf{b}(\mathbf{q})$ *is identically equal to zero. Its appearance in* (13) *enables us to treat more general boundary conditions.*

Let $\boldsymbol{\alpha} \in \mathbb{R}^D$ be the discrete design variable which is uniquely associated to $\alpha_h \in U_h^{ad}$. Let $\mathcal{U} \subset \mathbb{R}^D$ be a set, defined as follows:

$$\boldsymbol{\alpha} \in \mathcal{U} \text{ if and only if } \alpha_h \in U_h^{ad},$$

i.e. there is one-to-one correspondence between the discrete design variable $\boldsymbol{\alpha} \in \mathcal{U}$ and the parameter $\alpha_h \in U_h^{ad}$.

As $\mathbf{K}$, $\mathbf{b}$ and $\mathbf{f}$ depend on $\boldsymbol{\alpha}$ in general, let us write the system (13) in the form

$$\mathbf{K}(\boldsymbol{\alpha},\mathbf{q})\,\mathbf{q} + \mathbf{b}(\boldsymbol{\alpha},\mathbf{q}) = \mathbf{f}(\boldsymbol{\alpha}) \tag{14}$$

and the solution $\mathbf{q}$ as $\mathbf{q}(\boldsymbol{\alpha})$ to emphasize this dependence. Let $\boldsymbol{\beta} \in \mathbb{R}^D$ be given and assume the system (14) with $\boldsymbol{\alpha} := \boldsymbol{\alpha} + t\boldsymbol{\beta}$, where $t > 0$ is a positive parameter tending to zero. Our goal will be to compute

$$\delta\mathbf{q} \equiv \lim_{t \to 0+} \frac{\mathbf{q}(\boldsymbol{\alpha} + t\boldsymbol{\beta}) - \mathbf{q}(\boldsymbol{\alpha})}{t},$$

i.e. the directional derivative of $\mathbf{q}$ at $\boldsymbol{\alpha}$ and the direction $\boldsymbol{\beta}$. Analogously the symbol $\delta\mathbf{K}(\mathbf{q})$ used in what follows is defined as

$$\delta\mathbf{K}(\mathbf{q}) \equiv \lim_{t\to 0+} \frac{\mathbf{K}(\boldsymbol{\alpha}+t\boldsymbol{\beta}, \mathbf{q}(\boldsymbol{\alpha}+t\boldsymbol{\beta})) - \mathbf{K}(\boldsymbol{\alpha}, \mathbf{q}(\boldsymbol{\alpha}))}{t}.$$

For later use it is convenient to write the equations (2) and (6) in the matrix form

$$\boldsymbol{\sigma} = \mathbf{D}(\Gamma)\,\mathbf{e}, \tag{15}$$

$$\Gamma = \left\{\mathbf{e}^{\mathrm{T}}\mathbf{S}\mathbf{e}\right\}^{\frac{1}{2}}. \tag{16}$$

Here

$$\boldsymbol{\sigma} = [\sigma_{11}\ \sigma_{22}\ \ \sigma_{12}]^{\mathrm{T}}, \quad \mathbf{e} = [e_{11}\ e_{22}\ \ 2e_{12}]^{\mathrm{T}}, \tag{17}$$

$\mathbf{S}$ is a symmetric matrix with constant elements, and $\mathbf{D}(\Gamma)$ is a symmetric matrix with elements depending on Γ.

From (2) it follows that the matrix $\mathbf{D}(\Gamma)$ can be split as follows

$$\mathbf{D}(\Gamma) = \mathbf{D}^0 + \mu(\Gamma)\mathbf{D}^1, \tag{18}$$

where

$$\mathbf{D}^0 = \begin{pmatrix} \varkappa & \varkappa & 0 \\ \varkappa & \varkappa & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{D}^1 = \begin{pmatrix} 4/3 & -2/3 & 0 \\ -2/3 & 4/3 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{19}$$

Next we shall assume that material functions $\varkappa$ and μ *do not depend* on $x \in \hat{\Omega}$.

Differentiating (13) implicitly we obtain

$$\left[\delta\mathbf{K}(\mathbf{q})\right]\mathbf{q} + \mathbf{K}(\mathbf{q})\,\delta\mathbf{q} + \delta\left[\mathbf{b}(\mathbf{q})\right] = \delta\mathbf{f}. \tag{20}$$

The terms $\left[\delta\mathbf{K}(\mathbf{q})\right]\mathbf{q}$, $\delta\left[\mathbf{b}(\mathbf{q})\right]$ and $\delta\mathbf{f}$ can be computed element by element using the relations

$$\mathbf{K}(\mathbf{q})\mathbf{q} = \sum_e \mathbf{P}^e\mathbf{K}^e(\mathbf{q}^e)\,\mathbf{q}^e, \;\; \mathbf{b}(\mathbf{q}) = \sum_l \mathbf{P}^l\mathbf{b}^l(\mathbf{q}^l) \quad \text{and} \quad \mathbf{f} = \sum_e \mathbf{P}^e\mathbf{f}^e. \tag{21}$$

Here $\mathbf{P}^e$ is the "local-to-global" expanding matrix, $(\mathbf{P}^e)^{\mathrm{T}}$ is the "global-to-local" gathering matrix and $\mathbf{q}^e = (\mathbf{P}^e)^{\mathrm{T}}\mathbf{q}$ (nodal displacements associated to the e:th element).

