OPTIMAL CONTROL/ DUAL APPROACH FOR THE NUMERICAL SOLUTION OF A DAM PROBLEM

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Abstract. A dam problem is formulated as a state constrained optimal shape design problem. The primal and dual variational formulations of the state problem are used. Numerical examples are given.

1. Introduction

Let us assume a dam made from a *nonhomogeneous* material, separating two water levels of the height y_1, y_2, respectively. The aim is to find a curve φ, separating the wet and dry part of the dam and the velocity field of the water given by the vector $k(-u_x, -u_y)$, where u is the so-called piezometric head satisfying:

$$(1.1) \qquad \begin{cases} \nabla \cdot (k\nabla u) = 0 & \text{in } \Omega(\varphi), \\ u = y_1 & \text{on } \Gamma_1, \\ u = y_2 & \text{on } \Gamma_2, \\ u = y & \text{on } \Gamma(\varphi) \cup \Gamma_\sigma(\varphi), \\ k\dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma(\varphi) \cup \Gamma_0, \\ k\dfrac{\partial u}{\partial \nu} \leq 0 & \text{on } \Gamma_\sigma(\varphi). \end{cases}$$

The partition of the boundary $\partial\Omega(\varphi)$ into corresponding parts follows from Figure 1.

The function k is a permeability coefficient, $y = \varphi(x)$ is the free boundary, one of the unknowns of the problem. The dam problem has been extensively studied by many authors. If the separating walls are vertical and the dam is made from *homogeneous* material, then using the so-called Baiocchi transformation the problem can be converted into a variational inequality solved on a *fixed* domain $\hat{\Omega}$. The free boundary is then a curve separating the coincidence and the non-coincidence set. Using this approach, several properties, such as monotonicity and the concavity of φ, can be discovered, see [**1**], [**5**], [**7**]. If Γ_1 and Γ_2 are no longer vertical, this approach leads to a model, described by a *quasivariational* inequality [**3**]. Also the case of a non-homogeneous dam complicates this approach [**2**]. Another approach which can also be used for the numerical solution is presented in [**4**].

Here we shall deal with the optimal control approach which can also be used for other free boundary problems. This approach can be used with success in the case of a non-homogeneous media, a dam with non-vertical walls, etc. The idea is quite simple. On the free boundary $\Gamma(\varphi)$, boundary conditions are *overdetermined*, so that for a random choice of φ the problem (1.1) *is not well posed.* We define a new boundary value problem which will play the role of the *state problem* with one condition on $\Gamma(\varphi)$ only and the remaining boundary condition will be included in a suitable cost functional. This approach has been used in [**6**] in the case of a rectangular dam. Here the Dirichlet condition $u = y$ on $\Gamma(\varphi)$ was treated by means of the minimization of the cost functional $J(\varphi) = \int_{\Gamma(\varphi)}(u-\varphi)^2\,ds$. Authors used some à-priori known properties of the free boundary $\Gamma(\varphi)$, namely the fact that φ is decreasing and concave. The boundary condition $k\partial u/\partial\nu \leq 0$ on Γ_σ was also automatically satisfied, because of the fact that the walls are vertical.

For the nonhomogeneous dams however, the parametrization of the free boundary is not straightforward and assumptions like monotonicity or concavity are no longer true. Our aim was to define a class of possible candidates for the free boundary as large as possible, i.e. with the minimum of restrictions on the class of curves, among which the free boundary can be found. Nevertheless some compactness properties are still required. In order to get a free boundary of the problem, one has to take into account all physical conditions defining the problem, especially the condition $k\partial u/\partial n \leq 0$ on $\Gamma_\sigma(\varphi)$, which is not à-priori satisfied in a general dam problem. If this condition is not taken into account, the numerical procedure may give a nonphysical solution as will be illustrated on a model example. This condition on $\Gamma_\sigma(\varphi)$ will be treated as the state constraint. Contrary to [**6**], the Neumann condition on $\Gamma(\varphi)$ will be included in a cost functional J. This choice of J seems to be useful. Analysing optimality conditions, we deduce that in many situations a critical point of J (if it exists in a given class of domains) is the point of the absolute minimum of J (i.e. the solution of the dam problem) and not only a local minimum. As the cost functional and the state constraint are expressed in terms of fluxes, the dual variational formulation of the state problem (i.e. the formulation in terms of the cogradient) is used. Thus the divergence free finite elements have to be used for the numerical realization. Using them, one can very accurately approximate the required fluxes. Examples presented in this paper and our computational experiences confirm the superiority of this approach over the classical primal formulation of the state problem. Another important advantage of the dual approach is the fact that oscillations of the free boundary when its piecewise linear approximation is used does not occur.

2. Setting of the problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, with Lipschitz boundary [**10**]. Let the dam be represented by a bounded domain $\hat\Omega$ with a Lipschitz boundary $\partial\hat\Omega$. By Γ_1, Γ_2 we denote a part of $\partial\hat\Omega$, which comes in contact with water levels of the height y_1, y_2, respectively. Moreover $\partial\hat\Omega = \Gamma_0 \cup \tilde\Gamma_1 \cup \tilde\Gamma_2 \cup \tilde\Gamma_3$, where $\tilde\Gamma_1$, $\tilde\Gamma_2$ are lateral parts of $\partial\hat\Omega$ and Γ_0, $\tilde\Gamma_3$ are the base and the top of the dam, respectively (Γ_3 possibly empty). Let $\varphi : [0,1] \mapsto \mathbb{R}^2$ be a continuous curve the graph of which, denoted by $\Gamma(\varphi)$, lies in $\overline{\hat{\Omega}}$ and such that $\varphi(0) = A$, $\varphi(1) \in \tilde{\Gamma}_2 \setminus \Gamma_2$, where $A \in \Gamma_1$ is the point where the water surface of the height y_1 meets Γ_1.

Definition 2.1. By $\Omega(\varphi)$ we denote a domain bounded by Γ_0, Γ_1, Γ_2, $\Gamma(\varphi)$ and $\Gamma_\sigma(\varphi)$ being a part of $\tilde{\Gamma}_2$ between $\varphi(1)$ and $B \in \tilde{\Gamma}_2$. Here B is the point where the water surface of height y_2 meets $\tilde{\Gamma}_2$, see Figure 1. The family of all these $\Omega(\varphi)$ will be denoted by $\tilde{\mathcal{O}}$.

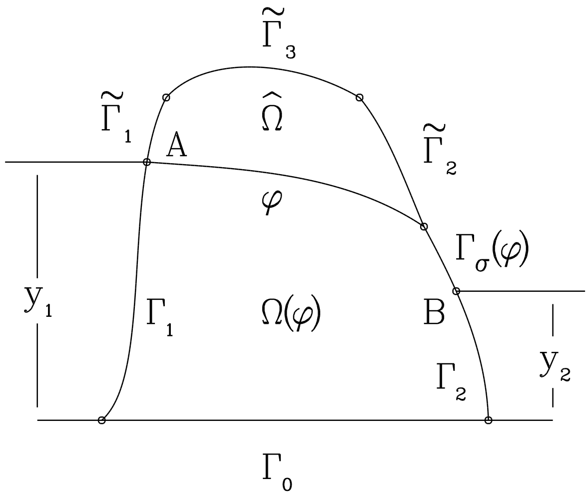

Figure 1: Geometry of the dam problem

Definition 2.2. By $\mathcal{O}$ we denote a subset of $\tilde{\mathcal{O}}$ which is compact with respect to the uniform convergence of boundaries and such that $\Omega(\varphi) \supseteq \hat{\Omega}_0$, where $\hat{\Omega}_0$ is a fixed domain in $\mathbb{R}^2$.

On any $\Omega(\varphi) \in \mathcal{O}$ we assume the state problem:

$$(P(\varphi)') \qquad \begin{cases} \nabla \cdot (k\nabla u) = 0 & \text{in } \Omega(\varphi) \\ u(\varphi) = \Phi(\varphi) & \text{on } \Sigma(\varphi) \\ k\dfrac{\partial u}{\partial \nu}(\varphi) = 0 & \text{on } \Gamma_0, \end{cases}$$

where $\Phi(\varphi) \in H^1(\Omega(\varphi))$ is a function, realizing nonhomogeneous Dirichlet boundary conditions on $\Sigma(\varphi) = \Gamma_1 \cup \Gamma_2 \cup \Gamma(\varphi) \cup \Gamma_\sigma(\varphi)$.

The weak form of $(P(\varphi)')$ is done by

$$(P(\varphi)) \qquad \begin{cases} \text{Find } u(\varphi) \in V_\Phi(\varphi) \text{ such that} \\ (k\nabla u, \nabla v)_{0,\Omega(\varphi)} = 0 \quad \forall v \in V(\varphi), \end{cases}$$

where

$$V(\varphi) = \{v \in H^1(\Omega(\varphi)) \mid v = 0 \text{ on } \Sigma(\varphi)\}$$
$$V_\Phi(\varphi) = \{v \in H^1(\Omega(\varphi)) \mid v = \Phi(\varphi) \text{ on } \Sigma(\varphi)\}.$$

For the following considerations, the dual form of $(P(\varphi))$ will be introduced. To this end we define

$$K_0(\varphi) = \{\mu \in (L^2(\Omega(\varphi))^2 \mid (\mu, \nabla v)_{0,\Omega(\varphi)} = 0 \quad \forall v \in H_0^1(\Omega(\varphi))\}$$
$$K_{00}(\varphi) = \{\mu \in K_0(\varphi) \mid (\mu, \nabla v)_{0,\Omega(\varphi)} = 0 \quad \forall v \in V(\varphi)\}.$$

It is easy to see that $\mu \in K_0(\varphi)$ iff $\nabla \cdot \mu = 0$ in $\Omega(\varphi)$ in the sense of distributions and $\mu \in K_{00}(\varphi)$ iff $\nabla \cdot \mu = 0$ in $\Omega(\varphi)$, $\mu \cdot \nu = 0$ on Γ_0, i.e. $K_0(\varphi)$ is the set of divergence free vector fields in $\Omega(\varphi)$ and $K_{00}(\varphi)$ is its subset, containing all functions, that have zero normal flux across Γ_0.

Remark 2.1. Let $\tilde{\Gamma} \subset \partial\Omega(\varphi)$ be an open set in $\partial\Omega(\varphi)$, $\mu \in K_0(\varphi)$. Then the flux $\mu \cdot \nu$ across $\tilde{\Gamma}$ is defined by

$$\langle \mu \cdot \nu, v \rangle \equiv (\mu, \nabla v)_{0,\Omega(\varphi)} \quad \forall v \in V_0(\tilde{\Gamma}), \tag{2.1}$$

where

$$V_0(\tilde{\Gamma}) = \{v \in H^1(\Omega(\varphi)) \mid v = 0 \text{ on } \partial\Omega(\varphi) \setminus \overline{\tilde{\Gamma}}\}.$$

It is easy to see that (2.1) defines a linear continuous functional over the space $H\ (\tilde{\Gamma})$ which is the space of traces of all functions belonging to $V_0(\tilde{\Gamma})$. $\langle\ ,\ \rangle$ stands for the corresponding duality. We say that $\mu \cdot \nu \leq 0$ on $\tilde{\Gamma}$ iff $\langle \mu \cdot \nu, v \rangle \leq 0$ for any $v \in V_0(\tilde{\Gamma})$, $v \geq 0$ on $\tilde{\Gamma}$.

By the *dual* variational formulation of $(P(\varphi))$ we call the problem

$$(P^*(\varphi)) \qquad \begin{cases} \text{Find } \lambda(\varphi) \in K_{00}(\varphi) \text{ such that} \\ (k^{-1}\lambda(\varphi), \mu)_{0,\Omega(\varphi)} = (\nabla\Phi(\varphi), \mu)_{0,\Omega(\varphi)} \quad \forall \mu \in K_{00}(\varphi) \end{cases}$$

The relation between $(P(\varphi))$ and $(P^*(\varphi))$ is done by

Lemma 2.1. *There exist a unique solution $\lambda(\varphi)$ of $(P^*(\varphi))$. Moreover*

$$\lambda(\varphi) = k\nabla u(\varphi) \quad \text{in } \Omega(\varphi),$$

where $u(\varphi)$ is the unique solution of $(P(\varphi))$.

