4 On numerical solution of hyperbolic PDEs

Consider a first oder hyperbolic PDE

$$\begin{cases} \dfrac{\partial u}{\partial t} + a(x,t,u)\dfrac{\partial u}{\partial x} = g(x,t,u), \quad t > 0 \\ u(x,0) = u_0(x), \quad -\infty < x < \infty. \end{cases} \tag{33}$$

Let us consider curves ("characteritics") in t-x-plane defined by the differential equation $dx/dt = a(x,t,u)$.
Along every characteristic $x(t)$ the solution of (33) satisfies

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}a(x,t,u) = g(x,t,u).$$

A characteristic x and the value of the solution u on it can be calculated by solving ordinary differential equation system

$$\begin{cases} x'(t) = a(x,t,u), \quad x(0) = x_0 \\ u'(t) = g(x,t,u), \quad u(0) = u_0(x_0). \end{cases} \tag{34}$$

Example 4.1. Consider the problem

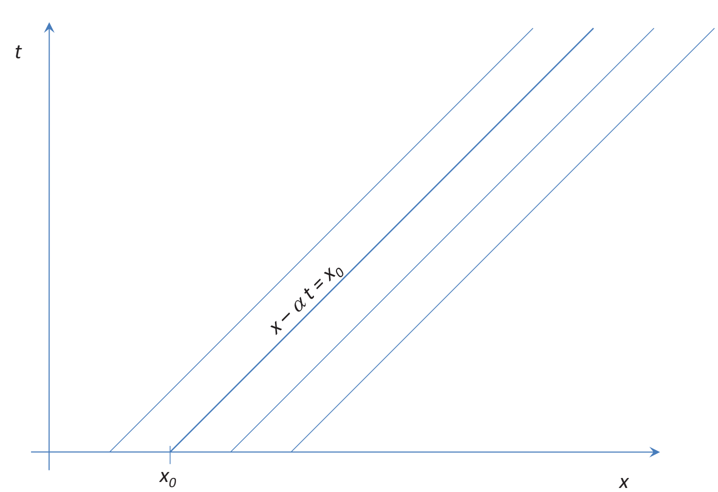
$$\frac{\partial u}{\partial t} + \alpha\frac{\partial u}{\partial x} = \beta, \quad \alpha \neq 0, \text{ and } \beta \text{ are constants.}$$

In this case (34) reads

$$\begin{cases} x'(t) = \alpha, \quad x(0) = x_0 \\ u'(t) = \beta, \quad u(0) = u_0(x_0). \end{cases}$$

This is easily solved resulting in $x(t) = \alpha t + x_0, \quad u(t) = \beta t + u_0(x_0)$. The solution of the original PDE along a characteristic is

$$\begin{aligned} &u(x,t)|_{x-\alpha t = x_0} = \beta t + u(x_0) \\ &\implies u(x,t) = \beta t + u_0(x - \alpha t) \quad \forall x,\ \forall t > 0. \end{aligned}$$



Example 4.2. Consider the problem

$$\frac{\partial u}{\partial t} + \alpha \frac{\partial u}{\partial x} = \beta u, \quad \alpha \neq 0, \text{ and } \beta \text{ are constants.}$$

In this case (34) reads

$$\begin{cases} x'(t) = \alpha, & x(0) = x_0 \\ u'(t) = \beta u, & u(0) = u_0(x_0). \end{cases}$$

This is again easily solved resulting in $x(t) = \alpha t + x_0, \quad u(t) = u_0(x_0)\, e^{\beta t}$. The solution of the original PDE along a characteristic is

$$\begin{aligned} & u(x,t)|_{x-\alpha t = x_0} = u(x_0)\, e^{\beta t} \\ & \Longrightarrow \; u(x,t) = u_0(x - \alpha t)\, e^{\beta t} \quad \forall x,\ \forall t > 0. \end{aligned}$$

4.1 Finite difference approximation

Consider the simple hyperbolic PDE

$$\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0$$

with constant $a > 0$.
Forward difference in time:

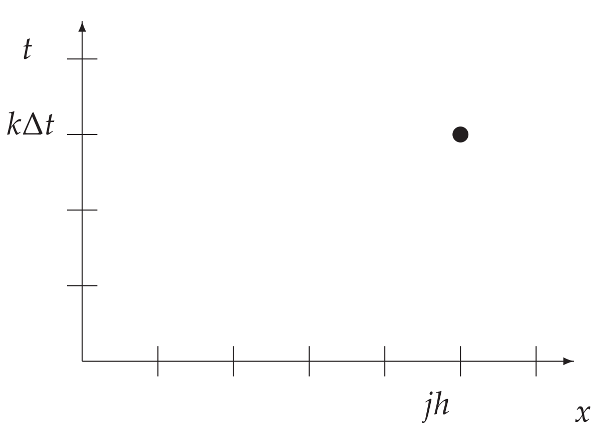
$$\frac{\partial u}{\partial t} = \frac{u_{k+1,j} - u_{k,j}}{\Delta t}.$$

Spatial discretization:

$$\frac{\partial u}{\partial x} = \frac{u_{k,j+\epsilon} - u_{k,j-\eta}}{(\epsilon + \eta) h}$$

Several formulas depending on the values of ϵ, η, for example

$$\begin{aligned} &\epsilon = 1,\ \eta = 1 \quad \text{central difference } \mathcal{O}(h^2) \\ &\epsilon = 1,\ \eta = 0 \quad \text{forward difference } \mathcal{O}(h) \\ &\epsilon = 0,\ \eta = 1 \quad \text{backward difference } \mathcal{O}(h). \end{aligned}$$



What about the stability of the above schemes? Let's perform the von Neumann stability analysis, i.e. we assume that $u_{k,j} = \xi^k e^{\mathrm{i}j\varphi}$. Substituting this into the difference scheme above we obtain

$$\frac{\xi^{k+1}e^{\mathrm{i}j\varphi} - \xi^k e^{\mathrm{i}j\varphi}}{\Delta t} + a \cdot \frac{\xi^k e^{\mathrm{i}(j+\epsilon)\varphi} - \xi^k e^{\mathrm{i}(j-\eta)\varphi}}{(\epsilon+\eta)h} = 0. \tag{35}$$

Let us define the *Courant number* $r := a \cdot \Delta t/h$. Solving ξ from (35) we obtain

$$\xi = 1 - \frac{r}{\epsilon+\eta}\left(e^{\mathrm{i}\epsilon\varphi} - e^{-\mathrm{i}\eta\varphi}\right).$$

Now, the choices of ϵ, η above result in:

$$\begin{aligned}
&\epsilon = \eta = 1 \implies \xi = 1 + \mathrm{i}r\sin\varphi \implies \|\xi\| \geq 1\ \forall r \\
&\epsilon = 1,\ \eta = 0 \implies \xi = 1 + r - re^{\mathrm{i}\varphi} \implies \|\xi\| \geq 1\ \forall r \\
&\epsilon = 0,\ \eta = 1 \implies \xi = 1 - r + re^{-\mathrm{i}\varphi} \implies \|\xi\| \leq 1 \ \text{ if } r \leq 1.
\end{aligned}$$

Thus, only the combination "forward difference in time / backward difference in space" works provided that $a \cdot \Delta t/h \leq 1$ (the Courant–Friedrichs–Levy condition, CFL).

Lax–Wendroff scheme

This scheme is a modification of the unsuccesfull $\epsilon = \eta = 1$ scheme. It reads as

$$\frac{u_{k+1,j} - u_{k,j}}{\Delta t} + \frac{a}{2h}\left(u_{k,j+1} - u_{k,j-1}\right) - \frac{a^2\Delta t}{2h^2}\left(u_{k,j+1} - 2u_{k,j} + u_{k,j-1}\right) = 0. \tag{36}$$

It can be shown that the accuracy of the scheme is $\mathcal{O}(h^2 + (\Delta t)^2$. Once again, the stability limits the step sizes, i.e. $r \leq 1$ should hold.
Notice that one can interpret this scheme by adding "artificial diffusion" to the original problem, i.e. we solve numerically the modified problem

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} - \frac{1}{2}a^2\Delta t\frac{\partial^2 u}{\partial x^2} = 0.$$

4.2 Finite difference approximation of the wave equation

Consider the followiing wave equation in one space dimension:

$$\begin{cases}
\dfrac{\partial^2 u}{\partial t^2} - c^2\dfrac{\partial^2 u}{\partial x^2} = 0, \quad a < x < b,\ t > 0 \\
u(a,t) = \alpha(t), \quad u(b,t) = \beta(t) \\
u(x,0) = f(x), \quad \dfrac{\partial u}{\partial t}(x,0) = g(x).
\end{cases} \tag{37}$$