In the case of isoparametric elements each element T_e is obtained from the parent element $\hat{T}$ ($[-1,1]^2$, for example) by the mapping $\hat{T} \to T_e :\ \xi \mapsto x(\xi)$. Let

$$\mathbf{N} = \begin{pmatrix} \hat{\varphi}_1 \\ \vdots \\ \hat{\varphi}_m \end{pmatrix} \quad \text{and} \quad \mathbf{L} = \begin{pmatrix} \partial\hat{\varphi}_1/\partial\xi_1 & \dots & \partial\hat{\varphi}_m/\partial\xi_1 \\ \partial\hat{\varphi}_1/\partial\xi_2 & \dots & \partial\hat{\varphi}_m/\partial\xi_2 \end{pmatrix} \tag{22}$$

be the matrices containing the values of the shape functions and their derivatives for the parent element. Let

$$\mathbf{G} = \begin{pmatrix} \partial\varphi_1/\partial x_1 \dots \partial\varphi_m/\partial x_1 \\ \partial\varphi_1/\partial x_2 \dots \partial\varphi_m/\partial x_2 \end{pmatrix} \tag{23}$$

be a matrix of shape function derivatives of a general element T_e evaluated at a point $x(\xi)$. Denote by $\mathsf{J} = [J_{ij}]_{i,j=1}^2 = \left[\frac{\partial x_j}{\partial \xi_i}\right]_{i,j=1}^2$ the Jacobian of the mapping $\xi \mapsto x(\xi)$ and $|\mathsf{J}|$ its determinant. We use the "engineering" Jacobian which is the transpose of the classical Jacobian. Finally let

$$\mathsf{X} = \begin{pmatrix} X_1^1 & X_2^1 \\ \vdots & \vdots \\ X_1^m & X_2^m \end{pmatrix} \tag{24}$$

be the matrix containing the nodal coordinates of the e:th element.

Matrices G and J are now given by the formulae $\mathsf{G} = \mathsf{J}^{-1}\mathsf{L}$ and $\mathsf{J} = \mathsf{LX}$. In T_e the approximate solution u_h and the vector of strain components at a point $x(\xi)$ are given in the form

$$u_h = \mathbf{\Phi}^{\mathrm{T}}\mathbf{q}^e \quad \text{and} \quad \mathbf{e}(u_h) = \mathsf{B}\mathbf{q}^e, \tag{25}$$

where

$$\mathbf{\Phi} = \begin{pmatrix} \Phi^1 \\ \Phi^2 \\ \Phi^3 \\ \Phi^4 \\ \vdots \\ \Phi^{2m-1} \\ \Phi^{2m} \end{pmatrix} = \begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_1 \\ \varphi_2 & 0 \\ 0 & \varphi_2 \\ \vdots & \\ \varphi_m & 0 \\ 0 & \varphi_m \end{pmatrix} \tag{26}$$

and

$$\mathsf{B} = [\mathbf{e}(\Phi^1)\dots\mathbf{e}(\Phi^{2m})]$$

$$= \begin{pmatrix} \partial\varphi_1/\partial x_1 & 0 & \partial\varphi_2/\partial x_1 & 0 & \dots & \partial\varphi_m/\partial x_1 & 0 \\ 0 & \partial\varphi_1/\partial x_2 & 0 & \partial\varphi_2/\partial x_2 & \dots & 0 & \partial\varphi_m/\partial x_2 \\ \partial\varphi_1/\partial x_2 & \partial\varphi_1/\partial x_1 & \partial\varphi_2/\partial x_2 & \partial\varphi_2/\partial\varphi_1 & \dots & \partial\varphi_m/\partial x_2 & \partial\varphi_m/\partial x_1 \end{pmatrix} \tag{27}$$

Elements of B are obtained from G.

The local stiffness matrix is now given by

$$\mathsf{K}^e = \int_{\hat{T}} \mathsf{B}^{\mathrm{T}}\mathsf{D}(\varGamma)\mathsf{B}|\mathsf{J}|\,d\xi. \tag{28}$$

The local force vector corresponding to the body force $F = (F_1, F_2)$ is given by

$$\mathbf{f}_V^e = \int_{\hat{T}} [F_1 \quad F_2]\mathbf{\Phi}^{\mathrm{T}}|\mathsf{J}|\,d\xi. \tag{29}$$

We assume that the normal and tangential components of the surface load $P = (P_n, P_t)$ are given. The local force term for a rectangular boundary element is then given by (see Hinton and Owen (1977), p. 152)

$$\mathbf{f}_S^e = \left(\int_{-1}^{1} [P_t J_{11} - P_n J_{12} \quad P_n J_{11} + P_t J_{12}]\mathbf{\Phi}^{\mathrm{T}} d\xi_1\right)_{\xi_2=-1}, \tag{30}$$

assuming that a part of ∂T_e on which the load P is presented is the image of $[-1,1] \times \{-1\} \subset \partial\hat{T}$ with respect to the mapping $\xi \to x(\xi),\ \xi \in \hat{T},\ x \in T_e$.

