p-HARMONIC APPROXIMATION OF FUNCTIONS OF LEAST GRADIENT

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Abstract. The purpose of this note is to establish a natural connection between the minimizers of two closely related variational problems. We prove global and local convergence results for the p-harmonic functions, defined as continuous local minimizers of the L^p norm of the gradient for $1 < p < \infty$, as $p \to 1$, and show that the limit function minimizes at least locally the total variation of the vector-valued measure ∇u in $BV(\Omega)$.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$, $n > 1$, be a bounded Lipschitz domain and $f : \partial\Omega \to \mathbb{R}$ a continuous function. The problem of minimizing the total variation of the vector-valued measure ∇u on Ω

$$\|\nabla u\|(\Omega) = \sup\left\{ \int_\Omega u \operatorname{div} \sigma \, dx : \sigma \in C_0^\infty(\Omega; \mathbb{R}^n), |\sigma(x)| \le 1 \text{ for } x \in \Omega \right\} \tag{1.1}$$

in the set

$$\{u : u \in BV(\Omega) \cap C(\overline{\Omega}),\ u = f \quad \text{on } \partial\Omega\}$$

has been studied in detail by Sternberg, Williams and Ziemer in [17]. The purpose of this note is to establish a connection between the minimizers of the above problem, called *functions of least gradient*, and the so-called *p-harmonic functions*, which are defined as (continuous) local minimizers of the L^p norm of the gradient in the Sobolev space $W^{1,p}_{loc}(\Omega)$ for $1 < p < \infty$ (see Definition 2.2 below).

Apart from the continuity requirement, these variational problems represent archetypal minimization problems in their respective spaces BV and $W^{1,p}$. The co-area formula

$$\|\nabla v\|(\Omega) = \int_{-\infty}^{\infty} P(\{v \ge t\}, \Omega)\, dt, \qquad v \in BV(\Omega),$$

where $P(\{v \ge t\}, \Omega) = \|\nabla \chi_{\{v \ge t\}}\|(\Omega)$ denotes the perimeter of the superlevel set $\{x \in \Omega : v(x) \ge t\}$ in Ω, readily connects functions of least gradient to the theory of parametric minimal surfaces [8]. Indeed, in [1] Bombieri, De Giorgi

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and Giusti have shown that the superlevel sets of a function of lest gradient are area-minimizing. Conversely, Sternberg et al. [17] prove the existence of a function of least gradient (under certain geometric conditions on the domain) by explicitly constructing each of its superlevel sets in such a way that they are area-minimizing and reflect the boundary condition. Another context in which functions of least gradient play an important role is the asymptotic behavior of solutions to the generalized motion by mean curvature, see [18], [11].

The theory of p-harmonic functions can in turn be seen as a natural generalization of the classical theory of harmonic functions, where the usual Dirichlet integral has been replaced by the functional

$$I_p(u) = \int_\Omega |\nabla u|^p \, dx, \qquad 1 < p < \infty,$$

and the Laplace equation by the nonlinear *p-Laplace equation*

$$-\Delta_p u := -\operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0. \tag{1.2}$$

Due to the nonlinearity and degeneracy of the equation, p-harmonic functions are, in general, not smooth and hence (1.2) has to be understood in a weak sense, see Definition 2.2. Notice that in the case $p = 2$, we recover the Dirichlet integral and the Laplace equation $-\Delta u = 0$. The p-Laplace equation is the prototype of a class of quasi-linear equations in the form

$$-\operatorname{div} \mathcal{A}_p(x, \nabla u(x)) = 0, \qquad \mathcal{A}_p(x, \xi) \approx |\xi|^p,$$

and it is fundamental in the nonlinear potential theory; cf. [10]. The p-harmonic operator $\Delta_p u$ also appears in many contexts in physics.

