7 System of (nonlinear) PDEs

7.1 Two-dimensional Navier–Stokes equations

Consider two-dimensional Navier–Stokes equations (in dimensionless form):

$$\begin{cases} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = -\dfrac{\partial p}{\partial x} + \dfrac{1}{\mathrm{Re}}\Delta u & \text{in } \Omega \\ \dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} = -\dfrac{\partial p}{\partial y} + \dfrac{1}{\mathrm{Re}}\Delta v & \text{in } \Omega \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 & \text{in } \Omega \\ \text{boundary conditions for } u, v & \text{on } \partial\Omega \end{cases} \tag{55}$$

Here (u, v) is the velocity and p is the pressure. The dimensionless parameter $\mathrm{Re} > 0$ is related to the viscosity, density and characteristic speed of the flow. In realistic simulations, Re is usually very large ($\sim 10^6$ or even more).
To simplify computations we introduce stream function $\Psi : \Omega \to \mathbb{R}$ by

$$u = \frac{\partial \Psi}{\partial y}, \quad v = -\frac{\partial \Psi}{\partial x}. \tag{56}$$

By differentiating (55) with respect to x and y we obtain

$$\begin{aligned} \frac{\partial^2 u}{\partial y \partial t} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} + u\frac{\partial^2 u}{\partial y \partial x} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} + v\frac{\partial^2 u}{\partial y^2} &= -\frac{\partial^2 p}{\partial y \partial x} + \frac{1}{\mathrm{Re}}\frac{\partial}{\partial y}(\Delta u) \\ \frac{\partial^2 v}{\partial x \partial t} + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + u\frac{\partial^2 v}{\partial y \partial x} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + v\frac{\partial^2 v}{\partial x \partial y} &= -\frac{\partial^2 p}{\partial x \partial y} + \frac{1}{\mathrm{Re}}\frac{\partial}{\partial x}(\Delta v) \end{aligned}$$

Subtracting the second equation from the first one results in

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} \\ + u\frac{\partial^2 u}{\partial y \partial x} + v\frac{\partial^2 u}{\partial y^2} - u\frac{\partial^2 v}{\partial x^2} - v\frac{\partial^2 v}{\partial x \partial y} = \frac{1}{\mathrm{Re}}\Delta\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) \end{aligned}$$

Taking into account $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ and (56) we obtain

$$\frac{\partial}{\partial t}(\Delta\Psi) + u\frac{\partial}{\partial x}(\Delta\Psi) + v\frac{\partial}{\partial y}(\Delta\Psi) = \frac{1}{\mathrm{Re}}\Delta^2\Psi.$$

Let ω satisfy $-\Delta\Psi = \omega$. Then we have

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \frac{1}{\mathrm{Re}}\Delta\omega. \tag{57}$$

Equation (57) is called the *stream function / vorticity* formulation of the 2D Navier–Stokes equations.

Example 7.1. Consider the classical "lid driven cavity" problem

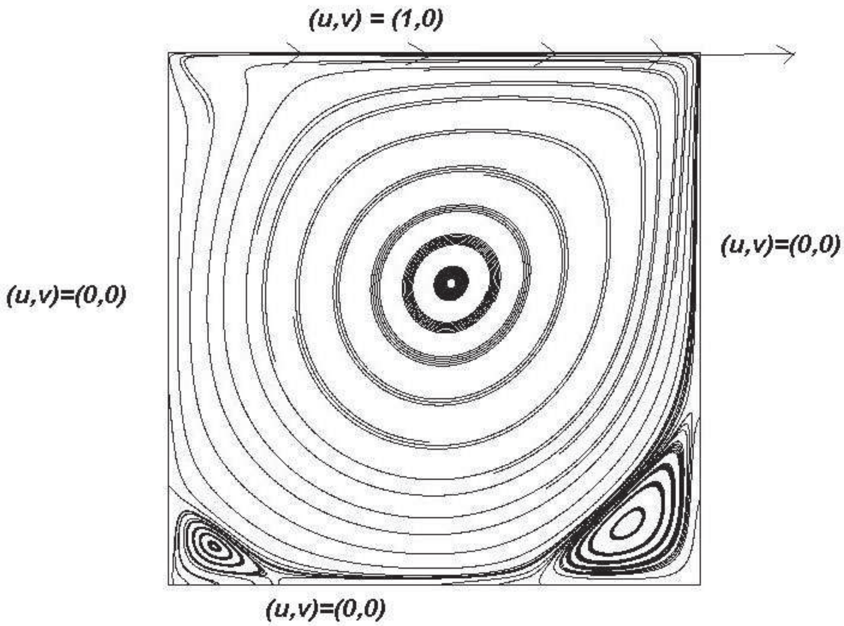

Figure 3: Physical boundary conditions for the velocity vector field (u, v)