The next step is to compute the local contributions $[\delta \mathbf{K}^e]\mathbf{q}^e,\ \delta[\mathbf{b}^e(\mathbf{q}^e)]$ and $\delta \mathbf{f}^e$ for equation (20). Differentiating (28) and multiplying it by $\mathbf{q}^e$ we obtain

$$(\delta \mathbf{K}^e)\mathbf{q}^e = \left[\int_{\hat{T}} \Big((\delta \mathbf{B})^{\mathrm{T}} \mathbf{D}\mathbf{B}|\mathbf{J}| + \mathbf{B}^{\mathrm{T}}\mathbf{D}(\delta \mathbf{B})|\mathbf{J}| + \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}(\delta|\mathbf{J}|)\Big)\, d\xi\right] \mathbf{q}^e + \left[\int_{\hat{T}} \mathbf{B}^{\mathrm{T}}(\delta \mathbf{D})\mathbf{B}|\mathbf{J}|\, d\xi\right] \mathbf{q}^e. \tag{31}$$

Now $\delta \mathbf{D}(\Gamma) = \mu'(\Gamma)(\delta\Gamma)\mathbf{D}^1$, where

$$\delta\Gamma = \frac{1}{\Gamma}\left((\mathbf{B}\mathbf{q}^e)^{\mathrm{T}}\mathbf{S}(\delta \mathbf{B})\mathbf{q}^e + (\mathbf{B}\mathbf{q}^e)^{\mathrm{T}}\mathbf{S}\mathbf{B}(\delta \mathbf{q}^e)\right), \tag{32}$$

making use of (25). By noting that $\delta\Gamma$ is a scalar, we get

$$\left[\int_{\hat{T}} \mathbf{B}^{\mathrm{T}}(\delta \mathbf{D})\mathbf{B}|\mathbf{J}|\, d\xi\right] \mathbf{q}^e = \left[\int_{\hat{T}} \frac{\mu'(\Gamma)}{\Gamma}|\mathbf{J}|\mathbf{B}^{\mathrm{T}}\mathbf{D}^1\mathbf{B}\mathbf{q}^e\mathbf{q}^{e\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{S}\mathbf{B}\, d\xi\right] \delta\mathbf{q}^e + \left[\int_{\hat{T}} \frac{\mu'(\Gamma)}{\Gamma}|\mathbf{J}|(\mathbf{B}\mathbf{q}^e)^{\mathrm{T}}\mathbf{S}(\delta \mathbf{B})\mathbf{q}^e\mathbf{B}^{\mathrm{T}}\mathbf{D}^1\mathbf{B}\, d\xi\right] \mathbf{q}^e \tag{33}$$

Thus the final expression for $\big(\delta \mathbf{K}^e(\mathbf{q})\big)\mathbf{q}^e$ is

$$\begin{aligned}\big(\delta \mathbf{K}^e(\mathbf{q})\big)\mathbf{q}^e = &\left[\int_{\hat{T}} \frac{\mu'(\Gamma)}{\Gamma}|\mathbf{J}|\mathbf{B}^{\mathrm{T}}\mathbf{D}^1\mathbf{B}\mathbf{q}^e\mathbf{q}^{e\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{S}\mathbf{B}\, d\xi\right] \delta\mathbf{q}^e \\ &+ \left[\int_{\hat{T}} \left(\frac{\mu'(\Gamma)}{\Gamma}|\mathbf{J}|(\mathbf{B}\mathbf{q}^e)^{\mathrm{T}}\mathbf{S}(\delta \mathbf{B})\mathbf{q}^e\mathbf{B}^{\mathrm{T}}\mathbf{D}^1\mathbf{B}\right.\right. \\ &\left.\left. + (\delta \mathbf{B})^{\mathrm{T}}\mathbf{D}\mathbf{B}|\mathbf{J}| + \mathbf{B}^{\mathrm{T}}\mathbf{D}(\delta \mathbf{B})|\mathbf{J}| + \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}(\delta|\mathbf{J}|)\right) d\xi\right] \mathbf{q}^e \\ &\equiv \mathbf{C}^e(\mathbf{q}^e)\,\delta\mathbf{q}^e + \mathbf{T}^e(\mathbf{q}^e)\,\mathbf{q}^e.\end{aligned} \tag{34}$$

Assuming that the body force F is constant as well as the normal and tangential components of P, we get

$$\delta \mathbf{f}_V^e = \int_{\hat{T}} [F_1 \quad F_2]\boldsymbol{\Phi}^{\mathrm{T}}\delta|\mathbf{J}|\, d\xi \tag{35}$$

and

$$\delta \mathbf{f}_S^e = \left(\int_{-1}^{1} [P_t(\delta J_{11}) - P_n(\delta J_{22}) \quad P_n(\delta J_{11}) + P_t(\delta J_{12})]\boldsymbol{\Phi}^{\mathrm{T}}\, d\xi_1\right)_{\xi_2=-1}. \tag{36}$$

All other terms, except $\delta\mathbf{B}$, $\delta\mathbf{J}$, $\delta|\mathbf{J}|$ and $\delta\mathbf{X}$ needed for computing $\mathbf{C}^e(\mathbf{q}^e)$, $\mathbf{T}^e(\mathbf{q}^e)$ and $\delta\mathbf{f}^e$ are available from the construction of $\mathbf{K}^e(\mathbf{q}^e)$ and $\mathbf{f}^e$. To construct $\delta\mathbf{B}$ one only needs $\delta\mathbf{G}$ which is given by $\delta\mathbf{G} = -\mathbf{G}(\delta\mathbf{X})\mathbf{G}$ (see Mäkinen (1990), Lemma 1). Derivatives of the Jacobian and its determinant are given by (see Brockman (1987)):

$$\delta\mathbf{J} = \mathbf{L}(\delta\mathbf{X}), \quad \delta|\mathbf{J}| = |\mathbf{J}|\sum_{j=1}^{m} \nabla\varphi_j(x)^{\mathrm{T}}(\delta X^j). \tag{37}$$

The matrix $\delta\mathbf{X}$ depends on how the finite element mesh is constructed and how the moving boundary is parametrized. The most important fact is to notice that the local matrices do not depend directly on the design vector but via $\mathbf{X}$.