Let $I(\lambda, \varphi)$ be *the objective functional*, defined as follows:

$$I(\lambda, \varphi) = \frac{1}{2}\|\lambda \cdot \nu\|^2_{-\ ,\Gamma(\varphi)}$$

with $\lambda \in K_0(\varphi)$, $\Omega(\varphi) \in \mathcal{O}$ and $\|\cdot\|_{-\ ,\Gamma(\varphi)}$ denoting the dual norm of the linear functional $\lambda \cdot \nu$ on $\Gamma(\varphi)$ (see Remark 2.1).

The *optimal control formulation* of the dam problem reads as follows:

$$(\tilde{P}) \qquad \begin{cases} \text{Find } \Omega(\varphi^*) \in \mathcal{O} \text{ such that} \\ J(\Omega(\varphi^*)) \leq J(\Omega(\varphi)) \quad \forall \Omega(\varphi) \in \mathcal{O}, \end{cases}$$

where

$$J(\Omega(\varphi)) = I(\lambda(\varphi), \varphi) = \frac{1}{2}\|\lambda(\varphi) \cdot \nu\|^2_{-\ ,\Gamma(\varphi)}$$

with $\lambda(\varphi) \in K_{00}(\varphi)$ being the solution of $(P^*(\varphi))$ and with *the additional constraint* $\lambda(\varphi) \cdot \nu \leq 0$ on $\Gamma_\sigma(\varphi)$, provided the one-dimensional measure of $\Gamma_\sigma(\varphi)$ is positive.

Remark 2.2. $(\tilde{P})$ is the optimal control problem in which the condition $\lambda(\varphi)\cdot\nu \le 0$ on $\Gamma_\sigma(\varphi)$ is considered as the state constraint of $(P^*(\varphi))$.

Remark 2.3. If $\Gamma(\varphi^*)$ is a free boundary in (1.1) and $\Omega(\varphi^*) \in \mathcal{O}$ then $J(\Omega(\varphi^*)) = 0$, i.e. $\Omega(\varphi^*)$ realizes the absolute minimum of J on $\mathcal{O}$.

Next we shall prove the existence of at least one absolute minimum of J on $\mathcal{O}$. As we are interested in zero values of J, the dual norm $\|\cdot\|_{-\ ,\Gamma(\varphi)}$ defining J can be replaced by an equivalent one. To this end let us introduce the set

$$K_{00}^{\lambda}(\varphi) = \{\mu \in K_{00}(\varphi) \mid (\mu, \nabla v)_{0,\Omega(\varphi)} = \langle \lambda(\varphi)\cdot\nu, v\rangle \quad \forall v \in V_0(\varphi)\},$$

where

$$V_0(\varphi) = \{v \in H^1(\Omega(\varphi)) \mid v = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_\sigma(\varphi)\}$$

and $\lambda(\varphi) \in K_{00}(\varphi)$ is the solution of $(P^*(\varphi))$.

It is easy to see that $\mu \in K_{00}^{\lambda}(\varphi)$ iff

$$\begin{cases} \nabla\cdot\mu = 0 & \text{in } \Omega(\varphi) \\ \mu\cdot\nu = 0 & \text{on } \Gamma_0 \\ \mu\cdot\nu = \lambda(\varphi)\cdot\nu & \text{on } \Gamma(\varphi). \end{cases}$$

Remark 2.4. Using Remark 2.1 we see that the equivalent expression of $K_{00}^{\lambda}(\varphi)$ is

$$K_{00}^{\lambda}(\varphi) = \{\mu \in K_{00}(\varphi) \mid (\mu, \nabla v)_{0,\Omega(\varphi)} = (\lambda(\varphi), \nabla v)_{0,\Omega(\varphi)} \quad \forall v \in V_0(\varphi)\}.$$

Now let us assume the problem

$$(A^*(\varphi)) \qquad \begin{cases} \text{Find } \chi(\varphi) \in K_{00}^{\lambda}(\varphi) \text{ such that} \\ (\chi(\varphi), \mu)_{0,\Omega(\varphi)} = 0 \quad \forall \mu \in K_{00}^{0}(\varphi), \end{cases}$$

where $K_{00}^{0}(\varphi)$ corresponds to $K_{00}^{\lambda}(\varphi)$ with $\lambda = 0$.

The problem $(A^*(\varphi))$ is nothing else than the dual formulation of $(A(\varphi))$ given by the following elliptic boundary value problem

$$(A(\varphi)) \qquad \begin{cases} \Delta z(\varphi) = 0 & \text{in } \Omega(\varphi) \\ z(\varphi) = 0 & \text{on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_\sigma(\varphi) \\ \dfrac{\partial z}{\partial \nu}(\varphi) = 0 & \text{on } \Gamma_0 \\ \dfrac{\partial z}{\partial \nu}(\varphi) = \lambda(\varphi)\cdot\nu = k\dfrac{\partial u}{\partial \nu}(\varphi) & \text{on } \Gamma(\varphi) \end{cases}$$

and $\chi(\varphi) = \nabla z(\varphi)$ in $\Omega(\varphi)$. It is easy to see that $\|\lambda(\varphi)\cdot\nu\|_{-\ ,\Gamma(\varphi)}$ and $\|\chi(\varphi)\|_{0,\Omega(\varphi)}$ are equivalent. Instead of J let us introduce the functional

$$F(\Omega(\varphi)) = \frac{1}{2}\|\chi(\varphi)\|_{0,\Omega(\varphi)}^2$$

where $\chi(\varphi) \in K_{00}^{\lambda}(\varphi)$ is the unique solution of $(A^*(\varphi))$ with $\lambda = \lambda(\varphi)$ being the solution of $(P^*(\varphi))$ and such that $\lambda\cdot\nu \le 0$ on $\Gamma_\sigma(\varphi)$ if the one dimensional measure of $\Gamma_\sigma(\varphi)$ is positive.

As a result, we shall study the problem

$$(\mathrm{P}) \qquad \begin{cases} \text{Find } \Omega(\varphi^*) \in \mathcal{O} \text{ such that} \\ F(\Omega(\varphi^*)) \le F(\Omega(\varphi)). \end{cases}$$

Remark 2.5. Let $\Omega(\tilde{\varphi}) \in \mathcal{O}$ be such that $F(\Omega(\tilde{\varphi})) = 0$. Then $J(\Omega(\tilde{\varphi})) = 0$, as well.

The main result of this Section is

Theorem 2.1. *There exists at least one solution of* (P).

Proof will be based on several auxiliary lemmas.

Lemma 2.2. *Let* $\Omega_n \to \Omega$, $\Omega_n = \Omega(\varphi_n)$, $\Omega = \Omega(\varphi) \in \mathcal{O}$ *and let* $\lambda_n \in K_{00}(\varphi_n)$ *be solutions of* $(P^*(\varphi_n))$. *Then there is a subsequence of* $\{\lambda_n\}$ *and an element* $\lambda \in K_{00}(\varphi)$ *such that*

$$\tilde{\lambda}_n \rightharpoonup \tilde{\lambda} \text{ in } (L^2(\hat{\Omega}))^2,$$

where the symbol $\tilde{}$ *denotes the extension by zero of the corresponding functions from the domain of their definition on* $\hat{\Omega}$. *Moreover,* λ *solves* $(P^*(\varphi))$.

Proof: Let $\lambda_n \in K_{00}(\varphi_n)$ be the solution of $(P^*(\varphi_n))$:

$$(k^{-1}\lambda_n, \mu)_{0,\Omega_n} = (\nabla\Phi(\varphi_n), \mu)_{0,\Omega_n} \quad \forall \mu \in K_{00}(\varphi_n). \tag{2.2}$$

As $\Omega_n \to \Omega$ ($\Longleftrightarrow \partial\Omega_n \partial\Omega$) one can construct $\Phi(\varphi_n) \in H^1(\Omega_n)$ in such a way that

$$\|\Phi(\varphi_n)\|_{1,\Omega_n} \le c \tag{2.3}$$

with a constant $c > 0$ which does not depend on n. Hence there exists a subsequence of $\{\Phi(\varphi_n)\}$ such that

$$\widetilde{\Phi}(\varphi_n) \rightharpoonup \Psi_1 \quad \text{in } L^2(\hat{\Omega}), \quad \widetilde{\nabla\Phi}(\varphi_n) \rightharpoonup \Psi_2 \quad \text{in } (L^2(\hat{\Omega}))^2. \tag{2.4}$$

Moreover, denoting $\Phi \equiv \Psi_1|_\Omega$ it is easy to prove that

$$\Phi = \Phi(\varphi) \quad \text{and} \quad \nabla\Phi = \Psi_2|_\Omega, \tag{2.5}$$

i.e. Φ realizes the nonhomogeneous Dirichlet boundary condition on $\Sigma(\varphi)$. From (2.2) and (2.3) it follows that $\{\|\lambda_n\|_{0,\Omega_n}\}$ is bounded. Hence there is a subsequence of $\{\tilde{\lambda}_n\}$ such that

$$\tilde{\lambda}_n \rightharpoonup \tilde{\lambda} \quad \text{in } (L^2(\hat{\Omega}))^2. \tag{2.6}$$

Denote $\lambda \equiv \tilde{\lambda}|_\Omega$. First we prove that $\lambda \in K_{00}(\varphi)$. Indeed, let $v \in V(\varphi)$ be an arbitrary function. Then there exists a sequence $\{v_j\}$, $v_j \in C^\infty(\overline{\Omega(\varphi)}) \cap V(\varphi)$ such that

$$v_j \to v \quad \text{in } H^1(\Omega(\varphi)) \tag{2.7}$$

and also

$$\tilde{v}_j \to \tilde{v} \quad \text{in } H^1(\hat{\Omega}). \tag{2.8}$$

Moreover, dist $(\operatorname{supp} v_j, \Sigma(\varphi)) > 0$ $\forall j$. Consequently,

$$v_j|_{\Omega_n} \in V(\varphi_n) \tag{2.9}$$

for n large enough. Let j be fixed and n such that (2.9) holds. Then

$$(\lambda_n, \nabla v_j)_{0,\Omega_n} = 0.$$

Taking into account (2.6) and the fact that $\Omega_n \to \Omega$ we see

$$0 = (\lambda_n, \nabla v_j)_{0,\Omega_n} = (\tilde{\lambda}_n I_n, \nabla \tilde{v}_j)_{0,\hat{\Omega}} \underset{n\to\infty}{\longrightarrow} (\tilde{\lambda} I, \nabla \tilde{v}_j)_{0,\hat{\Omega}} = (\lambda, \nabla \tilde{v}_j)_{0,\Omega}, \tag{2.10}$$

where I_n, I are characteristic functions of $\Omega(\varphi_n)$, $\Omega(\varphi)$, respectively. Letting $j \to \infty$ in (2.10) and using (2.8) we finally obtain

$$(\lambda, \nabla v)_{0,\Omega} = 0,$$

i.e. $\lambda \in K_{00}(\varphi)$. It remains to show that λ solves $(P^*(\varphi))$. Let $\mu \in K_{00}(\varphi)$ be given and $\hat{\mu} \in K_{00}(\hat{\Omega})$ such that $\mu = \hat{\mu}|_{\Omega(\varphi)}$. [0] As $\hat{\mu}|_{\Omega(\varphi_n)} \in K_{00}(\varphi_n)$ for any n, we have

$$(k^{-1}\lambda(\varphi_n), \hat{\mu})_{0,\Omega_n} = (\nabla \Phi(\varphi_n), \hat{\mu})_{0,\Omega_n}. \tag{2.11}$$

As before, we write (2.11) in the form

$$(k^{-1}\tilde{\lambda}(\varphi_n), I_n\hat{\mu})_{0,\hat{\Omega}} = (\widetilde{\nabla \Phi}(\varphi_n), I_n\hat{\mu})_{0,\hat{\Omega}}.$$

Letting $n \to \infty$ we obtain

$$(k^{-1}\lambda, \mu)_{0,\Omega} = (k^{-1}\tilde{\lambda} I, \hat{\mu})_{0,\hat{\Omega}} = (\Psi_2, \hat{\mu})_{0,\hat{\Omega}} = (\nabla \Phi(\varphi), \mu)_{\Omega},$$

making use of (2.4)–(2.6). Consequently, λ is a solution of $(P^*(\varphi))$.

