Misc. exercises and notes #2

1. Any linearly independent set $\{v_1, v_2, ..., v_m\}$ serves as a basis for an m-dimensional linear (sub)space V. Why it is necessary (or at least very desirable) that in numerics the basis is orthogonal?

   Hint: consider $V = \mathbb{R}^2$ and its basis $\{v_1, v_2\}$ with $v_1 = (1, 0),\ v_2 = (1, \epsilon)$, where $0 < \epsilon \ll 1$. Hint: what are the coordinates of a vector $x \in V$ in that basis?

2. Consider the two-dimensional subspace $V = \{x \in \mathbb{R}^3 \mid x = (\alpha, \alpha, \beta),\ \alpha, \beta \in \mathbb{R}\}$. Its basis is e.g. $v_1 = (0, 0, 1),\ v_2 = (\epsilon, \epsilon, 1)$, where $0 < \epsilon \ll 1$ is given. Build an orthonormal basis for V using the Gram–Schmidt algorithm to the set $\{v_1, v_2\}$.

3. One may wonder why Krylov subspace methods are based on the maybe not so obvious choice span$\{r_0, Ar_0, A^2 r_0, ..., A^{k-1} r_0\}$?

   Consider the Richardson method for $Ax = b$:

   $$x_{k+1} = x_k + \alpha(b - Ax_k), \quad \alpha > 0 \text{ given constant.}$$

   Assume (for simplicity) that $x_0 = 0$. Then

   $$\begin{aligned} x_1 &= \alpha(b - Ax_0) \\ x_2 &= x_1 + \alpha(b - Ax_1) = ... = 2\alpha r_0 - \alpha^2 A r_0 \\ x_3 &= x_2 + \alpha(b - Ax_2) = ... = 3\alpha r_0 - 3\alpha^2 A r_0 + \alpha^3 A^2 r_0 \\ &\ \vdots \\ x_k &= \sum_{j=1}^{k} \alpha_j A^{j-1} r_0. \end{aligned}$$

   We see that the approximate solution x_k is given as a linear combination of vectors $\{r_0, Ar_0, ..., A^{k-1} r_0\}$:-)

4. Consider solving a system $Ax = b$, $A = \text{diag}(11, 12, 13, 14, 15, 16)$, $b = [1, 1, 1, 1, 1, 1]^T$ using any Krylov-like iterative method starting from $x_0 = 0$. It seems odd that for such a trivial problem the methods take at least six iterations to converge to the exact solution. Why doesn't the method "see" that we are solving a trivial problem?