We have two main results in this paper. The first one, Theorem 3.1, shows that if Ω is a smooth domain whose boundary has positive mean curvature and $f \in C(\partial\Omega)$, then the sequence of the unique p-harmonic functions $u_p \in W^{1,p}_{loc}(\Omega) \cap C(\overline{\Omega})$ that agree with f on $\partial\Omega$ converges uniformly, as $p \to 1$, to a function $h \in BV(\Omega) \cap C(\overline{\Omega})$ that is the unique function of least gradient with boundary data f. The assumptions on the domain and boundary data guarantee that the problem of finding a function of least gradient with the given boundary values is well-posed, but this is not the case if Ω is an arbitrary bounded domain, see [17]. Therefore such a convergence result cannot hold in wider generality, and we must turn to local results. In this regard we show that if $\{u_p\}_{1<p<2}$ is a sequence of p-harmonic functions, bounded in L^s_{loc} for some $s > 1$, then, up to a subsequence, $u_p \to u_1$ in L^q_{loc} for $1 \le q < n/(n-1)$ and the limit function $u_1 \in BV_{loc}$ minimizes the total variation locally:

$$\|\nabla u_1\|(K) \le \|\nabla v\|(K)$$

whenever $v \in BV_{loc}(\Omega)$ is such that $v = u_1$ outside a compact subset $K \subset \Omega$. This is Theorem 4.1. It is easy to give examples showing that u_1 need not be continuous, but if it is, then it is of course locally a function of least gradient.

The above results appear quite natural, especially after noticing that

$$\|\nabla u\|(\Omega) = \int_\Omega |\nabla u|\, dx$$

for $u \in W^{1,1}(\Omega)$. However, some caution is needed in the proofs, because functions of least gradient differ from p-harmonic functions in many aspects. Most importantly, the characterization of p-harmonic functions in terms of equation (1.2) shows that the property of being p-harmonic is completely local. The same is not true for the functions of least gradient since their superlevel sets must be area-minimizing, and that is not a local property. From this it also follows that the functions of least gradient cannot be characterized by a differential equation. Furthermore, p-harmonic functions enjoy local $C^{1,\alpha}$ regularity and higher integrability, whereas functions of least gradient are, in general, merely continuous and need not have any Sobolev regularity.

Our results give yet another existence proof for the functions of least gradient, cf. [15], [17], [18]. Moreover, it is clear that all estimates for p-harmonic functions that are independent of p for p close to 1 remain valid for the functions of least gradient. As an example, we mention the sup-estimate in Remark 4.4. And although the class of functions of least gradient is not characterized by a differential equation, the approximation results provides an equation that all continuous functions of least gradient must satisfy in the viscosity sense, see Remark 3.3. The novelty here is that this necessary condition can be checked even if the function in question is not in $W^{1,1}$.

The asymptotic behavior as $p \to 1$ of the solutions to the non-homogeneous p-Laplace equation $-\Delta_p u = f(x)$ with zero boundary values has been studied by Cicalese and Trombetti [2], and in the special case $f \equiv 1$, which models torsional creep, by Kawohl [13]. Another related problem is to understand the limit, as $p \to 1$, of the p-Laplace eigenvalue problem $-\Delta_p u = \lambda |u|^{p-2} u$. This has been considered by Demengel [4] and Kawohl and Fridman [14].

2. PRELIMINARIES

In this section, we recall some definitions and results that are needed later, and also prove a uniform continuity estimate (Lemma 2.3 below) that plays an important role in the proof of Theorem 3.1.