Performing the usual assembly of local contributions $\mathbf{C}^e$, $\mathbf{T}^e$, $\mathbf{K}^e$, $\mathbf{f}^e$ and $\mathbf{b}^e$ we finally have:

Theorem (4.1)**.** *The sensitivity $\delta\mathbf{q}$ of the solution vector $\mathbf{q}$ with respect to design $\boldsymbol{\alpha}$ is given as the solution of the linear system of equations*

$$\left[\mathbf{K} + \mathbf{C}(\mathbf{q}) + \frac{\partial \mathbf{b}(\mathbf{q})}{\partial \mathbf{q}}\right] \delta\mathbf{q} = \delta\mathbf{f} - \delta\mathbf{b}(\mathbf{q}) - \mathbf{T}(\mathbf{q})\mathbf{q}. \tag{38}$$

Let $\mathcal{J} : [\boldsymbol{\alpha}, \mathbf{q}] \to \mathbb{R}$ be the algebraic form of the cost functional J_h. Next we shall suppose that $\mathcal{J}$ is sufficiently smooth so that the corresponding derivatives exist. Employing the standard adjoint equation technique of the optimal control theory to eliminate $\delta\mathbf{q}$ we obtain

$$\frac{d\mathcal{J}(\boldsymbol{\alpha})}{d\alpha_s} = \frac{\partial \mathcal{J}(\boldsymbol{\alpha}, \mathbf{q})}{\partial \alpha_s} + \mathbf{p}^{\mathrm{T}}\left[\frac{\partial \mathbf{f}}{\partial \alpha_s}\right] - \mathbf{p}^{\mathrm{T}}[\mathbf{T}(\mathbf{q})\mathbf{q}] - \mathbf{p}^{\mathrm{T}}\left[\frac{\partial \mathbf{b}(\mathbf{q})}{\partial \alpha_s}\right], \tag{39}$$

where $\mathbf{p}$ is the solution of the linear (adjoint) system

$$\left[\mathbf{K} + \mathbf{C}(\mathbf{q}) + \frac{\partial \mathbf{b}(\mathbf{q})}{\partial \mathbf{q}}\right] \mathbf{p} = \nabla_{\mathbf{q}} \mathcal{J}(\boldsymbol{\alpha}, \mathbf{q}). \tag{40}$$

5. Implementation

To solve the problem ($\mathbb{P}$) on computer it is preferred to use existing optimization routines. We have used sequential quadratic programming algorithm E04VCF from NAG library NAG (1990). E04VCF calls user written subroutine OBJFUN which must calculate the values of the cost function and its gradient at a given point.

The basic structure of the optimization program is the following:

1. Read necessary data for the optimization problem
2. Setup constraints for the optimization problem
3. Call optimization routine (Optimization routine calls OBJFUN until optimum found)
4. Make necessary post-processing

Subroutine OBJFUN is the major part of the program. It must perform mesh regeneration, solve state problem, solve adjoint problem and calculate the gradient of the cost function. The basic structure of the OBJFUN-routine is the following:

subroutine OBJFUN(n,$\boldsymbol{\alpha}$,$\mathcal{J}$,grad,first_call)
{*Given design parameter vector $\boldsymbol{\alpha}$ compute $\mathcal{J}(\boldsymbol{\alpha})$ and $\nabla_{\boldsymbol{\alpha}}\mathcal{J}(\boldsymbol{\alpha})$*}
Generate finite element mesh corresponding to $\boldsymbol{\alpha}$

if first_call **then**
  compute $\mathbf{f}$
end if
{ *Solve the state equation using Newton-Raphson method* }
do iter=1,itmax
  compute $\mathbf{K}(\mathbf{q})$, $\mathbf{C}(\mathbf{q})$ and $\frac{\partial \mathbf{b}(\mathbf{q})}{\partial \mathbf{q}}$
  compute $\mathbf{r} := \mathbf{K}(\mathbf{q})\mathbf{q} + \mathbf{b}(\mathbf{q}) - \mathbf{f}$
  solve $[\mathbf{K}(\mathbf{q}) + \mathbf{C}(\mathbf{q}) + \frac{\partial \mathbf{b}(\mathbf{q})}{\partial \mathbf{q}}]\, \Delta\mathbf{q} = \mathbf{r}$
  set $\mathbf{q} := \mathbf{q} - \Delta\mathbf{q}$
  if $\|\mathbf{r}\| < \tau$ **exitloop**
end do
{ *Solve the adjoint equation* }
compute $\mathcal{J} := \mathcal{J}(\boldsymbol{\alpha}, \mathbf{q})$ and $\nabla_{\mathbf{q}}\mathcal{J}(\boldsymbol{\alpha}, \mathbf{q})$
solve $[\mathbf{K}(\mathbf{q}) + \mathbf{C}(\mathbf{q}) + \frac{\partial \mathbf{b}(\mathbf{q})}{\partial \mathbf{q}}]\, \mathbf{p} = \nabla_{\mathbf{q}}\mathcal{J}(\boldsymbol{\alpha}, \mathbf{q})$
{ *Compute gradient* }
do i=1,n
  differentiate the finite-element mesh w.r.t. α_i
  compute $\partial\mathcal{J}(\boldsymbol{\alpha}, \mathbf{q})/\partial\alpha_i$
  grad(i) := $\partial\mathcal{J}/\partial\alpha_i$
  compute $\mathbf{T}(\mathbf{q})$
  grad(i) := grad(i) $-\mathbf{p}^{\mathrm{T}}[\mathbf{T}(\mathbf{q})\mathbf{q}]$
  compute $\partial\mathbf{b}/\partial\alpha_i$
  grad(i) := grad(i) $-\mathbf{p}^{\mathrm{T}}(\partial\mathbf{b}/\partial\alpha_i)$
  compute $\partial\mathbf{f}/\partial\alpha_i$
  grad(i) := grad(i) $+\mathbf{p}^{\mathrm{T}}[\partial\mathbf{f}/\partial\alpha_i]$
end do

