8 On the finite element method

8.1 Weighted residual methods

Consider the model problem

$$-u''(x) = f(x), \quad 0 < x < 1, \quad u(0) = u(1) = 0. \tag{59}$$

Multiply (59) by a *test function* $v \in V$ and integrating we obtain

$$\int_0^1 (-u'' - f)v \, dx = 0, \quad \forall v \in V. \tag{60}$$

Expressing the approximate solution by $u_h = \sum_{i=1}^N u_i \varphi_i$ and "testing" against N functions $\psi_i \in V$ we obtain

$$\sum_{j=1}^N u_j \int_0^1 -\varphi_j'' \psi_i \, dx = \int_0^1 f \phi_i \, dx, \quad i = 1, ..., N. \tag{61}$$

This is called *weighted residual method*. The realization of the method depends, of course, on the choise of the weight functions ψ_i.

8.2 Weak formulation of second order PDEs

Consider again the model problem (59). Let V contain those functions v that satisfy $v(0) = v(1) = 0$, are continuous, and whose derivative are square integrable, i.e. $\int_0^1 (v')^2 \, dx < \infty$. Multiply (59) by $v \in V$ and integrate by parts:

$$-\int_0^1 u'' v \, dx = -\Big/_0^1 u'v + \int_0^1 u'v' \, dx = \int_0^1 fv \, dx.$$

Taking into account the boundary conditions on v we get

$$\int_0^1 u'v' \, dx = \int_0^1 fv \, dx \quad \forall v \in V \tag{62}$$

as $v \in V$ was arbitrarily chosen.
Divide the interval $[0,1]$ into $n+1$ subintervals $0 = x_0 < x_1 < ... < x_n < x_{n+1} = 1$ of length $h = 1/(n+1)$. Let us define a subspace V_h of V such that V_h contains piecewise linear and continuous functions φ_i defined as

$$\varphi_i(x) = \begin{cases} (x - x_{i-1})/h, & x_{i-1} \le x \le x_i \\ (x_{i+1} - x)/h, & x_i \le x \le x_{i+1} \\ 0 & \text{otherwise.} \end{cases} \tag{63}$$

Substituting $u_h(x) = \sum_{j=1}^n u_j \varphi_j(x)$ and $v = \varphi_i$ into (62) we obtain

$$\sum_{j=1}^n u_j \int_0^1 \varphi_j'(x) \varphi_i'(x) \, dx = \int_0^1 f \varphi_i(x) \, dx, \quad i = 1, ..., n. \tag{64}$$

In matrix form this reads

$$\boldsymbol{Au} = \boldsymbol{f}, \tag{65}$$

where the matrix entries are easily found to be

$$a_{ij} = \begin{cases} \frac{2}{h} & \text{if } i = j \\ -\frac{1}{h} & \text{if } |i-j| = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{66}$$

If we use trapetzoidal rule $\int_a^b f\,dx \approx \frac{(b-a)}{2}(f(a)+f(b))$ to evaluate the integrals on the right hand side we get

$$f_i = hf(x_i).$$

In this way we obtain exactly the same algebraic system as with the (central) finite difference method!

The generalization of the integration by parts formula in higher dimensions is the *Green's formula*:
Let $\Omega \subset \mathbb{R}^d$ be a domain, and let $v, w \in C^1(\bar{\Omega})$, then

$$\int_\Omega w \frac{\partial v}{\partial x_j}\,dx = \int_{\partial\Omega} wvn_j\,ds - \int_\Omega \frac{\partial w}{\partial x_j} v\,dx, \quad 1 \le j \le d. \tag{67}$$