Lemma 2.3. *Let $\Omega_n \to \Omega$, $\Omega_n = \Omega(\varphi_n)$, $\Omega = \Omega(\varphi) \in \mathcal{O}$, $\lambda_n \in K_{00}(\varphi_n)$ be solutions of $(P^*(\varphi_n))$ and $\chi_n \in K_{00}^{\lambda_n}(\varphi_n)$ solutions of $(A^*(\varphi_n))$. Let*

$$\tilde{\lambda}_n \rightharpoonup \tilde{\lambda} \quad \text{in } (L^2(\hat{\Omega}))^2. \tag{2.12}$$

Then there is a subsequence of $\{\chi_n\}$ and an element $\chi \in K_{00}^{\lambda}(\varphi)$ such that

$$\tilde{\chi}_n \rightharpoonup \tilde{\chi} \quad \text{in } (L^2(\hat{\Omega}))^2. \tag{2.13}$$

Moreover, χ solves $(A^*(\varphi))$.

Proof: Let $\chi_n \in K_{00}^{\lambda_n}(\varphi_n)$ solve $(A^*(\varphi_n))$. Then $\chi_n = \nabla z_n$ in Ω_n, where $z_n \in V_0(\varphi_n)$ solves

$$\begin{aligned}(\nabla z_n, \nabla v)_{0,\Omega_n} &= \langle \lambda(\varphi_n)\cdot \nu, v\rangle = \langle k\frac{\partial u}{\partial \nu}(\varphi_n), v\rangle \\ &= (k\nabla u_n, \nabla v)_{0,\Omega_n} \quad \forall v \in V_0(\varphi_n),\end{aligned} \tag{$A(\varphi_n)$}$$

=It is easy to prove that any function $\mu \in K\varphi$ can be abtained as the restriction of a function $\mu \in K$.

where $(A(\varphi_n))$ is the weak formulation of $(A(\varphi))$, introduced in Remark 2.4. It is easy to see that $\{\|z_n\|_{1,\Omega_n}\}$ and $\{\|u_n\|_{1,\Omega_n}\}$ are bounded. Hence

$$(2.14) \qquad \begin{cases} \tilde{u}_n \rightharpoonup U, \quad \tilde{z}_n \rightharpoonup Z \quad \text{in } L^2(\hat{\Omega}) \\ \widetilde{\nabla u_n} \rightharpoonup \Psi_1, \quad \widetilde{\nabla z_n} \rightharpoonup \Psi_2 \quad \text{in } (L^2(\hat{\Omega}))^2. \end{cases}$$

(Let us recall that the symbol $\sim$ stands for the extension by zero on $\hat{\Omega}$). Repeating the standard process one can prove that setting

$$u \equiv U|_{\Omega(\varphi)}, \quad z \equiv Z|_{\Omega(\varphi)}$$

we have

$$\nabla u = \Psi_1|_{\Omega(\varphi)}, \quad \nabla z = \Psi_2|_{\Omega(\varphi)}$$

and u, z are solutions of $(P(\varphi))$, $(A(\varphi))$, respectively. Comparing (2.12) and (2.13) with (2.14) we see that

$$\tilde{\lambda} = \Psi_1, \quad \tilde{\chi} = \Psi_2.$$

Therefore $\chi \equiv \tilde{\chi}|_{\Omega(\varphi)} = \nabla z(\varphi) \in K_{00}^{\lambda}(\varphi)$ and χ solves $(A^*(\varphi))$.

Lemma 2.4. *Let $\Omega_n \to \Omega$, $\Omega_n = \Omega(\varphi_n)$, $\Omega = \Omega(\varphi) \in \mathcal{O}$ and one dimensional measure of $\Gamma_\sigma(\varphi)$ be positive. Let $\lambda_n \in K_0(\varphi_n)$ be such that $\lambda_n \cdot \nu \leq 0$ on $\Gamma_\sigma(\varphi_n)$ and*

$$\tilde{\lambda}_n \rightharpoonup \tilde{\lambda} \quad \text{in } (L^2(\hat{\Omega}))^2.$$

Then $\tilde{\lambda} \cdot \nu \leq 0$ on $\Gamma_\sigma(\varphi)$.

Proof: Let $\lambda_n \in K_0(\varphi_n)$ be such that $\lambda_n \cdot \nu \leq 0$ on $\Gamma_\sigma(\varphi_n)$. This means (see Remark 2.1):

$$(2.15) \qquad \langle \lambda_n \cdot \nu, v \rangle = (\lambda_n, \nabla v)_{0,\Omega_n} \leq 0 \quad \forall v \in V_0(\Gamma_\sigma(\varphi_n)), \quad v \geq 0 \text{ on } \Gamma_\sigma(\varphi_n)$$

where

$$V_0(\Gamma_\sigma(\varphi_n)) = \{v \in H^1(\Omega(\varphi_n)) \mid v = 0 \text{ on } \partial\Omega(\varphi_n) \setminus \bar{\Gamma}_\sigma(\varphi_n)\}.$$

Let $v \in V_0(\Gamma_\sigma(\varphi))$, $v \geq 0$ on $\Gamma_\sigma(\varphi)$ be given. Then one can find a sequence $\{v_j\}$, $v_j \in C^\infty(\overline{\Omega(\varphi)})$, $\operatorname{dist}(\operatorname{supp} v_j, \partial\Omega(\varphi) \setminus \bar{\Gamma}_\sigma(\varphi)) > 0$, $v_j \geq 0$ on $\Gamma_\sigma(\varphi)$ and such that $v_j \to v$ in $H^1(\Omega(\varphi))$. It is readily seen that $\tilde{v}_j|_{\Omega_n} \in V_0(\Gamma_\sigma(\varphi_n))$, for n large enough. Inserting v_j into (2.15) we have

$$0 \geq (\lambda_n, \nabla v_j)_{0,\Omega_n} = (\tilde{\lambda}_n I_n, \nabla v_j)_{0,\hat{\Omega}} \underset{n\to\infty}{\longrightarrow} (\tilde{\lambda} I, \nabla v_j)_{0,\hat{\Omega}} = (\lambda, \nabla v_j)_{0,\Omega}$$

Finally, letting $j \to \infty$ we arrive at

$$(\lambda, \nabla v) \leq 0 \quad \forall v \in V_0(\Gamma_\sigma(\varphi)), \ v \geq 0 \text{ on } \Gamma_\sigma(\varphi),$$

i.e. $\lambda \cdot \nu \leq 0$ on $\Gamma_\sigma(\varphi)$.

Proof of Theorem 2.1: Let $\{\Omega(\varphi_n)\}$, $\Omega(\varphi_n) \in \mathcal{O}$ be a minimizing sequence of (P):

$$q = \inf_{\mathcal{O}} F(\Omega(\varphi)) = \lim_{n\to\infty} F(\Omega(\varphi_n)).$$

We may assume that

$$\Omega(\varphi_n) \to \Omega(\varphi^*) \in \mathcal{O}.$$

Let λ_n, χ_n be solutions of $(P^*(\varphi_n))$, $(A^*(\varphi_n))$, respectively. Taking into account Lemmas 2.2 and 2.3, we see that there exist functions $\lambda^* \in K_{00}(\Omega(\varphi^*))$, $\chi^* \in K_{00}^{\lambda^*}(\Omega(\varphi^*))$ being solutions of $(P^*(\varphi^*))$, $(A^*(\varphi^*))$, respectively and

$$\tilde{\chi}_n \rightharpoonup \tilde{\chi}^* \quad \text{in } (L^2(\hat{\Omega}))^2.$$

From this and the fact that

$$F(\Omega(\varphi_n)) = \frac{1}{2}\|\chi(\varphi_n)\|^2_{0,\Omega_n} = \frac{1}{2}\|\tilde{\chi}(\varphi_n)\|^2_{0,\hat{\Omega}}$$

we obtain

$$q = \liminf_{n\to\infty} F(\Omega(\varphi_n)) \geq \frac{1}{2}\|\tilde{\chi}^*\|^2_{0,\hat{\Omega}} = \frac{1}{2}\|\chi^*\|^2_{0,\Omega(\varphi^*)} = F(\Omega(\varphi^*)).$$

Moreover, if the one dimensional measure of $\Gamma_\sigma(\varphi^*)$ is positive then $\lambda^* \cdot \nu \leq 0$ on $\Gamma_\sigma(\varphi^*)$ as follows from Lemma 2.4. Thus $\Omega(\varphi^*)$ is a solution of (P).

3. Sensitivity analysis and optimality conditions

The aim of the present section is to derive optimality conditions for the problem (P). To this end it will be convenient to assume the cost functional $F(\Omega(\varphi))$ in its "primal" form, namely

$$F(\Omega(\varphi)) = \frac{1}{2}\|z(\varphi)\|^2_{1,\Omega(\varphi)}, \tag{3.1}$$

where $z(\varphi) \in V_0(\varphi)$ is the solution of $(A(\varphi))$.

For the sake of simplicity, we shall analyze a homogeneous case only, with $k \equiv 1$.

Let $\Omega(\varphi) \in \mathcal{O}$ be given, $\mathcal{F}_t : \mathrm{R}^2 \to \mathrm{R}^2$ a mapping of the form

$$\mathcal{F}_t = \mathrm{id} + t\mathcal{V} \Longleftrightarrow \mathcal{F}_t(x_1, x_2) = (x_1, x_2) + t(\mathcal{V}_1(x_1, x_2), \mathcal{V}_2(x_1, x_2)),\ t \geq 0$$

such that $\Omega_t(\varphi_t) \equiv \mathcal{F}_t(\Omega(\varphi))$ belongs to $\mathcal{O}$ for $t > 0$ sufficiently small. $\Omega_t(\varphi_t)$ is a deformation of $\Omega(\varphi)$ defined by means of the mapping $\mathcal{F}_t$.

