

10 On the implementation of FEM

Assume that $d = 2$ and that we will use linear triangular elements. Consider element E shown in Figure 6. Let $x^{(i)}, x^{(j)}, x^{(k)}$ be the coordinates of its vertices. Inside E , the expressions of the basis functions related to nodes i, j, k read

$$\begin{cases} \varphi_i(x) = \frac{1}{2A}(a_i + b_i x_1 + c_i x_2) \\ \varphi_j(x) = \frac{1}{2A}(a_j + b_j x_1 + c_j x_2) \\ \varphi_k(x) = \frac{1}{2A}(a_k + b_k x_1 + c_k x_2), \end{cases} \quad (84)$$

where

$$\begin{cases} a_i = x_1^{(j)} x_2^{(k)} - x_1^{(k)} x_2^{(j)} \\ b_i = x_2^{(j)} - x_2^{(k)} \\ c_i = x_1^{(k)} - x_1^{(j)} \end{cases}$$

and A is the area of the element E given by

$$A = \frac{1}{2} \begin{vmatrix} 1 & x_1^{(i)} & x_2^{(i)} \\ 1 & x_1^{(j)} & x_2^{(j)} \\ 1 & x_1^{(k)} & x_2^{(k)} \end{vmatrix}.$$

The other coefficients in (84) are evaluated in an analogous way.

The gradients of basis functions inside E are easily obtained

$$\nabla \varphi_i = \frac{1}{2A} \begin{bmatrix} b_i \\ c_i \end{bmatrix}, \quad \nabla \varphi_j = \frac{1}{2A} \begin{bmatrix} b_j \\ c_j \end{bmatrix}, \quad \nabla \varphi_k = \frac{1}{2A} \begin{bmatrix} b_k \\ c_k \end{bmatrix}.$$

Consider now the following Poisson problem with mixed homogeneous boundary conditions

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_0 \\ \frac{\partial u}{\partial n} = 0 & \text{on } \Gamma_1, \quad \partial\Omega = \Gamma_0 \cup \Gamma_1. \end{cases} \quad (85)$$

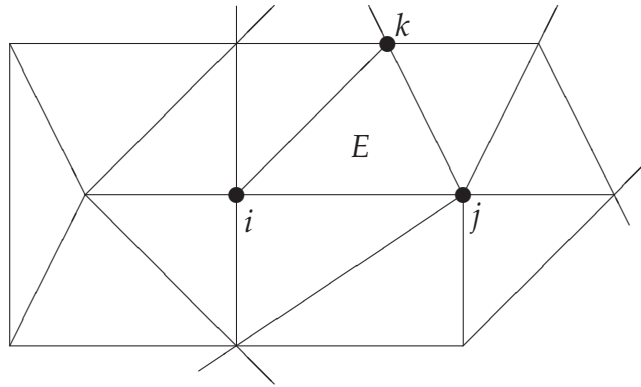


Figure 6:

The weak formulation of (85) reads

$$u \in V : \int_{\Omega} \nabla u \cdot \nabla v \, dx = \int_{\Omega} f v \, dx \quad \forall v \in V, \quad (86)$$

where

$$V = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_0\}.$$

The finite element approximation of (86) leads to the linear system of equations

$$Aq = f, \quad (87)$$

where

$$\begin{aligned} A &= \{a_{rs}\}_{r,s=1}^N, \quad f = \{f_s\}_{s=1}^N \\ a_{rs} &= \int_{\Omega} \nabla \varphi_r \cdot \nabla \varphi_s \, dx = \sum_E \int_E \nabla \varphi_r \cdot \nabla \varphi_s \, dx =: \sum_E a_{rs}^{(E)} \\ f_s &= \int_{\Omega} f \varphi_s \, dx = \sum_E \int_E f \varphi_s \, dx =: \sum_E f_s^{(E)}. \end{aligned}$$