Here $n = (n_1, ..., n_d)$ is the external unit normal vector defined on the boundary $\partial\Omega$ of Ω.
Green's formula implies e.g. the following formulas

$$\int_\Omega (\Delta u) v\,dx = \int_{\partial\Omega} \frac{\partial u}{\partial n} v\,ds - \int_\Omega \nabla u \cdot \nabla v\,dx$$
$$\int_\Omega (\nabla \cdot \boldsymbol{w})\,dx = \int_{\partial\Omega} \boldsymbol{w} \cdot n\,ds, \quad \boldsymbol{w} : \Omega \to \mathbb{R}^d$$

Consider the problem

$$\begin{cases} -\nabla \cdot (k\nabla u) + cu = f & \text{in } \Omega \\ \alpha u + k\nabla u \cdot n = 0 & \text{on } \partial\Omega. \end{cases} \tag{68}$$

Let us choose an arbitrary $v \in V = \{v : \Omega \to \mathbb{R} \mid \int_\Omega v^2\,dx < \infty, \quad \int_\Omega |\nabla v|^2\,dx < \infty\}$.
Multiplying (68) by v and integrating yields

$$-\int_\Omega \nabla \cdot (k\nabla u) v\,dx + \int_\Omega cuv\,dx = \int_\Omega fv\,dx.$$

Using Green's formula we obtain

$$\int_\Omega k\nabla u \cdot \nabla v\,dx - \int_{\partial\Omega} k\nabla u \cdot nv\,ds + \int_\Omega cuv\,dx = \int_\Omega fv\,dx. \tag{69}$$

Inserting the boundary condition of (68) into (69) we obtain

$$\int_\Omega k\nabla u \cdot \nabla v\,dx + \int_{\partial\Omega} \alpha uv\,ds + \int_\Omega cuv\,dx = \int_\Omega fv\,dx. \tag{70}$$

Equation (70) is defined even if the known coefficients k, c, f, α are only *piecewise smooth, or even discontinuous*.
Problem (70) is called the *weak formulation* of problem (68). The classical solution of a PDE always satisfies the weak formulation but not the opposite. Therefore the weak formulation is a generalization of the original problem.

8.3 Approximation of elliptic equations using the finite element method

Consider the abstract PDE in weak form

$$u \in V: \quad a(u,v) = F(v) \quad \forall v \in V. \tag{$\mathcal{P}$}$$

Here $a : V \times V \to \mathbb{R}$ is a continuous bilinear form and $F : V \to \mathbb{R}$ is a continuous linear form.
Let $V_h \subset V$ be a finite dimensional subspace. Define the approximate problem

$$u_h \in V_h: \quad a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h. \tag{$\mathcal{P}_h$}$$

Let us assume that the bilinear form $a(\cdot,\cdot)$ is symmetric and V-elliptic, i.e. $\exists c > 0 : \ a(v,v) \geq c\|v\|^2 \ \forall v \in V$. Then the problems $(\mathcal{P})$ and $(\mathcal{P}_h)$ are uniquely solvable.
Let $\{\varphi_i\}_{i=1}^N$ be a basis of V_h. Then the approximate solution u_h can be represented as

$$u_h = \sum_{j=1}^N c_j \varphi_j.$$

Inserting this into $(\mathcal{P}_h)$ results in

$$a(\sum_{j=1}^N c_j \varphi_j, \varphi_i) = F(\varphi_i), \quad i = 1, ..., N.$$

Taking into accout the (bi)linearity of $a(\cdot,\cdot)$ we obtain

$$\sum_{j=1}^N c_j a(\varphi_j, \varphi_i) = F(\varphi_i), \quad i = 1, ..., N. \tag{71}$$

As φ_j:s are known functions we write (71) in matrix form

$$\boldsymbol{A}\boldsymbol{c} = \boldsymbol{f}, \tag{72}$$

where $\boldsymbol{A} \in \mathbb{R}^{N\times N}$, $\boldsymbol{f} \in \mathbb{R}^N$, $\boldsymbol{c} \in \mathbb{R}^N$, and

$$a_{ij} = a(\varphi_i, \varphi_j), \quad f_i = F(\varphi_i).$$

If the bilinear form $a(\cdot,\cdot)$ is symmetric and V-elliptic, then the matrix $\boldsymbol{A}$ is symmetric and positive definite.