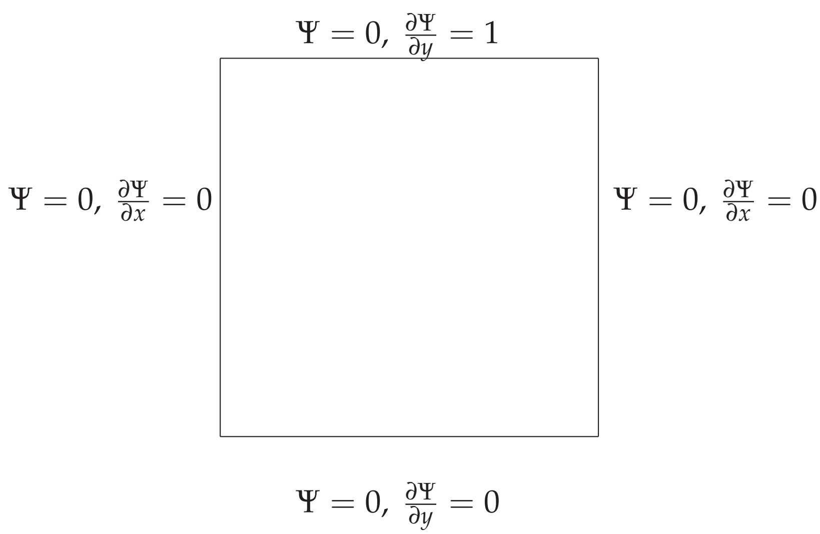

Figure 4: Boundary conditions for the stream funtion Ψ

Let us perform a finite difference discretization of (57). We assume that uniform stepsizes h_x, h_y in both spatial dimensions are used and explicit Euler discretization of the time derivative:

$$\frac{\omega_{ij}^{(k+1)} - \omega_{ij}^{(k)}}{\Delta t} + u_{ij}^{(k)} \frac{\omega_{i+1,j}^{(k)} - \omega_{i-1,j}^{(k)}}{2h_x} + v_{ij}^{(k)} \frac{\omega_{i,j+1}^{(k)} - \omega_{i,j-1}^{(k)}}{2h_y}$$
$$= \frac{1}{\mathrm{Re}} \left(\frac{\omega_{i+1,j}^{(k)} - 2\omega_{ij}^{(k)} + \omega_{i-1,j}^{(k)}}{h_x^2} + \frac{\omega_{i,j+1}^{(k)} - 2\omega_{i,j}^{(k)} + \omega_{i,j-1}^{(k)}}{h_y^2} \right).$$

On every time step we have to also solve a Poisson problem

$$\begin{cases} -\Delta\Psi^{(k+1)} = \omega^{(k+1)} & \text{in } \Omega \\ \Psi^{(k+1)} = 0 & \text{on } \partial\Omega. \end{cases} \tag{58}$$

On boundary nodes ω is approximated as follows. From Taylor expansion we get

$$\Psi(1-h_x, y) \approx \underbrace{\Psi(1,y)}_{=0} - h_x \underbrace{\frac{\partial\Psi(1,y)}{\partial x}}_{=0} + \frac{1}{2}h_x^2 \frac{\partial^2\Psi(1,y)}{\partial x^2}$$

Thus $\omega(1,y) = -\frac{\partial^2\Psi(1,y)}{\partial x^2} \approx -\frac{2}{h_x^2}\Psi(1-x_x, y)$.
Similarly

$$\Psi(x, 1-h_y) \approx \underbrace{\Psi(x,1)}_{=0} - h_y \underbrace{\frac{\partial\Psi(x,1)}{\partial y}}_{=1} + \frac{1}{2}h_y^2 \frac{\partial^2\Psi(x,1)}{\partial y^2}$$

and then $\omega(x,1) = -\frac{\partial^2\Psi(x,1)}{\partial y^2} \approx -\frac{2}{h_y^2}\left(\Psi(x,1-h_y) + h_y\right)$.