Thus the coefficients of the linear system (87) can be computed *element by element*. For example

$$\begin{aligned} a_{jk}^{(E)} &= \int_E \nabla \varphi_j \cdot \nabla \varphi_k \, dx = \int_E \frac{1}{4A^2} (b_j b_k + c_j c_k) \, dx \\ &= \frac{1}{4A^2} (b_j b_k + c_j c_k) \int_E dx = \frac{1}{4A} (b_j b_k + c_j c_k) \\ f_k^{(E)} &= \int_E f(x) \varphi_k(x) \, dx. \end{aligned}$$

If $f(x)$ is piecewise constant, then the integral on right hand side can be evaluated exactly using midpoint rule

$$\int_E f(x) \varphi_k(x) \, dx = A f(G) \varphi_k(G) = \frac{A}{3} f(G),$$

where G is the center of gravity of triangle E . In general case, *numerical integration (quadrature)* must be used.

The coefficients related to element E constitute matrix $A^{(E)} \in \mathbb{R}^{3 \times 3}$ and vector $f^{(E)} \in \mathbb{R}^3$ which are called *local stiffness matrix* and *force vector*.

The *assembly* of the system (87) can now be expressed as a simple algorithm:

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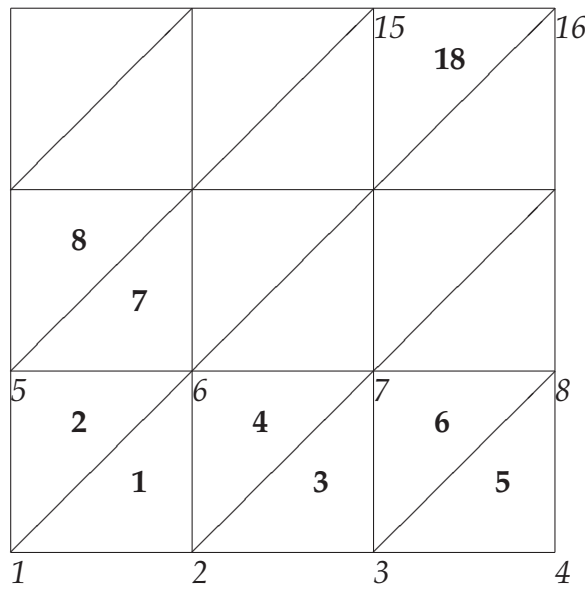
f := 0 ∈ ℝN
A := 0 ∈ ℝN×N
for each  $E \in \mathcal{T}_h$  do
    Compute local contributions  $A^{(E)}$  and  $f^{(E)}$ 
    Let  $\text{ind}(1:3) = \{\text{global node numbers of element } E\}$ 
    for  $i = 1:3$  do
        for  $j = 1:3$  do
             $A(\text{ind}(i), \text{ind}(j)) := A(\text{ind}(i), \text{ind}(j)) + A^{(E)}(i, j)$ 
        end for
    end for
end for

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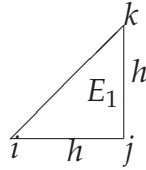
Until now we did not mention the boundary conditions. The homogenous Neumann boundary condition is a natural boundary condition and contributes nothing to the linear system. The Dirichlet condition $u = 0$ on Γ_0 , however, must be explicitly forced. The most common way to handle Dirichlet boundary conditions is elimination during assembly: If node s is on Γ_0 set $\text{ind}(s) = 0$. The unknown q_s is eliminated from the system and the corresponding row and column are not assembled in the global system.

Remark 10.1. If the unknowns (*degrees of freedom*) in the linear system of equation are nodal values of the approximate solution, then the respective element is called *Lagrangian finite element*.