Let $u_t(\varphi_t)$, $z_t(\varphi_t)$ be solutions of corresponding boundary value problems, defined on Ω_t:[0]

$$(P(\varphi_t)_t) \qquad \begin{cases} \text{Find } u_t(\varphi_t) \in V_\Phi(\varphi_t) \text{ such that} \\ (\nabla u_t(\varphi_t), \nabla v)_{0,\Omega_t} = 0 \quad v \in V(\varphi_t) \end{cases}$$

=We set $\varphi \equiv \varphi$, $P\varphi \equiv P\varphi$, $A\varphi \equiv A\varphi, \ldots$

and

$$(A(\varphi_t)_t) \qquad \begin{cases} \text{Find } z_t(\varphi_t) \in V_0(\varphi_t) \text{ such that} \\ (\nabla z_t, \nabla v)_{0,\Omega_t} = (\nabla u_t(\varphi_t), \nabla v)_{0,\Omega_t} \quad \forall v \in V_0(\varphi_t), \end{cases}$$

respectively, where

$$\begin{aligned} V(\varphi_t) &= \{v \in H^1(\Omega_t(\varphi_t)) \mid v = 0 \text{ on } \Sigma(\varphi_t) = \Gamma_1 \cup \Gamma_2 \cup \Gamma(\varphi_t) \cup \Gamma_\sigma(\varphi_t)\}, \\ V_\Phi(\varphi_t) &= \{v \in H^1(\Omega_t(\varphi_t)) \mid v = \Phi(\varphi_t) \text{ on } \Sigma(\varphi_t)\}, \\ V_0(\varphi_t) &= \{v \in H^1(\Omega_t(\varphi_t)) \mid v = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_\sigma(\varphi_t)\}. \end{aligned}$$

Denote by $u^t \equiv u_t \circ \mathcal{F}_t$, $z^t \equiv z_t \circ \mathcal{F}_t$, $\Phi^t \equiv \Phi(\varphi_t) \circ \mathcal{F}_t$ functions defined on $\Omega(\varphi)$ ($u^0 \equiv u(\varphi)$, $z^0 \equiv z(\varphi)$, $\Phi^0 \equiv \Phi(\varphi)$). Using Fubini's theorem in $(P(\varphi_t)_t)$, $(A(\varphi_t)_t)$ we see that u^t, z^t are solutions of

$$(P(\varphi)^t) \qquad \begin{cases} \text{Find } u^t \in V_{\Phi^t}(\varphi) \text{ such that} \\ (A(t)\nabla u^t, \nabla v)_{0,\Omega(\varphi)} = 0 \quad \forall v \in V(\varphi) \end{cases}$$

and

$$(A(\varphi)^t) \qquad \begin{cases} \text{Find } z^t \in V_0(\varphi) \quad \text{such that} \\ (A(t)\nabla z^t, \nabla v)_{0,\Omega(\varphi)} = (A(t)\nabla u^t, \nabla v)_{0,\Omega(\varphi)} \quad \forall v \in V_0(\varphi), \end{cases}$$

respectively, where the differential operator $A(t)$ is done by the expression

$$A(t) = (D\mathcal{F}_t)^{-1}(D\mathcal{F}_t)^{\mathrm{T}}\mathcal{I}_t,$$

with $D\mathcal{F}_t$ denoting the Jacobian of $\mathcal{F}_t$, $\mathcal{I}_t = \det(D\mathcal{F}_t)$. Let

$$\dot{u} = \lim_{t\to 0+} \frac{u^t - u^0}{t}, \quad \dot{z} = \lim_{t\to 0+} \frac{z^t - z^0}{t}, \quad \dot{\Phi} = \lim_{t\to 0+} \frac{\Phi^t - \Phi^0}{t}$$

be the material derivatives of $u(\varphi)$, $z(\varphi)$, $\Phi(\varphi)$ respectively (limits are considered in $H^1(\Omega(\varphi))$-norm). Differentiating $(P(\varphi)^t)$, $(A(\varphi)^t)$ with respect to t at $t = 0+$ we see that $\dot{u} \in H^1(\Omega(\varphi))$, $\dot{z} \in V_0(\varphi)$ are solutions of

$$(\dot{P}(\varphi)) \qquad \begin{cases} (\nabla \dot{u}, \nabla v)_{0,\Omega(\varphi)} = -(\mathcal{A}\nabla u(\varphi), \nabla v)_{0,\Omega(\varphi)} \quad \forall v \in V(\varphi) \\ \dot{u} = \dot{\Phi} \quad \text{on } \Sigma(\varphi) \end{cases}$$

and

$$(\dot{A}(\varphi)) \qquad \begin{cases} (\nabla \dot{z}, \nabla v)_{0,\Omega(\varphi)} = (\mathcal{A}\nabla u(\varphi), \nabla v)_{0,\Omega(\varphi)} \\ \quad + (\nabla \dot{u}, \nabla v)_{0,\Omega(\varphi)} - (\mathcal{A}\nabla z(\varphi), \nabla v)_{0,\Omega(\varphi)} \quad \forall v \in V_0(\varphi), \end{cases}$$

respectively, with

$$\mathcal{A} = \operatorname{div}\mathcal{V}\,\mathrm{id} - (D\mathcal{V})^{\mathrm{T}} - D\mathcal{V}, \quad \mathcal{V} = (\mathcal{V}_1, \mathcal{V}_2),$$

where

$$D\mathcal{V} = \left(\frac{\partial \mathcal{V}_i}{\partial x_j}\right)_{i,j=1}^2$$

and "id" denotes the identity matrix.

Let

$$F(\Omega_t(\varphi_t)) = \frac{1}{2}\|\nabla z_t\|_{1,\Omega_t(\varphi_t)} = \frac{1}{2}(A(t)\nabla z^t, \nabla z^t)_{0,\Omega(\varphi)},$$

where z^t solves $(A(\varphi)^t)$ and denote $\dot{F}(0) \equiv \frac{d}{dt}F(\Omega_t(\varphi_t))|_{t=0}$.

Lemma 3.1. *It holds that:*

$$\dot{F}(0) = -\frac{1}{2}(\mathcal{A}\nabla z(\varphi), \nabla z(\varphi))_{0,\Omega(\varphi)} + (\mathcal{A}\nabla u(\varphi), \nabla z(\varphi))_{0,\Omega(\varphi)} + (\nabla\dot{\Phi}, \nabla z(\varphi))_{0,\Omega(\varphi)}, \tag{3.2}$$

where $u(\varphi)$, $z(\varphi)$ *are solutions of* $(P(\varphi))$, $(A(\varphi))$, *respectively.*

Proof: A direct calculation yields:

$$\begin{aligned} \dot{F}(0) &= \frac{1}{2}(\mathcal{A}\nabla z(\varphi), \nabla z(\varphi))_{0,\Omega(\varphi)} + (\nabla\dot{z}, \nabla z(\varphi))_{0,\Omega(\varphi)} \\ &= -\frac{1}{2}(\mathcal{A}\nabla z(\varphi), \nabla z(\varphi))_{0,\Omega(\varphi)} \\ &\quad + (\mathcal{A}\nabla u(\varphi), \nabla z(\varphi))_{0,\Omega(\varphi)} + (\nabla\dot{u}, \nabla z)_{0,\Omega(\varphi)} \end{aligned} \tag{3.3}$$

using the definition of $(\dot{A}(\varphi))$ with $v := z(\varphi)$. The function $\dot{u}$ being the solution of $(\dot{P}(\varphi))$ can be split and written as $\dot{u} = \tilde{u} + \dot{\Phi}$, where $\tilde{u} = 0$ on $\Sigma(\varphi)$. Thus

$$\begin{aligned} (\nabla\dot{u}, \nabla z)_{0,\Omega(\varphi)} &= (\nabla\tilde{u}, \nabla z)_{0,\Omega(\varphi)} + (\nabla\dot{\Phi}, \nabla z(\varphi))_{0,\Omega(\varphi)} \\ &= (\nabla\dot{\Phi}, \nabla z(\varphi))_{0,\Omega(\varphi)} \end{aligned}$$

taking into account the definition of $(A(\varphi))$. From this and (3.3), the assertion follows.

Next we shall suppose that the solutions of $(A(\varphi))$ and $(P(\varphi))$ are smooth enough. As an easy exercise, one can prove

Lemma 3.2. *Let* f, g *be sufficiently smooth. Then*

$$\begin{aligned} (\mathcal{A}\nabla f, \nabla g)_{0,\Omega(\varphi)} &= ((\mathcal{V}\cdot\nabla g), \Delta f)_{0,\Omega(\varphi)} + ((\mathcal{V}\cdot\nabla f), \Delta g)_{0,\Omega(\varphi)} \\ &\quad + \int_{\partial\Omega(\varphi)} \left(\mathcal{V}_1 \frac{\partial g}{\partial x_2}\frac{\partial f}{\partial s} - \mathcal{V}_2 \frac{\partial g}{\partial x_1}\frac{\partial f}{\partial s}\right) ds - \int_{\partial\Omega(\varphi)} (\mathcal{V}\cdot\nabla g)\frac{\partial f}{\partial \nu}\, ds. \end{aligned} \tag{3.4}$$

Evaluating integrals in (3.2), making use of (3.4) and the fact that $\Delta u(\varphi) = \Delta z(\varphi) = 0$ in $\Omega(\varphi)$, we obtain the following expression for $\dot{F}(0)$, containing integrals on $\partial\Omega(\varphi)$ only.

Lemma 3.3. *It holds that:*

$$\begin{aligned} \dot{F}(0) &= -\frac{1}{2}\int_{\partial\Omega(\varphi)} \mathcal{V}_1 \frac{\partial z}{\partial x_2}\frac{\partial z}{\partial s}\, ds + \frac{1}{2}\int_{\partial\Omega(\varphi)} \mathcal{V}_2 \frac{\partial z}{\partial x_1}\frac{\partial z}{\partial s}\, ds \\ &\quad + \frac{1}{2}\int_{\partial\Omega(\varphi)} (\mathcal{V}\cdot\nabla z)\frac{\partial z}{\partial \nu}\, ds + \int_{\Omega(\varphi)} \mathcal{V}_1 \frac{\partial z}{\partial x_2}\frac{\partial u}{\partial s}\, ds \\ &\quad - \int_{\partial\Omega(\varphi)} \mathcal{V}_2 \frac{\partial z}{\partial x_1}\frac{\partial u}{\partial s}\, ds - \int_{\partial\Omega(\varphi)} (\mathcal{V}\cdot\nabla z)\frac{\partial u}{\partial \nu}\, ds + \int_{\partial\Omega(\varphi)} \mathcal{V}_2 \frac{\partial z}{\partial \nu}\, ds. \end{aligned} \tag{3.5}$$

Definition 3.1. We say that $\Omega(\bar{\varphi}) \in \mathcal{O}$ is a *stationary point* of F iff

$$\dot{F}(0) = \frac{d}{dt}F(\Omega_t(\bar{\varphi}_t))\big|_{t=0} = 0, \quad \Omega_t(\bar{\varphi}_t) = \mathcal{F}_t(\Omega(\bar{\varphi}))$$

holds for any vector field $\mathcal{V} = (\mathcal{V}_1, \mathcal{V}_2)$ with $\operatorname{supp} \mathcal{V}_i \subset \hat{\Omega}$, $i = 1, 2$.

Let $\Omega(\bar{\varphi})$ be a stationary point of F. Taking $\mathcal{V} = (\mathcal{V}_1, \mathcal{V}_2)$ such that $\operatorname{supp} \mathcal{V}_i \subset \hat{\Omega}$, $i = 1, 2$ only integrals on $\Gamma(\bar{\varphi})$ appear in (3.5). As $\partial u/\partial \nu = \partial z/\partial \nu$ on $\Gamma(\bar{\varphi})$, (3.5) leads to the following relations on $\Gamma(\bar{\varphi})$:

$$\frac{\partial z}{\partial x_2}\frac{\partial z}{\partial s} + \frac{\partial z}{\partial x_1}\frac{\partial z}{\partial \nu} = 2\frac{\partial z}{\partial x_2}\frac{\partial u}{\partial s} \tag{3.6}$$

$$\frac{\partial z}{\partial x_1}\frac{\partial z}{\partial s} - \frac{\partial z}{\partial x_2}\frac{\partial z}{\partial \nu} = 2\frac{\partial z}{\partial x_1}\frac{\partial u}{\partial s} - 2\frac{\partial z}{\partial \nu}. \tag{3.7}$$

A direct calculation shows that the left hand side of (3.6) and (3.7) is equal to $|\nabla z|^2 \nu_1$, $-|\nabla z|^2 \nu_2$, respectively, i.e.

$$|\nabla z|^2 \nu_1 = 2\frac{\partial z}{\partial x_2}\frac{\partial u}{\partial s} \tag{3.8}$$

$$-|\nabla z|^2 \nu_2 = 2\frac{\partial z}{\partial x_1}\frac{\partial u}{\partial s} - 2\frac{\partial z}{\partial \nu} \tag{3.9}$$

on $\Gamma(\bar{\varphi})$.