Remark (5.1)**.** *Another method, commonly used for the numerical solution of* (13) *(*$\mathbf{b} \equiv 0$*) is the Kachanov or secant modulus method, based on the following iterative scheme:*

$$\mathbf{K}(\mathbf{q}^n)\, \mathbf{q}^{n+1} = \mathbf{f}, \quad n = 0, 1, ...$$

Sufficient conditions for its convergence are studied in Nečas and Hlaváček (1981).

6. Numerical examples

We assume that μ is of the form

$$\mu(\Gamma) = \begin{cases} \mu_1, & \Gamma \le \Gamma_0 \\ \mu_1\Gamma_0(\ln\Gamma + 1 - \ln\Gamma_0)/\Gamma, & \Gamma > \Gamma_0 \end{cases}$$

i.e. the material behaves linearly for sufficiently small strains. We choose $\mu_1 = \mathrm{E}/(2 + 2\nu)$, $\varkappa = \mathrm{E}/(3 - 6\nu)$ and $\Gamma_0 = 0.05$ with Young modulus $\mathrm{E} = 1.0$ and Poisson's ratio $\nu = 0.3$.

In numerical computations four-noded isoparametric element was used. The resulting nonlinear system was solved iteratively using Newton-Raphson method. The stopping criterion for the Newton-Raphson iteration was

$$\sum_i |r_i^{(k)}| < 10^{-8} \sum_i |q_i^{(k)}|,$$

where $r_i^{(k)}$, $q_i^{(k)}$ are components of the residual and nodal displacement vectors $\mathbf{r}^{(k)}$, $\mathbf{q}^{(k)}$, respectively, at the k:th iteration. The linear systems of equations were solved using Cholesky method. All computations were done on HP9000/835-computer using double precision arithmetic.

Example (6.1)**.** The geometry of the domain is of the form

$$\Omega(\alpha) = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_1 < \alpha(x_2),\ x_2 \in (0,1),\ \alpha \in U^{ad}\},$$

with

$$U^{ad} = \{\alpha \in C^{0,1}([0,1]) \mid 0.2 \leq \alpha \leq 1.5,\ |\alpha'| \leq 2\}.$$

On $\Gamma_\alpha \cup \Gamma_u$, $\Gamma_u = (0, \alpha(0)) \times \{0\}$ we assume zero displacements. On the remaining part of the boundary we have pressure loading

$$P = \begin{cases} (0.05, 0), & x_1 = 0 \\ (0, 0), & \text{otherwise.} \end{cases}$$

As the cost functional we take

$$J(\alpha) = \frac{1}{2} \int_0^1 [u_1(0, x_2) - \varphi(x_2)]^2 \, dx_2,$$

where $\varphi(x_2) = 0.03\, x_2^2$.

The domain was discretized by using 128 elements. The initial guess was $\alpha \equiv 1$. The initial cost was 5.84×10^{-4}. After 12 optimization iterations and 200 CPU-seconds the cost was reduced to 0.124×10^{-4}. The finite element mesh of the optimal body is shown in Figure 6.1 The decrease of the cost functional versus the number of optimization iterations is presented in Figure 6.2.

Example (6.2)**.** As the cost functional we take the "compliance" $J(\alpha) = \langle L, u(\alpha) \rangle_{\Omega(\alpha)}$. The geometry of the domain is of the form

$$\Omega(\alpha) = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_1 < \alpha(x_2),\ x_2 \in (0,4),\ \alpha \in U^{ad}\},$$

with

$$U^{ad} = \{\alpha \in C^{0,1}([0,4]) \mid 0.1 \leq \alpha \leq 2,\ |\alpha''| \leq 4,\ \text{meas}\, \Omega(\alpha) = 4\}.$$

Because in the finite dimensional case α_h is piecewise linear, the constraint on the second derivative must be understood in the sense of finite differences. This constraint is included to prevent the oscillating design which may be obtained when coarse mesh is used.

On $\Gamma_u = (0, \alpha(0)) \times \{0\}$ we assume zero displacements. On the remaining part of the boundary pressure load

$$P = \begin{cases} (0.005, 0), & x_1 = 0,\ x_2 \in (2,4) \\ (0, 0), & \text{otherwise} \end{cases}$$