Example 10.1. Let us study the assembling of the stiffness matrix of a simple model problem. Let the unit square be divided into regular linear triangular elements as shown below:



All triangles in the mesh are either of type E_1 or E_2 . Let us consider a triangle of type E_1 (length of base = height = h):



Inside E_1 only basis functions $\varphi_i, \varphi_j, \varphi_k$ are nonvanishing and their gradients are

$$\nabla \varphi_i = \left(-\frac{1}{h}, 0\right), \quad \nabla \varphi_j = \left(\frac{1}{h}, -\frac{1}{h}\right), \quad \nabla \varphi_k = \left(0, \frac{1}{h}\right)$$

The coefficients of the local stiffness matrix are thus

$$\begin{aligned}\int_{E_1} \nabla \varphi_i \cdot \nabla \varphi_i dx &= \int_{E_1} \frac{1}{h^2} dx = \frac{1}{2} \\ \int_{E_1} \nabla \varphi_i \cdot \nabla \varphi_j dx &= \int_{E_1} -\frac{1}{h^2} dx = -\frac{1}{2} \\ \int_{E_1} \nabla \varphi_i \cdot \nabla \varphi_k dx &= \int_{E_1} 0 dx = 0 \\ \int_{E_1} \nabla \varphi_j \cdot \nabla \varphi_j dx &= \int_{E_1} \frac{2}{h^2} dx = 1 \\ \int_{E_1} \nabla \varphi_j \cdot \nabla \varphi_k dx &= \int_{E_1} -\frac{1}{h^2} dx = -\frac{1}{2} \\ \int_{E_1} \nabla \varphi_k \cdot \nabla \varphi_k dx &= \int_{E_1} \frac{1}{h^2} dx = \frac{1}{2}\end{aligned}$$

The local stiffness matrix related to E_1 in “compressed” form reads

$$\mathbf{A}^{(1)} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Similarly the local matrix corresponding triangles of type E_2 is

$$A^{(2)} = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}.$$

For the element E_1 the array containing node numbers is $\text{ind}=(/ 1, 2, 6 /)$, for E_2 $\text{ind}=(/ 1, 6, 5 /)$, and so on.

We go through the elements E_1, E_2, E_3, \dots one by one and form and assemble the local matrices into the global one A :

[illegible]

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0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0