Let us assume the simplest case, when $\Gamma(\bar{\varphi})$ is done by a function $\bar{\varphi} = \bar{\varphi}(x_1)$. Then

$$\nu = (\nu_1, \nu_2) = \left(\frac{-\bar{\varphi}'}{\sqrt{1 + (\bar{\varphi}')^2}}, \frac{1}{\sqrt{1 + (\bar{\varphi}')^2}} \right),$$

$s = (-\nu_2, \nu_1)$ and

$$\frac{\partial u}{\partial s} = -\frac{\bar{\varphi}'}{\sqrt{1 + (\bar{\varphi}')^2}} \quad \text{on } \Gamma(\bar{\varphi}), \tag{3.10}$$

using the fact that $u(x, \bar{\varphi}(x)) = \bar{\varphi}(x)$ on $\Gamma(\bar{\varphi})$. Now, as a deformation field we use a vector $\mathcal{V}$, the first component of which is identically equal to zero in $\Omega(\bar{\varphi})$ so that $\Gamma(\bar{\varphi}_t)$ is given by the function

$$\bar{\varphi}_t(x_1) = \bar{\varphi}(x_1) + t\mathcal{V}_2(x_1, \bar{\varphi}(x_1)), \ t \geq 0.$$

Replacing $\partial u/\partial s$ in (3.8) by (3.10), we finally obtain

$$|\nabla z(\bar{\varphi})|^2 = 2\frac{\partial z(\bar{\varphi})}{\partial x_2} \quad \text{on } \Gamma(\bar{\varphi}). \tag{3.11}$$

Consequently $\partial z(\bar{\varphi})/\partial x_2 \geq 0$ on $\Gamma(\bar{\varphi})$. Next we shall suppose that there is no point $(x_1, \bar{\varphi}(x_1))$ lying in $\Omega(\bar{\varphi})$, in which $\bar{\varphi}'(x_1) = \pm\infty$. From this and the Hopf maximum principle, the maximum of $z(\bar{\varphi})$ is attained on $\Gamma(\bar{\varphi})$. We shall show that only the function $z(\bar{\varphi}) \equiv 0$ on $\Gamma(\bar{\varphi})$ (and consequently $z(\bar{\varphi}) \equiv 0$ in $\Omega(\bar{\varphi})$) satisfies (3.11). Indeed, let $x^* \in \Gamma(\bar{\varphi})$ be such that

$$z(\bar{\varphi})(x^*) = \max_{x \in \Gamma(\bar{\varphi})} z(\bar{\varphi})(x).$$

Let us suppose that $z(\bar{\varphi})(x^*) > 0$. Thus also $\partial z(\bar{\varphi})(x^*)/\partial\nu > 0$. From (3.11) it follows:

$$|\nabla z(x^*)|^2 = \left|\frac{\partial z(\bar{\varphi})}{\partial \nu}(x^*)\right|^2 = 2\frac{\partial z(\bar{\varphi})}{\partial x_2}(x^*). \tag{3.12}$$

On the other hand

$$\frac{\partial z(\bar{\varphi})}{\partial x_2} = \nu_2 \frac{\partial z(\bar{\varphi})}{\partial \nu} + \nu_1 \frac{\partial z}{\partial s}$$

holds on $\Gamma(\bar{\varphi})$. Especially at $x = x^*$ one has

$$\frac{\partial z(\bar{\varphi})}{\partial x_2}(x^*) = \frac{1}{\sqrt{1 + (\bar{\varphi}'(x^*))^2}} \frac{\partial z(\bar{\varphi})}{\partial \nu}(x^*).$$

This, together with (3.12) leads to

$$\frac{\partial z(\bar{\varphi})}{\partial \nu}(x^*) = \frac{2}{\sqrt{1 + (\bar{\varphi}'(x^*))^2}} \tag{3.13}$$

Let $u(\bar{\varphi})$ be the solution of $(P(\bar{\varphi}))$. Then $u(\bar{\varphi})$ can be split as follows

$$u(\bar{\varphi})(x, y) = f(x, y) + y,$$

where f is harmonic in $\Omega(\bar{\varphi})$, $f = 0$ on $\Gamma(\bar{\varphi}) \cup \Gamma_\sigma(\bar{\varphi})$, $f = y_i - y$ on Γ_i, $i = 1, 2$ and $\partial f/\partial\nu = -1$ on Γ_0. Hence $f > 0$ in $\Omega(\bar{\varphi})$ and

$$\frac{\partial f}{\partial \nu} \leq 0 \quad \text{on } \Gamma(\bar{\varphi}). \tag{3.14}$$

As $\partial u(\bar{\varphi})/\partial\nu = \partial z(\bar{\varphi})/\partial\nu$ on $\Gamma(\bar{\varphi})$, the relation (3.13) yields:

$$\frac{\partial u(\bar{\varphi})}{\partial \nu}(x^*) = \frac{2}{\sqrt{1 + (\bar{\varphi}'(x^*))^2}}.$$

On the other hand

$$\frac{\partial u(\bar{\varphi})}{\partial \nu}(x^*) = \frac{\partial f}{\partial \nu}(x^*) + \frac{\partial y}{\partial \nu} = \frac{\partial f}{\partial \nu} + \frac{1}{\sqrt{1 + (\bar{\varphi}'(x^*))^2}} = \frac{2}{\sqrt{1 + (\bar{\varphi}'(x^*))^2}}$$

so that

$$\frac{\partial f}{\partial \nu}(x^*) = \frac{1}{\sqrt{1 + (\bar{\varphi}'(x^*))^2}} > 0$$

which contradicts (3.14).

Thus we have proved:

Theorem 3.1. *Let $\Omega(\bar{\varphi}) \in \mathcal{O}$ be a stationary point of F in the sense of definition 3.1. Let $\Gamma(\bar{\varphi})$ be described by a function $\bar{\varphi} = \bar{\varphi}(x_1)$ admitting no vertical tangent at the interior of $\hat{\Omega}$. Let $u(\bar{\varphi})$, $z(\bar{\varphi})$ be solutions of $(P(\bar{\varphi}))$, $(A(\bar{\varphi}))$, respectively on $\Omega(\bar{\varphi})$. Then $\partial u(\bar{\varphi})/\partial n = 0$ on $\Gamma(\bar{\varphi})$.*

Proof: From the above considerations, it follows that

$$\max_{x \in \Gamma(\bar{\varphi})} z(\bar{\varphi})(x) = 0.$$

Hence $z(\bar{\varphi}) = 0$ on $\Gamma(\bar{\varphi})$ and also in $\Omega(\bar{\varphi})$. As $\partial u(\bar{\varphi})/\partial\nu = \partial z(\bar{\varphi})/\partial\nu$ on $\Gamma(\bar{\varphi})$, we arrive at the assertion of the Theorem.

4. Approximation of (P)

The present section deals with the approximation of (P) when the divergence free finite elements are used for the realization of $(P^*(\varphi))$ and $(A^*(\varphi))$. For the sake of simplicity, we shall assume that $\hat{\Omega}$ is polygonal and $\Gamma(\varphi)$ is approximated by piecewise linear functions. More precisely, let $D_h :\ 0 = d_0 < d_1 < \dots < d_{n(h)} = 1$ be a partition of $[0,1]$, the norm of which tends to zero when $h \to 0+$ and

$$P_1^h = \{\varphi_h : [0,1] \to \mathbb{R}^2 \,|\varphi_h(0) = A,\ \varphi_h(1) \in \tilde{\Gamma}_2 \setminus \Gamma_2,$$
$$\varphi_h|_{d_i - d_i} \text{ is linear and the graph } \Gamma(\varphi_h) \text{ lies in } \hat{\Omega}\}.$$

Let

$$\mathcal{O}_h = \{\Omega(\varphi_h) \in \mathcal{O} \mid \varphi_h \in P_1^h\}.$$

Any $\Omega(\varphi_h) \in \mathcal{O}_h$ is a domain with the polygonal boundary $\partial\Omega(\varphi_h)$. Let $\{\mathcal{T}(h,\varphi_h)\}$ be a family of triangulations of $\Omega(\varphi_h) \in \mathcal{O}_h$ satisfying the usual requirements for the mutual position of elements belonging to $\mathcal{T}(h,\varphi_h)$. Moreover, we shall assume that

- A1. for any $h > 0$ fixed, the triangulation $\mathcal{T}(h,\varphi_h)$ depends continuously on $\varphi_h \in P_h^1$, i.e. the position of nodes of $\mathcal{T}(h,\varphi_h)$ depends continuously on changes of $\varphi_h \in P_1^h$;
- A2. for any $h > 0$ fixed, the triangulations $\mathcal{T}(h,\varphi_h)$ are "topologically" equivalent, i.e. the neighbours of any node of $\mathcal{T}(h,\varphi_h)$ remain the same for any $\varphi_h \in P_1^h$;
- A3. the family $\{\mathcal{T}(h,\varphi_h)\}$, $h > 0$, $\varphi_h \in P_1^h$ is uniformly regular in the following sense: $\exists\vartheta_0 > 0$ such that $\vartheta(h,\varphi_h) \geq \vartheta_0$ for any $h > 0$ and for any $\Omega(\varphi_h) \in \mathcal{O}_h$, where $\vartheta(h,\varphi_h)$ is the minimum interior angle of elements, belonging to $\mathcal{T}(h,\varphi_h)$.

The domain $\Omega(\varphi_h)$ with a given triangulation $\mathcal{T}(h,\varphi_h) \in \{\mathcal{T}(h,\varphi_h)\}$ satisfying A1–A3 will be denoted by Ω_h.

With any $\mathcal{T}(h,\varphi_h)$, finite dimensional subspaces $K_{00}^h(\varphi_h)$, $K_{00}^{0h}(\varphi_h)$ of $K_{00}(\varphi_h)$, $K_{00}^0(\varphi_h)$, respectively will be associated:

$$K_{00}^h(\varphi_h) \subset K_{00}(\varphi_h), \quad \dim K_{00}^h(\varphi_h) = m_1(h) \to \infty \text{ if } h \to 0+;$$
$$K_{00}^{0h}(\varphi_h) \subset K_{00}^0(\varphi_h), \quad \dim K_{00}^{0h}(\varphi_h) = m_2(h) \to \infty \text{ if } h \to 0+.$$

Remark 4.1. In practice, functions belonging to $K_{00}^h(\varphi_h)$, $K_{00}^{0h}(\varphi_h)$ are piecewise polynomial on $\mathcal{T}(h,\varphi_h)$ (see next section).

Moreover, we shall assume families of $\{K_{00}^0(\varphi_h)\}$, $\{K_{00}^{0h}(\varphi_h)\}$ having the following approximating properties:

- A4. for any $\mu \in K_{00}(\varphi)$, $K_{00}^0(\varphi)$, respectively, $\Omega(\varphi) \in \mathcal{O}$ and any sequence $\{\Omega_h\}$, $\Omega_h \in \mathcal{O}_h$ such that $\Omega_h \to \Omega$ there is a sequence $\{\mu_h\}$, $\mu_h \in K_{00}^h(\varphi_h)$, $K_{00}^{0h}(\varphi_h)$ respectively, such that $\tilde{\mu}_h \to \tilde{\mu}$ in $(L^2(\hat{\Omega}))^2$, where the symbol $\tilde{}$ denotes the extension by zero on $\hat{\Omega}$.

The approximations of $(P^*(\varphi))$, $(A^*(\varphi))$ are defined as follows:

$$(P_h^*(\varphi_h)) \qquad \begin{cases} \text{Find } \lambda_h(\varphi_h) \in K_{00}^h(\varphi_h) \text{ such that} \\ (k^{-1}\lambda_h(\varphi_h), \mu_h)_{0,\Omega_h} = (\nabla\Phi_h(\varphi_h), \mu_h)_{0,\Omega_h} \quad \forall\mu \in K_{00}^h(\varphi_h) \end{cases}$$

and

$$(A_h^*(\varphi_h)) \qquad \begin{cases} \text{Find } \chi_h(\varphi_h) \in K_{00}^{\lambda_h, h}(\varphi_h) \text{ such that} \\ (\chi_h(\varphi_h), \mu_h)_{0,\Omega_h} = 0 \quad \forall \mu_h \in K_{00}^{0h}(\varphi_h). \end{cases}$$

Here $\Phi_h(\varphi_h)$ is a function from $H^1(\Omega(\varphi_h))$ realizing the nonhomogeneous Dirichlet boundary conditions on $\Sigma(\varphi_h)$ and

$$K_{00}^{\lambda_h, h}(\varphi_h) = \{\mu_h \in K_{00}^h(\varphi_h) \mid \mu_h \cdot \nu = \lambda_h \cdot \nu \text{ on } \Gamma(\varphi_h), \quad \text{where } \lambda_h \in K_{00}^{0h}(\varphi_h)\}.$$

Finally, denote by $F_{h,\varepsilon}(\Omega_h)$ the cost functional defined by

$$F_{h,\varepsilon}(\Omega_h) = \frac{1}{2}\|\chi_h(\varphi_h)\|_{0,\Omega_h}^2 + \frac{1}{2\varepsilon}\int_{\Gamma_\sigma(\varphi_h)} \left([\lambda_h \cdot \nu]^+\right)^2 ds,$$

where $\lambda_h,\ \chi_h$ are solutions of $(P_h^*(\varphi_h))$, $(A_h^*(\varphi_h))$, respectively, $\varepsilon > 0$ is a penalty parameter and $[\cdot]^+$ stands for the positive part of a real number.