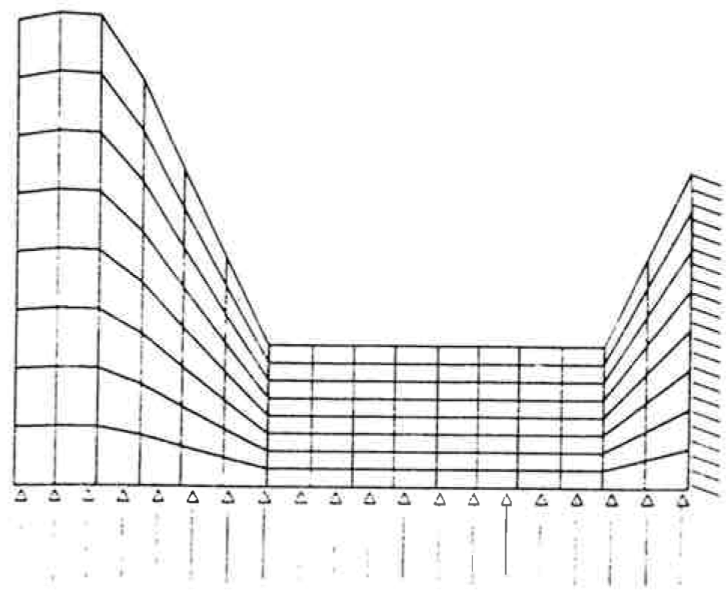

Fig. 6.1 Finite element mesh of the optimal structure

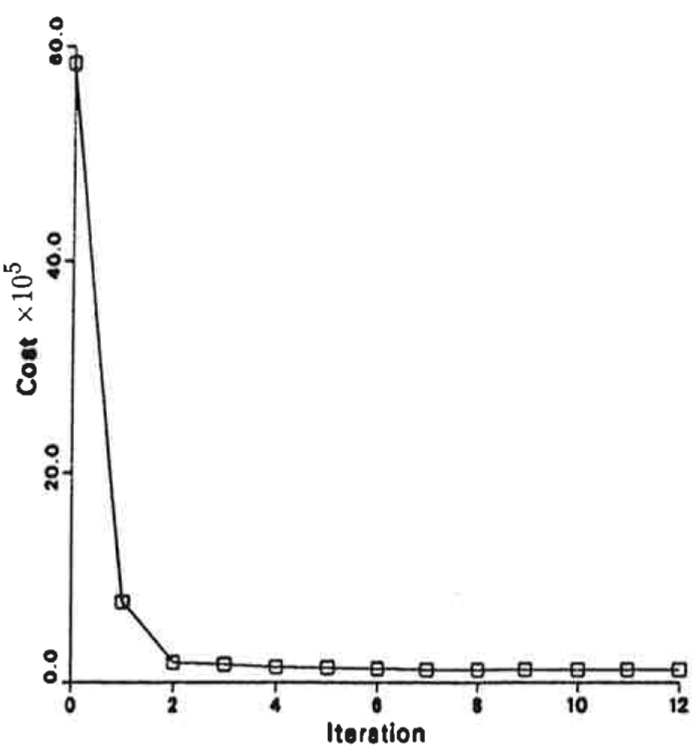

Fig. 6.2 Evolution of the cost

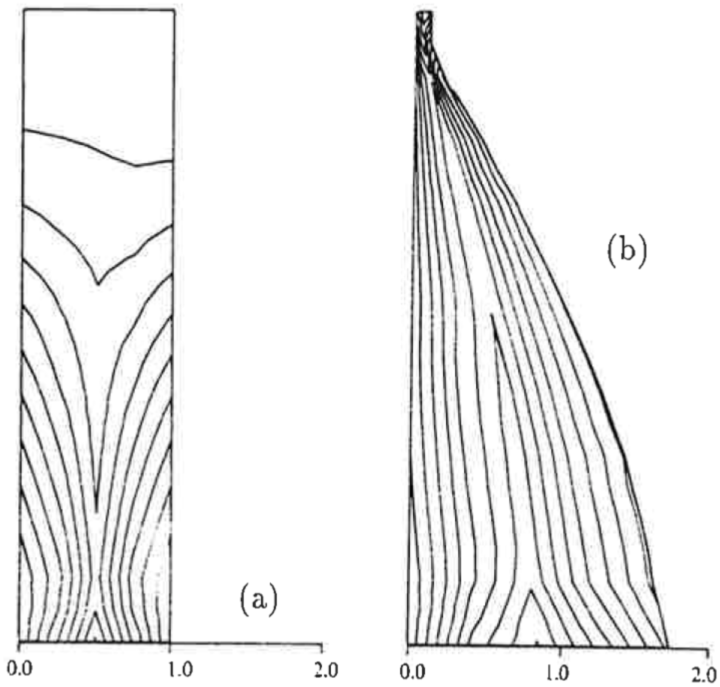

Fig. 6.3 Equivalent stress distribution: (a) initial structure; (b) optimized structure

is applied.

The domain $\Omega(\alpha)$ was discretized by using 80 elements. As the initial guess the value $\alpha \equiv 1$ was used. The initial cost was equal to 1.18×10^{-2}. After 10 optimization iterations and 250 CPU-seconds the cost was reduced to 0.424×10^{-2}.

The isolines of the distribution of equivalent stress in the initial and final structures are shown in Figure 6.3. In the optimal structure the stress is more uniformly distributed. The decrease of the cost functional versus the number of optimization iterations is presented in Figure 6.4.

Example (6.3)**.** Let $\Omega(\alpha)$ be as above. On Γ_α we assume the following contact conditions

$$\sigma_{1j}n_j = -\frac{1}{\varepsilon}\big[(u_1 - g_\alpha)_+\big]^2, \quad \sigma_{2j}n_j = 0,$$

where g_α is a function describing the gap between the elastic body $\Omega(\alpha)$ and a rigid obstacle represented by a halfplane $x_1 \geq 1.05$. The symbol $(\cdot)_+$ denotes the positive part of a number. As the cost functional we take the L^p-norm of the contact pressure

$$J(\alpha) = \left(\int_0^4 \{[(u_1(\alpha(x_2), x_2) - g_\alpha(x_2))_+]^2\}^p \, dx_2\right)^{1/p}$$

as for large p it approximates the maximum contact pressure. We take

$$U^{ad} = \{\alpha \in C^{0,1}([0,4]) \mid 0.9 \leq \alpha \leq 1.05, \text{ meas } \Omega(\alpha) = 4\},$$

$g_\alpha = 1.05 - \alpha,\ p = 8$ and $\varepsilon = 10^{-2}$.