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***** After element # 3 A=

```

1.0 -0.5 0 0 -0.5 0 0 0 0 0 0 0 0 0 0 0
-0.5 1.5 -0.5 0 0 -0.5 0 0 0 0 0 0 0 0 0 0
0 -0.5 1.0 0 0 0 -0.5 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
-0.5 0 0 0 1.0 -0.5 0 0 0 0 0 0 0 0 0 0
0 -0.5 0 0 -0.5 1.0 0 0 0 0 0 0 0 0 0 0
0 0 -0.5 0 0 0 0.5 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0

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***** After element # 7 A=

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1.0 -0.5 0 0 -0.5 0 0 0 0 0 0 0 0 0 0 0
-0.5 2.0 -0.5 0 0 -1.0 0 0 0 0 0 0 0 0 0 0
0 -0.5 2.0 -0.5 0 0 -1.0 0 0 0 0 0 0 0 0 0
0 0 -0.5 1.0 0 0 0 -0.5 0 0 0 0 0 0 0 0
-0.5 0 0 0 1.5 -1.0 0 0 0 0 0 0 0 0 0 0
0 -1.0 0 0 -1.0 3.0 -0.5 0 0 -0.5 0 0 0 0 0 0
0 0 -1.0 0 0 -0.5 2.0 -0.5 0 0 0 0 0 0 0 0
0 0 0 -0.5 0 0 -0.5 1.0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 -0.5 0 0 0 0.5 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0

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***** After element # 18 A=

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1.0 -0.5 0 0 -0.5 0 0 0 0 0 0 0 0 0 0 0
-0.5 2.0 -0.5 0 0 -1.0 0 0 0 0 0 0 0 0 0 0
0 -0.5 2.0 -0.5 0 0 -1.0 0 0 0 0 0 0 0 0 0
0 0 -0.5 1.0 0 0 0 -0.5 0 0 0 0 0 0 0 0
-0.5 0 0 0 2.0 -1.0 0 0 -0.5 0 0 0 0 0 0 0
0 -1.0 0 0 -1.0 4.0 -1.0 0 0 -1.0 0 0 0 0 0 0
0 0 -1.0 0 0 -1.0 4.0 -1.0 0 0 -1.0 0 0 0 0 0
0 0 0 -0.5 0 0 -1.0 2.0 0 0 -0.5 0 0 0 0 0
0 0 0 0 -0.5 0 0 2.0 -1.0 0 0 -0.5 0 0 0 0
0 0 0 0 0 -1.0 0 0 -1.0 4.0 -1.0 0 0 -1.0 0 0
0 0 0 0 0 0 -1.0 0 0 -1.0 4.0 -1.0 0 0 -1.0 0
0 0 0 0 0 0 0 -0.5 0 0 -1.0 2.0 0 0 0 -0.5
0 0 0 0 0 0 0 0 -0.5 0 0 1.0 -0.5 0 0 0
0 0 0 0 0 0 0 0 -1.0 0 0 -0.5 2.0 -0.5 0
0 0 0 0 0 0 0 0 0 -1.0 0 0 -0.5 2.0 -0.5
0 0 0 0 0 0 0 0 0 0 -0.5 0 0 -0.5 1.0

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As we see, the matrix A is sparse and banded (due to the nodal numbering similar to natural numbering in the finite difference method). Actually, if we pose Dirichlet boundary condition $u = 0$, then the degrees of freedoms corresponding to boundary nodes are eliminated. The resulting matrix would then be exactly the same as in the 5-point finite difference method.

11 Solving PDE systems with FEM

11.1 Introduction

Consider the simple system of two Helmholtz-type PDE's:

$$\begin{cases} -\Delta u_1 - cu_2 = f_1 & \text{in } \Omega \\ -\Delta u_2 + cu_1 = f_2 & \text{in } \Omega \\ u_1 = u_2 = 0 & \text{on } \partial\Omega. \end{cases} \quad (88)$$

Let $v_1, v_2 \in H_0^1(\Omega)$. Multiply the first equation by v_1 and the second by v_2 and use Green's formula. Then we get

$$\begin{cases} \int_{\Omega} \nabla u_1 \cdot \nabla v_1 dx - \int_{\Omega} cu_2 v_1 dx = \int_{\Omega} f_1 v_1 dx & \text{in } \Omega \\ \int_{\Omega} \nabla u_2 \cdot \nabla v_2 dx + \int_{\Omega} cu_1 v_2 dx = \int_{\Omega} f_2 v_2 dx & \text{in } \Omega \\ u_1 = u_2 = 0 & \text{on } \partial\Omega. \end{cases} \quad (89)$$

System (89) can be represented in vector form

$$\text{Find } \mathbf{u} \in \mathbb{V} : \quad \int_{\Omega} \sum_{i=1}^2 \nabla u_i \cdot \nabla v_i dx + \int_{\Omega} (-cu_2, cu_1) \cdot \mathbf{v} dx = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} dx \quad \forall \mathbf{v} \in \mathbb{V}, \quad (90)$$