Let $h, \varepsilon > 0$ be given. By the approximation of (P) we call the problem

$$(\mathrm{P}_{h\varepsilon}) \qquad \begin{cases} \text{Find } \Omega_{h\varepsilon}^* \in \mathcal{O}_h \text{ such that} \\ F_{h\varepsilon}(\Omega_{h\varepsilon}^*) \le F_{h\varepsilon}(\Omega_h) \quad \forall \Omega_h \in \mathcal{O}_h. \end{cases}$$

Remark 4.2. The state constraint $\lambda(\varphi) \cdot \nu \le 0$ on $\Gamma_\sigma(\varphi)$ appearing in the continuous case is treated by means of the penalty approach in the discrete case.

It is not difficult to prove:

Theorem 4.1. *Let A1–A2 be satisfied. Then there exists a solution of* $(\mathrm{P}_{h\varepsilon})$ *for any* $h, \varepsilon > 0$.

Next we shall study the relationship between the continuous case and its discretization. For the sake of simplicity, the state constraint $\lambda(\varphi) \cdot \nu \le 0$ on $\Gamma_\sigma(\varphi)$ will be omitted. The failure of this constraint can be justified in cases when the dam is such that $\nu_2 \le 0$ along $\tilde{\Gamma}_2$ (especially when the outflow part of the dam is vertical). In such a case, the condition $\lambda(\varphi) \cdot \nu \le 0$ is the consequence of the maximum principle. When the condition $\lambda(\varphi) \cdot \nu \le 0$ on $\Gamma_\sigma(\varphi)$ is omitted, the discrete cost functional takes a simpler form, namely:

$$F_h(\Omega_h) = \frac{1}{2}\|\chi_h(\varphi_h)\|_{0,\Omega_h}^2$$

and the approximation of (P) reads as follows:

$$(\mathrm{P}_h) \qquad \begin{cases} \text{Find } {\Omega_h}^* \in \mathcal{O}_h \text{ such that} \\ F_h({\Omega_h}^*) \le F_h(\Omega_h) \quad \forall \Omega_h \in \mathcal{O}_h. \end{cases}$$

Remark 4.3. In practical applications however it is recommended to use the penalized form $F_{h\varepsilon}$ forcing the solution ${\Omega_h}^*$ to be physically admissible (let us recall that no assumptions, concerning the monotonicity of $\Gamma(\varphi)$ are included into the definition of $\mathcal{O}$). The automatic satisfaction of the condition $\lambda(\varphi) \cdot \nu \le 0$ on $\Gamma_\sigma(\varphi)$ as a consequence of the maximum principle does not imply the same for $\lambda_h(\varphi_h) \cdot \nu$ on $\Gamma_\sigma(\varphi_h)$, in general.

Next we shall study the relation between (P) and (P_h) for $h \to 0+$. First of all we start with several lemmas.

Lemma 4.1. *Let* $\Omega_h \to \Omega(\varphi)$, $\Omega_h \in \mathcal{O}_h$, $\Omega(\varphi) \in \mathcal{O}$. *Let* $\{\mu_h\}$, $\mu_h \in K^h_{00}(\varphi_h)$, $K^{0h}_{00}(\varphi_h)$, *respectively be such that*

$$\tilde{\mu}_h \rightharpoonup \tilde{\mu} \quad in\ (L^2(\hat{\Omega}))^2.$$

Then $\mu \equiv \tilde{\mu}|_{\Omega(\varphi)}$ *belongs to* $K_{00}(\varphi)$, $K^0_{00}(\varphi)$, *respectively.*

Proof: is contained in the proof of Lemma 2.2.

Lemma 4.2. *Let* $\Omega_h \to \Omega(\varphi)$, $\Omega_h \in \mathcal{O}_h$, $\Omega(\varphi) \in \mathcal{O}$. *Let* $\{\lambda_h\}$, $\{\mu_h\}$, $\lambda_h \in K^h_{00}(\varphi_h)$, $\mu_h \in K^{\lambda_h,h}_{00}(\varphi_h)$ *be such that*

$$\tilde{\lambda}_h \rightharpoonup \tilde{\lambda} \quad in\ (L^2(\hat{\Omega}))^2, \tag{4.1}$$

$$\tilde{\mu}_h \rightharpoonup \tilde{\mu} \quad in\ (L^2(\hat{\Omega}))^2. \tag{4.2}$$

Then $\mu \equiv \tilde{\mu}|_{\Omega(\varphi)} \in K^{\lambda}_{00}(\varphi)$, *where* $\lambda \equiv \tilde{\lambda}|_{\Omega(\varphi)}$

Proof: Let $\mu_h \in K^{\lambda_h,h}_{00}(\varphi_h)$, i.e.

$$(\mu_h, \nabla v)_{0,\Omega_h} = (\lambda_h, \nabla v)_{0,\Omega_h} \tag{4.3}$$

holds for any $v \in V_0(\varphi_h) = \{v \in H^1(\Omega(\varphi_h)) \mid v = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_\sigma(\varphi_h)\}$. Let $v \in V_0(\varphi)$ be given and $\hat{v} \in H^1(\hat{\Omega})$ its extension such that $\hat{v} = 0$ on $\partial\hat{\Omega} \setminus \bar{\Gamma}_0$. Then $\hat{v}|_{\Omega_h} \in V_0(\varphi_h)$ for any $\Omega_h \in \mathcal{O}_h$ and it can be inserted into (4.3), written equivalently in the form

$$(\tilde{\mu}_h I_h, \nabla\hat{v})_{0,\hat{\Omega}} = (\tilde{\lambda}_h I_h, \nabla\hat{v})_{0,\hat{\Omega}},$$

where I_h is the characteristic function of Ω_h. Passing to the limit with $h \to 0+$ and taking into account (4.1), (4.2) we arrive at

$$(\tilde{\mu}, \nabla v)_{0,\Omega(\varphi)} = (\tilde{\lambda}, \nabla v)_{0,\Omega(\varphi)},$$

i.e. $\mu \equiv \tilde{|}_{\Omega(\varphi)} \in K^{\lambda}_{00}(\varphi)$.

Lemma 4.3. *Let* $\Omega_h \to \Omega(\varphi)$, $\Omega_h \in \mathcal{O}_h$, $\Omega(\varphi) \in \mathcal{O}$, $\lambda_h(\varphi_h) \in K^h_{00}(\varphi_h)$, $\chi_h(\varphi_h) \in K^{\lambda_h,h}_{00}(\varphi_h)$ *be solutions of* $(P^*_h(\varphi_h))$, $(A^*_h(\varphi_h))$, *respectively. Moreover, let A4 be satisfied. Then there exist subsequences of* $\{\lambda_h(\varphi_h)\}$ *and* $\{\chi_h(\varphi_h)\}$ *such that*

$$\tilde{\lambda}_h(\varphi_h) \rightharpoonup \tilde{\lambda} \quad in\ (L^2(\hat{\Omega}))^2 \tag{4.4}$$

$$\tilde{\mu}_h(\varphi_h) \rightharpoonup \tilde{\mu} \quad in\ (L^2(\hat{\Omega}))^2. \tag{4.5}$$

Moreover, $\lambda(\varphi) \equiv \tilde{\lambda}|_{\Omega(\varphi)}$, $\chi(\varphi) \equiv \tilde{\chi}|_{\Omega(\varphi)}$ *are solutions of* $(P^*(\varphi))$, $(A^*(\varphi))$, *respectively.*

Proof: Let $\lambda_h(\varphi_h) \in K^h_{00}(\varphi_h)$ be a solution of $(P^*_h(\varphi_h))$:

$$(k^{-1}\lambda_h(\varphi_h), \mu_h)_{0,\Omega_h} = (\nabla\Phi_h(\varphi_h), \mu_h)_{0,\Omega_h} \quad \forall \mu_h \in K^h_{00}(\varphi_h). \tag{4.6}$$

Arguing in the same way as in Lemma 2.2, one can assume that the sequence $\{\|\Phi_h(\varphi_h)\|_{1,\Omega_h}\}$ is bounded and

$$\widetilde{\Phi_h}(\varphi_h) \rightharpoonup \Psi_1, \quad \widetilde{\nabla\Phi_h}(\varphi_h) \rightharpoonup \Psi_2 \quad \text{in}(L^2(\hat{\Omega}))^2.$$

Denoting by $\Phi \equiv \Psi_1|_{\Omega(\varphi)}$, one has $\Psi_2|_{\Omega(\varphi)} = \nabla\Phi$ and $\Phi = \Phi(\varphi)$, i.e. Φ realizes the nonhomogeneous Dirichlet boundary condition on $\Sigma(\varphi)$.

From (4.6) it follows that $\{\|\lambda_h(\varphi_h)\|_{0,\Omega_h}\}$ is bounded. Hence there is a function $\tilde\lambda \in (L^2(\hat\Omega))^2$ such that

$$\tilde\lambda_h(\varphi_h) \rightharpoonup \tilde\lambda \quad \text{in } (L^2(\hat\Omega))^2.$$

Let us show that $\lambda \equiv \tilde\lambda|_{\Omega(\varphi)}$ solves $(P^*(\varphi))$. The fact that $\lambda \in K_{00}(\varphi)$ follows from Lemma 4.1. Let $\mu \in K_{00}(\varphi)$ be given. Accordingly to A4 there is a sequence $\{\mu_h\}$, $\mu_h \in K_{00}^h(\varphi_h)$ and such that

$$\tilde\mu_h \to \tilde\mu \quad \text{in } (L^2(\hat\Omega)).$$

Inserting μ_h into (4.6) and passing to the limit with $h \to 0+$ we finally obtain:

$$(k^{-1}\lambda, \mu)_{0,\Omega(\varphi)} = (k^{-1}\tilde\lambda, \tilde\mu)_{0,\hat\Omega} = (\widetilde{\nabla\Phi}(\varphi), \tilde\mu)_{0,\hat\Omega} = (\nabla\Phi(\varphi), \mu)_{0,\hat\Omega}$$

i.e. λ solves $(P^*(\varphi))$. The fact that χ is a solution of $(A^*(\varphi))$ can be verified in the same way.

The question is: if and when the weak convergence in (4.4)–(4.5) can be replaced by the strong one, which will be needed for establishing the relation between (P) and (P_h).

Let $\Omega(\varphi) \in \mathcal{O}$ be given. Then its variable part $\Gamma(\varphi)$ can be rectified, i.e. $\Gamma(\varphi)$ can be approximated by a sequence $\{\Gamma(\varphi_h)\}$, where $\Gamma(\varphi_h)$ is given by the graph of the piecewise linear function φ_h, the lagrange interpolation of φ. Moreover if $\Omega(\varphi) \in \mathcal{O}$ then $\Omega(\varphi_h) \in \mathcal{O}_h$ and the boundaries of $\Omega(\varphi)$ and $\Omega(\varphi_h)$ are close even in H^1-norm. Then one can construct functions $\{\Phi_h(\varphi_h)\}$ and $\Phi(\varphi)$ in such a way that

$$\hat\Phi_h(\varphi_h) \to \hat\Phi(\varphi) \quad \text{in } H^1(\hat\Omega), \tag{4.7}$$

where $\widehat{\ }$ denotes the suitable extension of Φ_h, Φ, respectively, on $\hat\Omega$.

