9 Time dependent problems

9.1 Parabolic problem

Consider the following parabolic PDE ("heat equation") on time interval $I :=]0, T]$

$$\begin{cases} \gamma \dfrac{\partial u}{\partial t} - \nabla \cdot (k \nabla u) = f & \text{in } \Omega \times I \\ u = 0 & \text{on } \partial\Omega \times I \\ u(x, 0) = u_0(x) & x \in \bar{\Omega} \text{ (initial condition)} \end{cases} \tag{73}$$

For a fixed t we can define a weak formulation of (73)

$$\begin{cases} \text{Find } u(t) \in V,\ t \in I \text{ such that} \\ \displaystyle\int_\Omega \gamma \frac{\partial u(t)}{\partial t} v \, dx + \int_\Omega k \nabla u(t) \cdot \nabla v \, dx = \int_\Omega f v \, dx \quad \forall v \in V \\ u(0) = u_0 \end{cases} \tag{74}$$

The semidiscrete Galerkin formulation of (74) reads

$$\begin{cases} \text{Find } u_h(t) \in V_h,\ t \in I \text{ such that} \\ \displaystyle\int_\Omega \gamma \frac{\partial u_h(t)}{\partial t} v_h \, dx + \int_\Omega k \nabla u_h(t) \cdot \nabla v_h \, dx = \int_\Omega f v_h \, dx \quad \forall v_h \in V_h \\ u_h(0) = u_0 \end{cases} \tag{75}$$

If $\{\varphi_i\}$ is the basis of V_h then

$$u_h(x, t) = \sum_{i=1}^{N} q_i(t) \varphi_i(x).$$

Problem (75) is then a system of ordinary differential equations

$$\begin{cases} \displaystyle\sum_{j=1}^{N} \frac{dq_i(t)}{dt} \int_\Omega \gamma \varphi_i \varphi_j \, dx + \sum_{j=1}^{N} q_j(t) \int_\Omega k \nabla \varphi_i \cdot \nabla \varphi_j \, dx = \int_\Omega f \varphi_i \, dx \quad i = 1, ..., N \\ q_i(0) = u_0(N_i), \quad i = 1, ...N. \end{cases} \tag{76}$$

Equation (76) can be written in matrix form

$$\begin{cases} \boldsymbol{M} \boldsymbol{q}'(t) + \boldsymbol{A} \boldsymbol{q}(t) = \boldsymbol{f}(t) \\ \boldsymbol{q}(0) = \boldsymbol{q}^0. \end{cases} \tag{77}$$

The time derivatives in (77) can be discretized using standard methods, like

$$\boldsymbol{M} \frac{\boldsymbol{q}^{n+1} - \boldsymbol{q}^n}{\Delta t} + \boldsymbol{A} \boldsymbol{q}^n = \boldsymbol{f}^n \quad \text{(Euler)} \tag{78}$$

$$\boldsymbol{M} \frac{\boldsymbol{q}^{n+1} - \boldsymbol{q}^n}{\Delta t} + \boldsymbol{A} \boldsymbol{q}^{n+1} = \boldsymbol{f}^{n+1} \quad \text{(implicit Euler)} \tag{79}$$

$$\boldsymbol{M} \frac{\boldsymbol{q}^{n+1} - \boldsymbol{q}^n}{\Delta t} + \boldsymbol{A} \frac{\boldsymbol{q}^{n+1} + \boldsymbol{q}^n}{2} = \frac{\boldsymbol{f}^{n+1} + \boldsymbol{f}^n}{2} \quad \text{(Crank–Nicolson)} \tag{80}$$

At each time step one has to solve linear system of equations with coefficient matrix $\boldsymbol{M}$ (Euler), $\boldsymbol{M} + \Delta t \boldsymbol{A}$ (implicit Euler), or $\boldsymbol{M} + \frac{1}{2}\Delta t \boldsymbol{A}$ (Crank–Nicolson). If one uses constant time step and direct method to solve the linear system of equations (e.g. Cholesky method), then triangular factorization needs to be done only once, for example

$$\boldsymbol{M} + \frac{1}{2}\Delta t \boldsymbol{A} = \boldsymbol{L}\boldsymbol{L}^{\mathrm{T}}.$$

9.2 Hyperbolic problem

A hyperbolic problem

$$\begin{cases} \gamma \dfrac{\partial^2 u}{\partial t^2} - \nabla \cdot (k \nabla u) = f & \text{in } \Omega \times I \\ u = 0 & \text{on } \partial\Omega \times I \\ u(x,0) = u_0(x), \quad \dfrac{\partial u(x,0)}{\partial t} & x \in \bar{\Omega} \text{ (initial conditions)} \end{cases} \tag{81}$$

can be semidiscretized in the same way as the parabolic problem. Here we get the following system of second order ordinary differential equations:

$$\begin{cases} \boldsymbol{M}\boldsymbol{q}''(t) + \boldsymbol{A}\boldsymbol{q}(t) = \boldsymbol{f}(t) \\ \boldsymbol{q}(0) = \boldsymbol{q}^0, \quad \boldsymbol{q}'(0) = \tilde{\boldsymbol{q}}^0. \end{cases} \tag{82}$$

The time discretization of (82) can be done using e.g. the following family of formulas:

$$\boldsymbol{M}\frac{\boldsymbol{q}^{n+1} - 2\boldsymbol{q}^n + \boldsymbol{q}^{n-1}}{(\Delta t)^2} + \boldsymbol{A}\boldsymbol{q}^{n,\theta} = \boldsymbol{f}^{n,\theta}, \tag{83}$$

where $\boldsymbol{q}^{n,\theta} := \theta \boldsymbol{q}^{n+1} + (1 - 2\theta)\boldsymbol{q}^n + \theta \boldsymbol{q}^{n-1}$.
If one chooses $\theta = \frac{1}{4}$ in (83), then one gets an implicit method that is stable for all time step lengths and is $\mathcal{O}((\Delta t)^2)$ accurate in time.
The choice $\theta = 0$ yields the conditionally stable "leap-frog" method.
To start the time stepping, one needs the solution on time level $-\Delta t$. This can be obtained from Taylor expansion and assuming that the wave equation is satisfied at $t = 0$ too:

$$\begin{aligned} \boldsymbol{q}^{-1} &= \boldsymbol{q}(-\Delta t) \approx \boldsymbol{q}(0) - \boldsymbol{q}'(0)\Delta t + \frac{1}{2}(\Delta t)^2 \boldsymbol{q}''(0) \\ &= \boldsymbol{q}^0 - \Delta t \tilde{\boldsymbol{q}}^0 + \frac{1}{2}(\Delta t)^2 \boldsymbol{M}^{-1}(\boldsymbol{f}^0 - \boldsymbol{A}\boldsymbol{q}^0). \end{aligned}$$