Here α, β, f, g are known functions.
We discretize both derivatives using central differences:

$$\frac{u_{k+1,j} - 2u_{k,j} + u_{k-1,j}}{(\Delta t)^2} - c^2\frac{u_{k,j+1} - 2u_{k,j} + u_{k,j-1}}{h^2} = 0. \tag{38}$$

The numerical solution is obtained by marching in time by solving $u_{k+1,j}$ from equation (38). This is depicted in Figure 2 (the black nodal value is computed using the white ones).
In order to start the marching process, a small trick is needed. Artifical values $u_{-1,j}$ are obtained by taking into account the initial conditions:

$$\frac{u_{1,j} - u_{-1,j}}{2\Delta t} = g(x_j), \quad u_{0,j=f(x_j)}.$$

The accuracy of the leapfrog method is $\mathcal{O}(h^2 + (\Delta t)^2)$ and the CFL condition $c\Delta t/h \leq 1$ must hold.

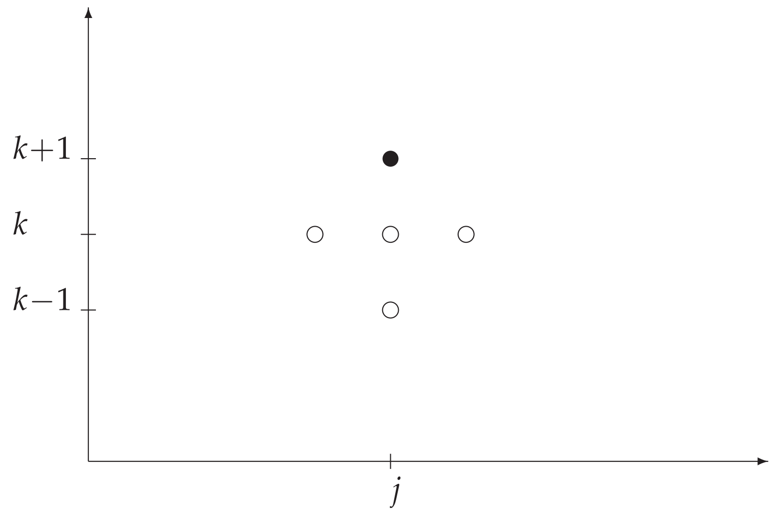


Figure 2: Leap frog scheme

4.3 On the nature of the solution of the wave equation

Consider the following wave equation in an unbounded domain

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2} = 0, \quad -\infty < x < \infty, \ t > 0 \\ u(x,0) = f(x), \quad \dfrac{\partial u}{\partial t}(x,0) = g(x). \end{cases} \tag{39}$$

The analytical solution to (39) is

$$u(x,t) = \frac{1}{2} f(x+ct) + \frac{1}{2} f(x-ct) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s)\, ds.$$

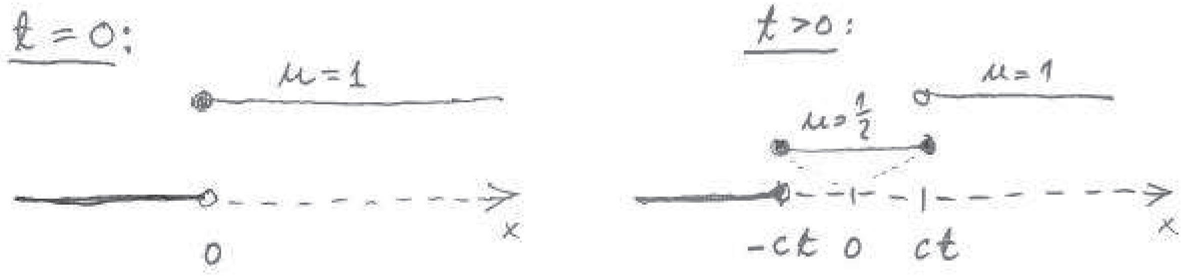
If $g \equiv 0$ then

$$u(x,t) = \frac{1}{2} f(x+ct) + \frac{1}{2} f(x-ct).$$

Thus the information of the initial condition spreads with constant speed c along *two* characteristics.

Example 4.3. Let $g \equiv 0$ and

$$f(x) = \begin{cases} 1, & x > 0 \\ 0, & \text{otherwise.} \end{cases}$$



From the picture
we see that the “information” (discontinuity at $x = 0$) travels to two directions at speed c.