where $\mathbb{V} = H_0^1(\Omega) \times H_0^1(\Omega) = \{\mathbf{v} = (v_1, v_2) \mid v_i \in H_0^1(\Omega), i = 1, 2\}$. The corresponding bilinear and linear forms read

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &:= \int_{\Omega} \sum_{i=1}^2 \nabla u_i \cdot \nabla v_i dx + \int_{\Omega} (-cu_2, cu_1) \cdot \mathbf{v} dx \\ F(\mathbf{v}) &:= \int_{\Omega} \mathbf{f} \cdot \mathbf{v} dx. \end{aligned}$$

With these notations, problem (90) reads in abstract form:

$$\text{Find } \mathbf{u} \in \mathbb{V} : \quad a(\mathbf{u}, \mathbf{v}) = F(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbb{V}.$$

The finite element discretization is performed analogously to the scalar case. The unknowns are approximated in the form $u_1(x) \approx \sum_{i=1}^N q_{1i} \varphi_i(x)$, $u_2(x) \approx \sum_{j=1}^N q_{2j} \varphi_j(x)$, where $\{\varphi_i\}$ are the piecewise polynomial and continuous basis functions, and N is the number of nodes not belonging to the boundary.

Numbering the unknowns as follows $q_{11}, q_{12}, \dots, q_{1N}, q_{21}, q_{22}, \dots, q_{2N}$ the resulting linear system has the following block structure

$$\begin{bmatrix} \mathbf{K} & -\mathbf{M} \\ \mathbf{M} & \mathbf{K} \end{bmatrix} \begin{bmatrix} \mathbf{q}^{(1)} \\ \mathbf{q}^{(2)} \end{bmatrix} = \begin{bmatrix} \mathbf{f}^{(1)} \\ \mathbf{f}^{(2)} \end{bmatrix}.$$

11.2 Plane strain problem of linear elasticity

Consider a two-dimensional homogeneous and isotropic elastic solid Ω under internal (body) forces \mathbf{f} and external forces (surface tractions) \mathbf{g} on part Γ_1 of $\partial\Omega$. Let $\mathbf{u} = (u_1, u_2)$ be

the displacement of the solid under the applied forces. We define so-called strain tensor $\mathbf{e}(\mathbf{u}) = \{e_{ij}(\mathbf{u})\}_{i,j=1}^2$ and stress tensor $\boldsymbol{\sigma}(\mathbf{u}) = \{\sigma_{ij}(\mathbf{u})\}_{i,j=1}^2$ as follows¹

$$e_{ij}(\mathbf{u}) = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad i, j = 1, 2.$$

$$\sigma_{ij}(\mathbf{u}) = \lambda \delta_{ij} \nabla \cdot \mathbf{u} + 2\mu e_{ij}(\mathbf{u}), \quad i, j = 1, 2.$$

Here $\lambda, \mu > 0$ are material parameters (Lame's constants).

The equations of plane strain then read

$$\begin{cases} -\frac{\partial \sigma_{11}(\mathbf{u})}{\partial x_1} - \frac{\partial \sigma_{12}(\mathbf{u})}{\partial x_2} = f_1 & \text{in } \Omega \\ -\frac{\partial \sigma_{21}(\mathbf{u})}{\partial x_1} - \frac{\partial \sigma_{22}(\mathbf{u})}{\partial x_2} = f_2 & \text{in } \Omega \\ u_1 = u_2 = 0 & \text{on } \Gamma_0 \\ \sigma_{11}(\mathbf{u})n_1 + \sigma_{12}n_2 = g_1 & \text{on } \Gamma_1 \\ \sigma_{21}(\mathbf{u})n_1 + \sigma_{22}n_2 = g_2 & \text{on } \Gamma_1. \end{cases} \quad (91)$$

The weak form of (91) is: Find $\mathbf{u} \in \mathbb{V}$ such that

$$\int_{\Omega} \left(\sigma_{11}(\mathbf{u}) \frac{\partial v_1}{\partial x_1} + \sigma_{12}(\mathbf{u}) \frac{\partial v_1}{\partial x_2} \right) dx = \int_{\Omega} f_1 v_1 dx + \int_{\Gamma_1} g_1 v_1 ds$$

$$\int_{\Omega} \left(\sigma_{21}(\mathbf{u}) \frac{\partial v_2}{\partial x_1} + \sigma_{22}(\mathbf{u}) \frac{\partial v_2}{\partial x_2} \right) dx = \int_{\Omega} f_2 v_2 dx + \int_{\Gamma_1} g_2 v_2 ds$$