The pressure loading is of the form

$$P = \begin{cases} (0.005, 0), & x_1 = 0,\ x_2 \in (2,4) \\ (0,0), & \text{otherwise.} \end{cases}$$

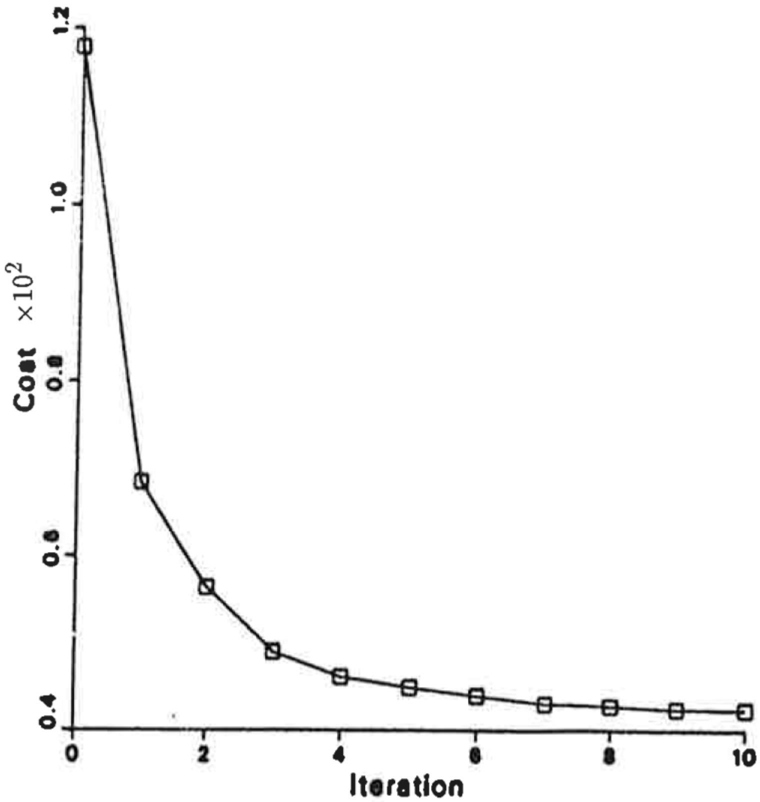

Fig. 6.4 Evolution of the cost

The domain $\Omega(\alpha)$ is discretized by using 64 elements. As the initial guess the value $\alpha \equiv 1$ was used. The initial cost was equal to 6.70×10^{-3}. After 36 optimization iterations and 1200 CPU-seconds the cost was reduced to 4.27×10^{-3}. The initial and final contact stress distributions are shown in Figure 6.5. The final domain is shown in Figure 6.6. The decrease of the cost functional versus the number of iterations is presented in Figure 6.7. Slower convergence might be explained by the fact that for large values of p, the cost functional J behaves like a nonsmooth one.

Remark (6.1). *The variational formulation of contact problems leads to the so called variational inequalities, i.e. to the minimization of the total potential energy over a convex set of functions, satisfying unilateral boundary conditions along the contact part. The shape optimization for structures, the state of which is given by variational inequalities is more involved than in the case of classical boundary conditions. The main difficulty consists of the fact that the mapping control $\mapsto$ state is usually only directionally differentiable. Thus the whole minimization problem is nonsmooth, in general. One way how to overcome this difficulty is to use a penalty approach in order to regularize our state problem. This approach was used in the previous example.*

Acknowledgement

This research was supported by the Academy of Finland.

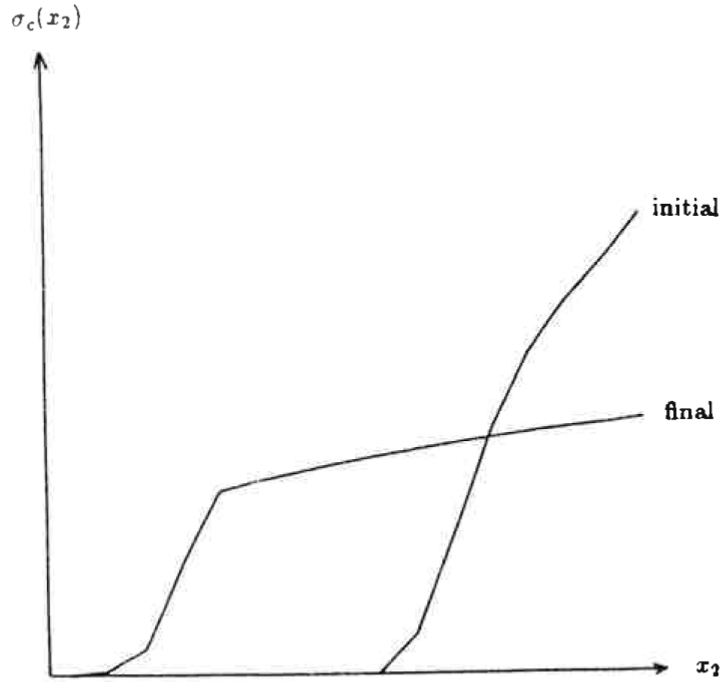

Fig.6.5 Contact stress distribution

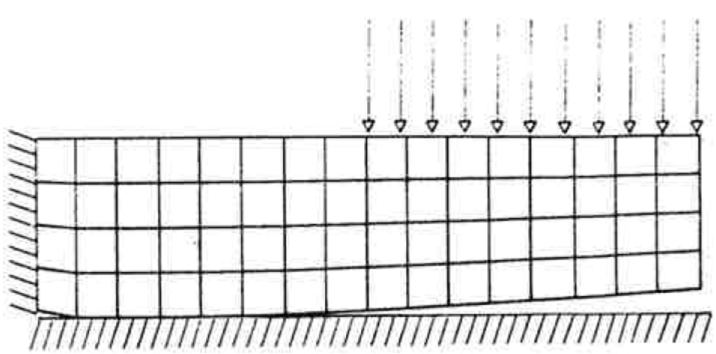
Fig.6.6 Finite element mesh of optimal structure