Lemma 4.4. *Let $\Omega(\varphi) \in \mathcal{O}$ be given and $\{\Omega_h(\varphi_h)\}$, $\Omega_h(\varphi_h) \in \mathcal{O}_h$ be a sequence with $\Gamma(\varphi_h)$ being the linear interpolation of $\Gamma(\varphi)$. Under the same assumptions as in Lemma 4.3, the weak convergence in (4.4)–(4.5) can be replaced by the strong one.*

Proof: Substituting $\mu_h = \lambda_h(\varphi_h)$ into (4.6) we have

$$(k^{-1}\tilde\lambda_h(\varphi_h), \tilde\lambda_h(\varphi_h))_{0,\hat\Omega} = (\nabla\hat\Phi_h(\varphi_h), \tilde\lambda_h(\varphi_h))_{0,\hat\Omega}.$$

Hence

$$\begin{aligned}\lim_{h\to 0+} (k^{-1}\tilde\lambda_h(\varphi_h), \tilde\lambda_h(\varphi_h))_{0,\hat\Omega} &= (\nabla\hat\Phi_h(\varphi_h), \tilde\lambda_h(\varphi_h))_{0,\hat\Omega} \\ &= (\nabla\hat\Phi(\varphi), \tilde\lambda(\varphi))_{0,\hat\Omega} = (k^{-1}\tilde\lambda(\varphi), \tilde\lambda(\varphi))_{0,\hat\Omega}\end{aligned}$$

taking into account (4.6) and (4.7). From this (4.4) follows.

Let $\chi_h(\varphi_h) \in K_{00}^{\lambda_h,h}(\varphi_h)$ be a solution of $(A_h^*(\varphi_h))$:

$$(\chi_h(\varphi_h), \mu_h)_{0,\Omega_h} = 0 \quad \forall \mu_h \in K_{00}^{0h}(\varphi_h). \tag{4.8}$$

Inserting $\mu_h = \lambda_h(\varphi_h) - \chi_h(\varphi_h) \in K_{00}^{0h}(\varphi_h)$ into (4.8) we obtain:

$$(\tilde{\chi}_h(\varphi_h), \tilde{\chi}_h(\varphi_h))_{0,\hat{\Omega}} = (\tilde{\chi}_h(\varphi_h), \tilde{\lambda}_h(\varphi_h))_{0,\hat{\Omega}}.$$

Hence

$$\begin{aligned}\lim_{h\to 0+} \|\tilde{\chi}_h(\varphi_h)\|_{0,\hat{\Omega}}^2 &= (\tilde{\chi}(\varphi), \tilde{\lambda}(\varphi))_{0,\hat{\Omega}} \\ &= (\tilde{\chi}(\varphi), \tilde{\chi}(\varphi))_{0,\hat{\Omega}} = \|\tilde{\chi}(\varphi)\|_{0,\hat{\Omega}}^2.\end{aligned}$$

Now we are ready to prove the main result of this section.

Theorem 4.2. *Let $\Omega_h{}^* = \Omega(\varphi_h^*)$ be a solution of (P_h) and $\lambda_h(\varphi_h{}^*)$, $\chi_h(\varphi_h{}^*)$ be solutions of $(P_h^*(\varphi_h{}^*))$ and $(A_h^*(\varphi_h{}^*))$, respectively. Then there exist subsequences of $\{\Omega_h^*\}$, $\{\lambda_h(\varphi_h{}^*)\}$, $\{\chi_h(\varphi_h{}^*)\}$ and elements $\Omega(\varphi^*) \in \mathcal{O}$, $\lambda(\varphi^*) \in K_{00}(\varphi^*)$, $\chi(\varphi^*) \in K_{00}^{\lambda(\varphi^*)}(\varphi^*)$ such that*

$$(4.9) \qquad \begin{aligned} &\Omega_h^* \to \Omega(\varphi^*) \\ &\left.\begin{aligned}\tilde{\lambda}_h(\varphi_h^*) &\rightharpoonup \tilde{\lambda}(\varphi^*) \\ \tilde{\chi}_h(\varphi_h{}^*) &\rightharpoonup \tilde{\chi}(\varphi^*)\end{aligned}\right\} \quad \text{in } (L^2(\hat{\Omega}))^2.\end{aligned}$$

Moreover, $\Omega(\varphi^*)$ is a solution of (P) and $\lambda(\varphi^*)$ and $\chi(\varphi^*)$ are solutions of $(P^*(\varphi^*))$ and $(A^*(\varphi^*))$ respectively.

Proof: We may already assume that

$$\Omega_h{}^* \to \Omega(\varphi^*) \in \mathcal{O}.$$

According to Lemma 4.3, functions $\lambda(\varphi^*)$ and $\chi(\varphi^*)$ from (4.9) are solutions of $(P(\varphi^*))$ and $(A(\varphi^*))$. From the definition of (P_h) and F_h it follows

$$(4.10) \qquad \frac{1}{2}\|\chi_h(\varphi_h{}^*)\|_{0,\Omega_h{}^*}^2 \le \frac{1}{2}\|\chi_h(\varphi_h)\|_{0,\Omega_h}^2 \quad \forall \Omega_h \in \mathcal{O}_h.$$

Let $\Omega(\varphi) \in \mathcal{O}$ be given and $\lambda(\varphi)$, $\chi(\varphi)$ be corresponding solutions of $(P^*(\varphi))$ and $(A^*(\varphi))$, respectively. Then there exists a sequence $\{\Omega_h(\varphi_h)\}$, $\Omega_h(\varphi_h) \in \mathcal{O}_h$ satisfying all assumptions of Lemma 4.4, especially

$$\begin{aligned}\tilde{\lambda}_h(\varphi_h) &\to \tilde{\lambda}(\varphi) \quad \text{in } (L^2(\hat{\Omega}))^2, \\ \tilde{\chi}_h(\varphi_h) &\to \tilde{\chi}(\varphi) \quad \text{in } (L^2(\hat{\Omega}))^2.\end{aligned}$$

Inserting these Ω_h and $\chi_h(\varphi_h)$ into the right hand side of (4.10) and passing to the limit in (4.10) with $h \to 0+$ we arrive at

$$\begin{aligned}\frac{1}{2}\|\chi(\varphi^*)\|_{0,\Omega(\varphi^*)}^2 &\le \liminf_{h\to 0+} \frac{1}{2}\|\chi_h(\varphi_h{}^*)\|_{0,\Omega_h{}^*}^2 \\ &\le \lim_{h\to 0+} \frac{1}{2}\|\chi_h(\varphi_h)\|_{0,\Omega_h}^2 = \frac{1}{2}\|\chi(\varphi)\|_{0,\Omega(\varphi)}^2.\end{aligned}$$

As $\Omega(\varphi) \in \mathcal{O}$ is arbitrary, $\Omega(\varphi^*)$ is a solution of (P).

5. Numerical realization

For the numerical solution of the dual state problem one must construct a divergence-free finite element space. In two dimensions this can be done by using the so-called stream function. For any $\varphi_h \in P_1^h$ the space $K_{00}(\varphi_h)$ can be identified with

$$\operatorname{curl} W(\varphi_h) = \{\mu \in (L^2(\Omega(\varphi_h)))^2 \mid \exists w \in W(\varphi_h) \text{ such that } \mu = \operatorname{curl} w\}.$$

Here $\operatorname{curl} w = (\partial w/\partial y, -\partial w/\partial x)$ and

$$W(\varphi_h) = \{w \in H^1(\Omega(\varphi_h)) \mid w = 0 \text{ on } \Gamma_0\}.$$

Let $W_h(\varphi_h) \subset W(\varphi_h)$ be a finite element space of functions over $\mathcal{T}(h, \varphi_h)$. Define the space of solenoidal finite elements

$$K_{00}^h(\varphi_h) = \operatorname{curl} W_h(\varphi_h), \tag{5.1}$$

which is an internal approximation of $K_{00}(\varphi_h)$.

From the bijectivity of the operator curl and the definition of (5.1) it follows that $\dim K_{00}^h(\varphi_h) = \dim W_h(\varphi_h)$. Moreover if $\{\psi_i\}_{i=1}^n$ is the basis of $W_h(\varphi_h)$ then

$$\{\Psi_i\}_{i=1}^n, \ \ \Psi_i = \operatorname{curl} \psi_i \tag{5.2}$$

is the basis of $K_{00}^h(\varphi_h)$.

On $\Omega_h(\varphi_h) \in \mathcal{O}_h$ we define the approximation of the dual state problem as follows:

$$(P_h(\varphi_h)) \qquad \begin{cases} \lambda_h(\varphi_h) \in K_{00}^h(\varphi_h) : \\ (\lambda_h(\varphi_h), \mu)_{(L(\Omega(\varphi_h)))} = (\nabla\Phi, \mu)_{(L(\Omega(\varphi_h)))} \quad \forall \mu \in K_{00}^h(\varphi_h). \end{cases}$$

The approximation of the problem $(\mathrm{P}_{h\varepsilon})$ is defined as follows:

$$(\tilde{\mathrm{P}}_{h\varepsilon}) \qquad \begin{cases} \text{Find } \Omega_{h\varepsilon}^* \in \mathcal{O}_h \text{ such that} \\ \mathcal{N}_{\varepsilon h}(\Omega_{h\varepsilon}^*) \le \mathcal{N}_{\varepsilon h}(\Omega_h) \quad \forall \Omega_h \in \mathcal{O}_h, \end{cases}$$

where

$$\mathcal{N}_{\varepsilon h}(\Omega_h) = \frac{1}{2}\|\lambda_h(\varphi_h)\cdot\nu\|_{0,\Gamma(\varphi_h)}^2 + \frac{1}{2\varepsilon}\|[\lambda_h(\varphi_h)\cdot\nu]^+\|_{0,\Gamma_\sigma}^2.$$

Taking into account (5.2), we get a linear system of algebraic equations

$$\sum_{i=1}^n c_i(\Psi_i, \Psi_j)_{(L(\Omega(\varphi_h)))} = (\nabla\Phi, \Psi_j)_{(L(\Omega(\varphi_h)))}$$

which can be written in a compact matrix form as

$$\mathbf{K}(\mathbf{a})\,\mathbf{c} = \mathbf{f}(\mathbf{a}). \tag{5.3}$$

The symbol $\mathbf{a}$ denotes the vector of the discrete design variables. Components of this vector are equal to $\varphi_h(d_i),\ i = 1, ..., n(h),\ \varphi_h \in P_1^h$.