for all $\mathbf{v} \in \mathbb{V} = \{\mathbf{v} \in H^1(\Omega) \times H^1(\Omega) \mid \mathbf{v} = 0 \text{ on } \Gamma_0\}$.

Remark 11.1. Sometimes engineers prefer to express $\boldsymbol{\sigma}$ and \mathbf{e} as vectors

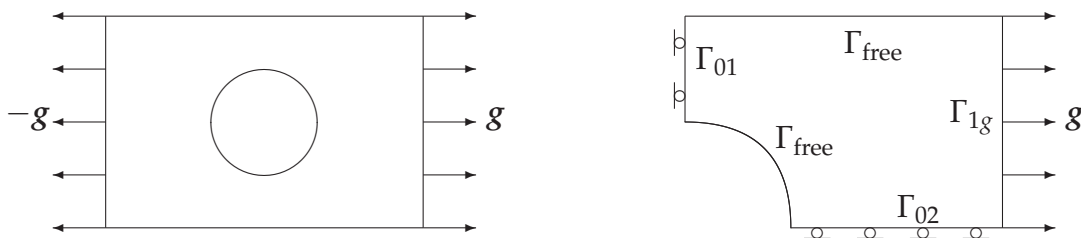
$$\boldsymbol{\sigma}(\mathbf{u}) = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}, \quad \mathbf{e}(\mathbf{u}) = \begin{bmatrix} e_{11} \\ e_{22} \\ 2e_{12} \end{bmatrix}.$$

Then the linear stress-strain relationship can be written as matrix-vector product $\boldsymbol{\sigma}(\mathbf{u}) = \mathbf{D}\mathbf{e}(\mathbf{u})$ (the matrix \mathbf{D} being defined solely by λ and μ).

Then the weak formulation of (91) can be written shortly as

$$\int_{\Omega} [\mathbf{e}(\mathbf{u})]^T \mathbf{D} \mathbf{e}(\mathbf{v}) dx = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} dx + \int_{\Gamma_1} \mathbf{g} \cdot \mathbf{v} ds.$$

Example 11.1. Consider the following “pure tension” problem for perforated solid. The original problem is a pure Neumann problem but taking into account the obvious symmetry we get a mixed boundary condition problem for the 1/4 geometry.



¹Here “tensors” can simply be identified with 2×2 symmetric matrix functions

The weak formulation for the 1/4 problem reads

$$\mathbf{u} \in \mathbb{V} : \quad \int_{\Omega} [\mathbf{e}(\mathbf{u})]^T \mathbf{D} \mathbf{e}(\mathbf{v}) \, dx = \int_{\Gamma_{1g}} \mathbf{g} \cdot \mathbf{v} \, ds \quad \forall \mathbf{v} \in \mathbb{V},$$

where

$$\mathbb{V} = \{\mathbf{v} \in [H^1(\Omega)]^2 \mid v_1|_{\Gamma_{01}} = 0, \quad v_2|_{\Gamma_{02}} = 0\}.$$

Thus, on Γ_{0i} the displacement in the i -th direction is restricted ("roller contact").

11.3 Stokes problem

The simplest model of a steady-state viscous incompressible fluid is the Stokes problem

$$\left\{ \begin{array}{ll} -\mu \Delta u_1 + \frac{\partial p}{\partial x_1} = f_1 & \text{in } \Omega \\ \vdots & \\ -\mu \Delta u_d + \frac{\partial p}{\partial x_d} = f_d & \text{in } \Omega \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u} = \mathbf{u}_0 & \text{on } \partial\Omega. \end{array} \right. \quad (92)$$

Here \mathbf{u} , p are unknown velocity and pressure distributions in a domain $\Omega \subset \mathbb{R}^d$. We assume, for simplicity, that the velocity field is known on the boundary. Note that there are no boundary conditions for the pressure and therefore it is unique up to a constant.

Let us now derive a weak formulation for the Stokes problem (92). Assume for simplicity that $d = 2$ and $\mathbf{u}_0 = \mathbf{0}$. Let $\mathbb{V} = [H_0^1(\Omega)]^2$ and $Q = L^2(\Omega)/\mathbb{R}$. Multiply the momentum equations by components of $\mathbf{v} \in \mathbb{V}$ and the continuity equation by $q \in Q$, and use Green's formula. Then we obtain the weak formulation

$$\left\{ \begin{array}{l} \int_{\Omega} \nabla u_1 \cdot \nabla v_1 \, dx - \int_{\Omega} p \frac{\partial v_1}{\partial x_1} \, dx = \int_{\Omega} f_1 v_1 \, dx \\ \int_{\Omega} \nabla u_2 \cdot \nabla v_2 \, dx - \int_{\Omega} p \frac{\partial v_2}{\partial x_2} \, dx = \int_{\Omega} f_2 v_2 \, dx \\ - \int_{\Omega} \nabla \cdot \mathbf{u} \, q \, dx = 0. \end{array} \right. \quad (93)$$