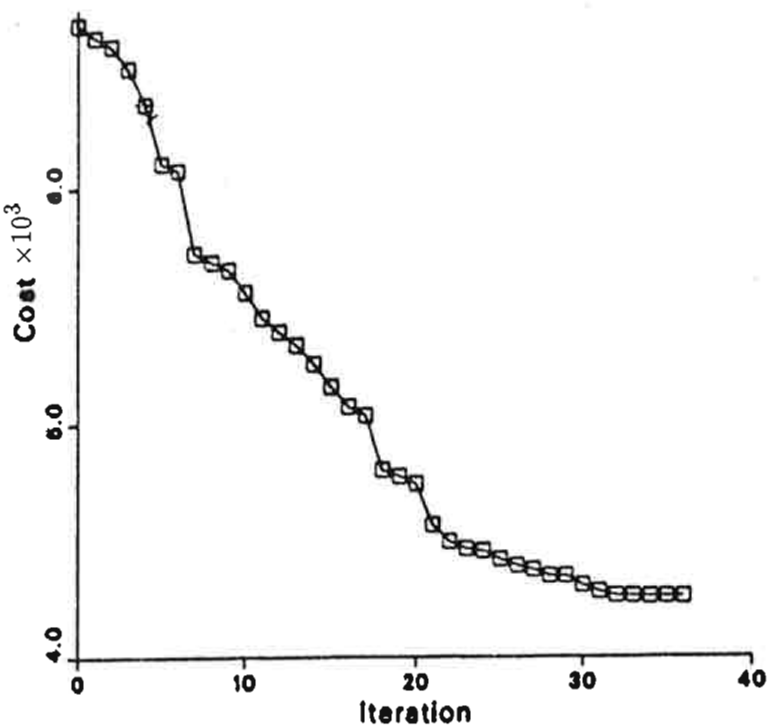

Fig. 6.7 Evolution of the cost

References

Banichuk, N.V. 1990: *Introduction to Optimization of Structures.* New York, Berlin, Heidelberg, London, Paris, Tokyo, Hong Kong: Springer-Verlag

Brockman, R.A. 1987: Geometric sensitivity analysis with isoparametric finite elements. *Commun. appl. numer. methods* 3, 495–499

Haftka, R.T. and Adelman, H.M. 1989: Recent developments in structural sensitivity analysis. *Struct. Optim.* 1, 137–151

Haftka, R.T.; Gürdal, Z. and Kamat, M.P. 1990: *Elements of Structural Optimization (second revised edition).* Dordrecht: Kluwer

Haslinger, J. and Dimitrovova, Z. 1991: Shape optimization of elasto-plastic bodies: mathematical theory. *Apl. Mat.* (to appear)

Haslinger, J. and Neittaanmäki, P. 1988: *Finite Element Approximation for Optimal Shape Design.* Chichester: John Wiley & Sons

Haug, E.J.; Choi, K.K. and Komkov, V. 1986: *Design Sensitivity Analysis of Structural Systems.* Orlando: Academic Press

Hinton, E. and Owen, D.R.J. 1977: *Finite Element Programming.* London, New York, San Francisco: Academic Press

Mäkinen, R. 1990: Finite element design sensitivity analysis for nonlinear potential problems. *Commun. appl. numer. methods* 6, 343–350

NAG 1990: *NAG Fortran library Mark 14.* Oxford: Numerical Algorithms Group Ltd

Nečas, J. and Hlaváček, I. 1981: *Mathematical Theory of Elastic and Elastico-Plastic bodies: An introduction.* Amsterdam: Elsevier

Neittaanmäki, P. and Salmenjoki, K. 1989: Sensitivity analysis for optimal shape design problems. *Struct. Optim.* 1, 241–251

Pironneau, O. 1984: *Optimal Shape Design for Elliptic Systems.* New York, Berlin: Springer-Verlag

Washizu, K. 1974: *Variational Methods in Elasticity and Plasticity* (second edition). Oxford: Pergamon Press

Appendix

Let V be a Hilbert space, equiped with the norm $\|\cdot\|$ and $\Phi : V \to \mathbb{R}$ a given functional. Φ is said to be *strictly convex on* V if and only if

$$\Phi(\lambda u + (1-\lambda)v) < \lambda\Phi(u) + (1-\lambda)\Phi(v)$$

holds for any $\lambda \in (0,1)$ and $u \neq v,\ u, v \in V$. Φ is said to be *coercive on* V if and only if

$$\lim_{\|v\|\to\infty} \Phi(v) = \infty,\ v \in V.$$

Let V' denote the space of linear continuous functionals defined on V. We say that a sequence $\{x_n\}$ of elements of V *tends weakly* to an element $x \in V$ if and only if

$$\lim_{n\to\infty} \ell(x_n) = \ell(x)$$

for any $\ell \in V'$.