8.4 Approximation using piecewise linear elements

Let us divide the domain Ω into set of nonoverlapping triangles $\mathcal{T}_h$ (tetrahedrons in 3D) such that

$$\bar{\Omega} = \bigcup_{T \in \mathcal{T}_h} T.$$

We call $\mathcal{T}_h$ the triangulation (or the finite element mesh) of Ω. We define the approximation V_h of V as follows:

$$V_h = \{v : \bar{\Omega} \to \mathbb{R} \mid v \text{ is continuous and piecewise linear}\}.$$

A basis of V_h is then simply defined by piecewise linear continuous functions $\varphi_i : \bar{\Omega} \to \mathbb{R}$ satisfying

$$\varphi_i(x^{(j)}) = \delta_{ij}.$$

Here $\{x^{(j)}\}_{j=1}^N$ is the set of nodes of the triangulation.

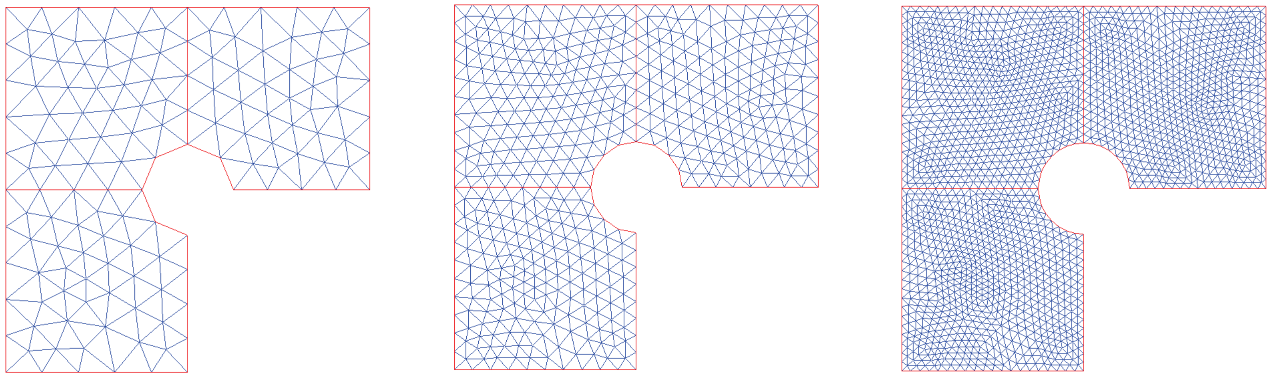

Figure 5: Example of regular refinement of an unstructured triangulation

Theorem 8.1. *Let $\bar{\Omega} \subset \mathbb{R}^2$ be a polygon and let the solution to $(\mathcal{P})$ be "sufficiently" regular. Let $\{\mathcal{T}_h\}$ be a regular (see Figure 5) collection of triangulations (i.e. there are no arbitrary large or small angles in triangles as $h \to 0$). Then there exists $C > 0$ such that for $h > 0$ sufficiently small*

$$\sqrt{\int_\Omega (u - u_h)^2 \, dx} = \mathcal{O}(h^2).$$

Often one is more interested in the gradient of the solution than the solution itself. One can derive the following error estimate

$$\sqrt{\int_\Omega |\nabla(u - u_h)|^2 \, dx} = \mathcal{O}(h).$$

Instead of piecewise linear approximation, higher order elements are often used in practical computations. For piecewise quadratic (and C^0 continuous) approximation we have the following estimations

$$\sqrt{\int_\Omega (u - u_h)^2 \, dx} = \mathcal{O}(h^3)$$
$$\sqrt{\int_\Omega |\nabla(u - u_h)|^2 \, dx} = \mathcal{O}(h^2).$$