Problem (93) can be written in abstract form

$$\left\{ \begin{array}{ll} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = F(\mathbf{v}) & \forall \mathbf{v} \in \mathbb{V} \\ b(\mathbf{u}, q) = 0 & \forall q \in Q. \end{array} \right. \quad (94)$$

The solvability of the abstract problem (94) can be proved provided that the bilinear forms $a(\cdot, \cdot)$, $b(\cdot, \cdot)$ and the spaces \mathbb{V} , Q satisfy the following conditions: $\exists c, \beta > 0$ such that

$$a(\mathbf{v}, \mathbf{v}) \geq c \|\mathbf{v}\|_{\mathbb{V}}^2 \quad \forall \mathbf{v} \in \mathbb{V} \quad (95)$$

$$\sup_{\mathbf{0} \neq \mathbf{v} \in \mathbb{V}} \frac{|b(\mathbf{v}, q)|}{\|\mathbf{v}\|_{\mathbb{V}}} \geq \beta \|q\|_Q \quad \forall q \in Q. \quad (96)$$

The latter condition is called *LBB condition* or *inf-sup condition* as it can be expressed in an equivalent form

$$\inf_{q \in Q} \sup_{\mathbf{v} \in \mathbf{V}} \frac{|b(\mathbf{v}, q)|}{\|\mathbf{v}\|_{\mathbf{V}} \|q\|_Q} \geq \beta > 0.$$

The difficulty in finite element approximations is that the discrete finite element spaces \mathbf{V}_h , Q_h must also satisfy the discrete versions of (95), (96). Unfortunately, some obvious choices e.g. piecewise linear continuous velocity and pressure do not satisfy the discrete inf-sup condition. In order to satisfy it, the element for the pressure must be lower order (or “simpler”) than for the velocity. For example quadratic element for velocity and linear element for pressure is a working combination.

Let $\mathbf{u}_h \in \mathbf{V}_h$, $p_h \in Q_h$,

$$u_{hi}(x) = \sum_{j=1}^N q_{ij} \varphi_j(x), \quad i = 1, 2$$

$$p_h(x) = \sum_{k=1}^M q_{3k} \psi_k(x).$$

Then the approximate problem in algebraic form reads

$$\left\{ \begin{array}{l} \sum_{j=1}^N q_{1j} \int_{\Omega} \mu \nabla \varphi_j \cdot \nabla \varphi_i dx - \sum_{k=1}^M q_{3k} \int_{\Omega} \psi_j \frac{\partial \varphi_i}{\partial x_1} dx = \int_{\Omega} f_1 \varphi_i dx, \quad i = 1, \dots, N \\ \sum_{j=1}^N q_{2j} \int_{\Omega} \mu \nabla \varphi_j \cdot \nabla \varphi_i dx - \sum_{k=1}^M q_{3k} \int_{\Omega} \psi_k \frac{\partial \varphi_i}{\partial x_2} dx = \int_{\Omega} f_2 \varphi_i dx, \quad i = 1, \dots, N \\ - \sum_{j=1}^N q_{1j} \int_{\Omega} \frac{\partial \varphi_j}{\partial x_1} \psi_i dx - \sum_{j=1}^N q_{2j} \int_{\Omega} \frac{\partial \varphi_j}{\partial x_2} \psi_i dx = 0, \quad i = 1, \dots, M. \end{array} \right.$$

Using similar numbering for unknowns as in Subsection 11.1 the coefficient matrix has a block structure

$$\begin{bmatrix} \mathbf{K} & \mathbf{0} & \mathbf{B} \\ \mathbf{0} & \mathbf{K} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{B}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{q}^{(1)} \\ \mathbf{q}^{(2)} \\ \mathbf{q}^{(3)} \end{bmatrix} = \begin{bmatrix} \mathbf{f}^{(1)} \\ \mathbf{f}^{(2)} \\ \mathbf{0} \end{bmatrix}. \quad (97)$$

The numerical solution of (97) is complicated by the fact that the coefficient matrix is not positive definite but indefinite (i.e. it has both negative and positive eigenvalues).