As $\lambda_h(\varphi_h) = \sum_{i=1}^n c_i\Psi_i = \mathbf{B}(\mathbf{a})\,\mathbf{c}$, the cost functional can be written in a matrix form

$$\tilde{\mathcal{N}}(\mathbf{a}, \mathbf{c}) = \frac{1}{2}\sum_{\Gamma^e\subset\Gamma(\varphi_h)}\int_{\Gamma^e}(\mathbf{B}(\mathbf{a})\,\mathbf{c}\cdot\nu)^2\,ds + \frac{1}{2\varepsilon}\sum_{\Gamma^e\subset\Gamma_\sigma}\int_{\Gamma^e}\left([\mathbf{B}(\mathbf{a})\,\mathbf{c}\cdot\nu]^+\right)^2\,ds.$$

To be able to use efficient optimization algorithms in solving $(\tilde{\mathrm{P}}_{h\varepsilon})$ one must calculate the gradient of the cost functional $\tilde{\mathcal{N}}$. The following result is standard:

Theorem 5.1. *Partial derivatives $\partial\tilde{\mathcal{N}}/\partial a_j$ are given by*

$$\frac{\partial\tilde{\mathcal{N}}(\mathbf{a},\mathbf{c}(\mathbf{a}))}{\partial a_j} = \frac{\partial\tilde{\mathcal{N}}(\mathbf{a},\mathbf{c})}{\partial a_j} + \mathbf{p}(\mathbf{a})^{\mathrm{T}}\left(\frac{\partial\mathbf{f}(\mathbf{a})}{\partial a_j} - \frac{\partial\mathbf{K}(\mathbf{a})}{\partial a_j}\mathbf{c}(\mathbf{a})\right), \tag{5.4}$$

where $\mathbf{c}(\mathbf{a})$ is the solution of (5.3) and $\mathbf{p}$ is the solution of the adjoint equation

$$\mathbf{K}(\mathbf{a})\,\mathbf{p}(\mathbf{a}) = \nabla_{\mathbf{c}}\tilde{\mathcal{N}}(\mathbf{a},\mathbf{c}(\mathbf{a})).$$

For the construction of $K^h_{00}(\varphi_h)$, $\varphi_h \in P^1_h$ we use the space $W_h(\varphi_h)$ made from piecewise linear triangular and piecewise bilinear quadrilateral finite elements. As λ_h is constant on element boundaries the cost functional can be written as follows

$$\tilde{\mathcal{N}}(\mathbf{a},\mathbf{c}) = \sum_{\Gamma^e\subset\Gamma(\varphi_h)\cup\Gamma_\sigma} \operatorname{meas}(\Gamma^e)\rho_e(\lambda^e\cdot\nu^e),$$

where $\lambda^e \equiv \lambda^e_h = \mathbf{B}^e\mathbf{c}$ and

$$\rho_e(s) = \begin{cases} \frac{1}{2}s^2, & \Gamma^e \subset \Gamma(\varphi_h) \\ \frac{1}{2}(s^+)^2, & \Gamma^e \subset \Gamma_\sigma(\varphi_h). \end{cases}$$

Then we have

$$\nabla_{\mathbf{c}}\tilde{\mathcal{N}}(\mathbf{a},\mathbf{c}) = \sum_{\Gamma^e} \operatorname{meas}(\Gamma^e)\rho'_e(\lambda^e\cdot\nu^e)(\mathbf{B}^e)^{\mathrm{T}}\nu^e$$

and

$$\frac{\partial\tilde{\mathcal{N}}(\mathbf{a},\mathbf{c})}{\partial a_j} = \sum_{\Gamma^e}\Big\{ \operatorname{meas}(\Gamma^e)\rho'_e(\lambda^e\cdot\nu^e)\frac{\partial\mathbf{B}^e}{\partial a_j}\mathbf{c} + \frac{\partial\operatorname{meas}(\Gamma^e)}{\partial a_j}\rho_e(\lambda^e\cdot\nu^e) + \operatorname{meas}(\Gamma^e)\rho'_e(\lambda^e\cdot\nu^e)\lambda^e\cdot\frac{\partial\nu^e}{\partial a_j}\Big\}.$$

The terms $\partial\mathbf{f}/\partial a_j$, $\partial\mathbf{K}/\partial a_j$ and $\partial\mathbf{B}^e/\partial a_j$ can be calculated using techniques described in [**8**].

6. Numerical examples

In this Section we report our experiences in solving the dam problem numerically. In optimization we have used a Sequential Quadratic Programming (SQP) algorithm E04VCF from the NAG-library [**9**]. In SQP-methods a new approximation for the optimization parameter $\mathbf{a}$ is found from the equation

$$\mathbf{a}^{(k+1)} = \mathbf{a}^{(k)} + \eta_k\mathbf{b}^{(k)},$$

where a direction of descent $\mathbf{b}^{(k)}$ is found as a solution of a quadratic programming subproblem and η_k is the step length obtained via line search.

The state problem and adjoint problem were solved using band-Cholesky method. All computations were done in double precision using HP 9000/835-computer.

The numerical value of the cost has no physical meaning. Instead we use the ratio R between the final and the initial cost. In the numerical computations one may expect that $R \neq 0$. Thus it may be difficult to say if the result obtained is really a global optimum or just a local one. Therefore one is recommended to use graphical post-processing to plot flow vectors, streamlines, etc. to ensure the correct solution.

Example 6.1. In order to make a comparison with published numerical results we first solve the problem with a rectangular homogeneous dam $\hat{\Omega} = (0, L) \times (0, y_1)$. As in [**6**] we choose $L = 1.62$, $y_1 = 3.22$ and $y_2 = 0.84$. In [**6**] the position of the free boundary on the right side of the dam was $\varphi(L) = 2.089$ when 400 elements were used and $\varphi(L) = 2.070$ with 800 elements. For the coarser mesh $R = 0.975 \times 10^{-5}$ was reported.

We solved the problem with 72 elements using the dual approach. After 21 SQP-iterations and 36 CPU-seconds a solution with $R = 1.65 \times 10^{-6}$ was obtained. The position of the free boundary on the right side of the dam was $\varphi(L) = 2.072$. The final mesh and streamlines are shown in Figures 2–3.

In [**6**] the free boundary was constrained to be concave. Our computations were done without any additional constraints. However, in the case of primal formulation of the state problem we could not get a physical solution without assuming the free boundary to be decreasing. This is a very common situation when a simple piecewise linear parametrization for the free boundary is used. More complicated spline approximation would yield much better results. In the case of dual variational formulation the approximate state solution is more "physical" and seems to force the boundary to be smoother without additional restrictions.

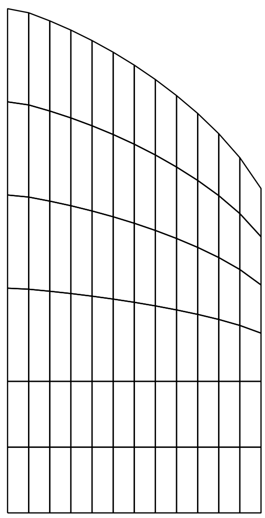

Figure 2: Finite element mesh

Example 6.2. We consider a nonhomogeneous dam $\hat{\Omega} = (0, 1/2) \times (0, 1)$ with $y_1 = 1$, $y_2 = 0$ and

$$k(x, y) = \begin{cases} 1/10, & y > 1/2 \\ 10, & y \leq 1/2. \end{cases}$$

This problem was also discussed in [**2**]. We solved this problem using the dual approach with 176 finite elements. As the initial guess a completely wet dam was used. After 60 SQP-iterations and 365 CPU-seconds a solution with $R = 7.19 \times 10^{-8}$ was obtained. The streamlines are shown in Figure 4. Note that in this example the free boundary is not given as a graph of a function $y = \varphi(x)$.

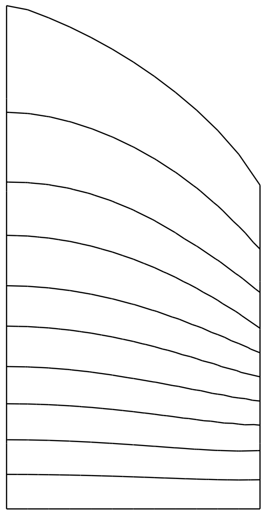

Figure 3: Streamlines

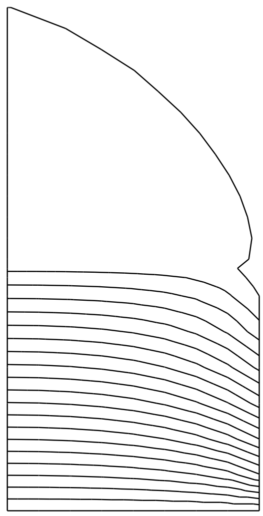

Figure 4: Streamlines

Example 6.3. We assume the case of a trapetzoidal dam, the geometry of which is given in Figure 5, with $y_1 = 1$ and $y_2 = 0.1$. We shall solve the problem by using the primal and the dual formulations of the state problem in order to show the superiority of the dual approach. We demonstrate also an important role of the state constraint $\partial u/\partial \nu \leq 0$ on $\Gamma_\sigma(\varphi)$. Contrary to previous examples, this condition is not satisfied à priori. The finite element mesh consists of 90 elements. The number of optimization variables is 13. In all cases the initial guess of the free boundary corresponds to a completely wet dam.

a) We solve the problem by using the primal formulation of the state problem. The constraint $\partial u/\partial \nu \leq 0$ on Γ_σ was applied only on the element lying on the junction of $\Gamma(\varphi)$ and Γ_σ. After 7 SQP-iterations and 25 CPU-seconds a solution with $R = 2.565 \times 10^{-4}$ was obtained. The equipotential lines of the flow are shown in Figure 6. At the end of the free boundary some oscillation is visible.

b) We solve the problem by using the dual formulation of the state problem. We do not include the condition $\partial u/\partial\nu \leq 0$ on Γ_σ. As the result we got $R = 8.124 \times 10^{-3}$ but the solution shown in Figure 7 is clearly unphysical.

c) We solve the problem by using the dual formulation and the penalization of the state constraint with $\varepsilon = 10^{-2}$. After 24 SQP-iterations and 82 CPU-seconds a solution with $R = 1.193 \times 10^{-6}$ was obtained. The streamlines of the flow are shown in Figure 8.

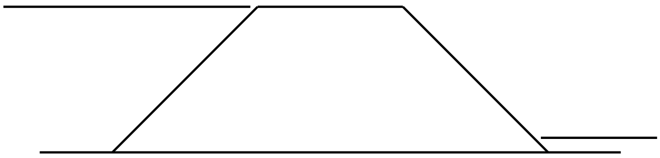

Figure 5: Geometry of the trapetzoidal dam

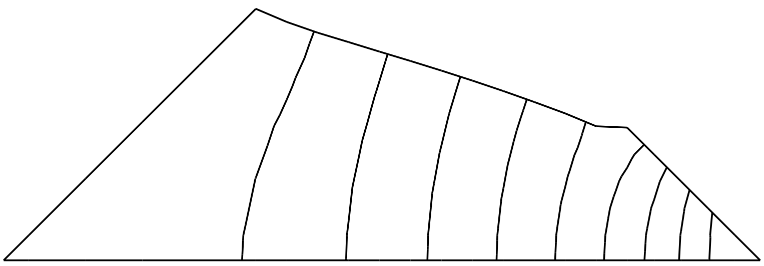

Figure 6: Primal formulation of the state problem

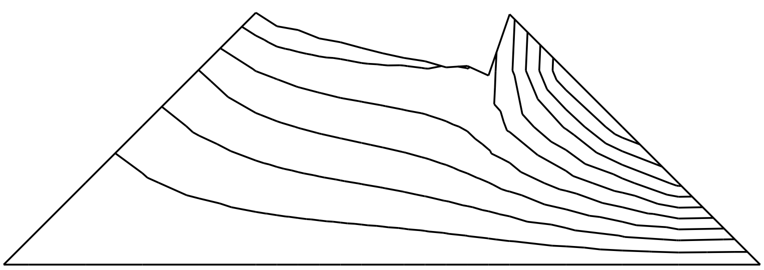

Figure 7: Unphysical solution

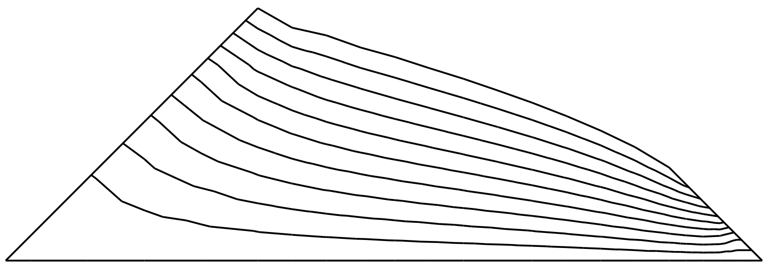

Figure 8: Dual formulation of the state problem

====7. Conclusions

The numerical calculations indicate that the dual variational formulation is suitable for the numerical solution of the dam problem. The free boundary obtained is quite smooth even for rather coarse meshes. In two dimensions the construction of a divergence free finite element space is easily done with the aid of a stream function. In non-rectangular geometries it is necessary to include the state constraint to ensure a physical solution.

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