We begin with the functions of least gradient that were already defined in the introduction. Regarding existence and uniqueness, we recall the following result due to Sternberg et al. [17]:

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain such that $\partial\Omega$ has non-negative mean curvature (in a weak sense) and is not locally are-minimizing. If $f \in C(\partial\Omega)$, then there exists a unique function of least gradient $h \in BV(\Omega) \cap C(\overline{\Omega})$ such that $h = f$ on $\partial\Omega$.*

Examples of the form $h(x,y) = f(x/y)$ in two dimensions show that, in general, functions of least gradient have regularity in the interior no better than that at the boundary. Fortunately it is also true that better regularity of the boundary data induces better regularity in the interior. For us it is particularly important to know that if Ω is a smooth domain having positive mean curvature and the boundary data $f \in C^{1,1}(\partial\Omega)$, then the function of least gradient h that agrees with f on $\partial\Omega$ is Lipschitz continuous in $\overline{\Omega}$, see [17], Theorem 5.9.

Next we briefly review some aspects of the theory of p-harmonic functions.

Definition 2.2. A continuous function $u \in W^{1,p}_{loc}(\Omega)$ is *p-harmonic* in Ω if

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi \, dx = 0 \tag{2.1}$$

for every $\varphi \in C_0^\infty(\Omega)$. Here $1 < p < \infty$.

By the regularity theory of elliptic partial differential equations, the continuity is redundant in the definition. According to a theorem of Ural'tseva, in the case $p > 2$, later extended by DiBenedetto and Lewis to all $p > 1$, $u \in C^{1,\alpha}_{loc}(\Omega)$ for some $\alpha = \alpha(p, n) > 0$; see e.g. [5] and the references therein. Furthermore, if a p-harmonic function u belongs to $W^{1,p}(\Omega)$, then (2.1) holds also for every $\varphi \in W_0^{1,p}(\Omega)$.

The p-Laplace equation is the Euler–Lagrange equation for the variational integral

$$I_p(u) = \int_\Omega |\nabla u(x)|^p \, dx. \tag{2.2}$$

More precisely, a continuous function $u \in W^{1,p}_{loc}(\Omega)$ is p-harmonic in Ω if and only if

$$\int_{\Omega_0} |\nabla u|^p \, dx \le \int_{\Omega_0} |\nabla v|^p \, dx \qquad \text{whenever } \Omega_0 \subset\subset \Omega \text{ and } u - v \in W_0^{1,p}(\Omega_0).$$

Given a function $g \in W_0^{1,p}(\Omega)$, it readily follows from the strict convexity of the functional I_p that there exists a unique p-harmonic function u_p such that $u_p - g \in W_0^{1,p}(\Omega)$. However, the solvability of the boundary value problem

$$\begin{cases} -\Delta_p v = 0 & \text{in } \Omega, \\ \ \ v(x) = f(x) & \text{for all } x \in \partial\Omega, \end{cases} \tag{2.3}$$

where $f \in C(\partial\Omega)$, is a more subtle issue, cf. [10]. In this paper we will use the fact that the problem (2.3) is well-posed for any $1 < p < \infty$ if $\partial\Omega$ is Lipschitz. Moreover, since we are interested in the convergence of p-harmonic functions as $p \to 1$, we need continuity estimates that are independent of the parameter p at least for p close to 1. These follow by a quite standard application of the barrier method:

Lemma 2.3. *Suppose that Ω is a bounded smooth domain whose boundary everywhere has positive mean curvature. Let $f \in C(\partial\Omega)$ and for $1 < p < 2$ let $u_p \in W^{1,p}_{loc}(\Omega) \cap C(\overline{\Omega})$ be the unique p-harmonic function satisfying $u_p = f$ on $\partial\Omega$. Then the sequence $\{u_p\}_{1<p<2}$ is equicontinuous in $\overline{\Omega}$.*

Proof. Let $\rho : [0,\infty[\to [0,\infty[$ be a smooth, concave and increasing modulus of continuity for f, i.e.,

$$|f(x) - f(y)| \leq \rho(|x-y|) \qquad \text{for } x, y \in \partial\Omega;$$

it is not difficult to see that such a function ρ always exists. We will show that there exists a constant $C > 0$, depending only on f and Ω, such that

$$|u_p(x) - u_p(y)| \leq C\rho(|x-y|^{1/2}) \tag{2.4}$$

for $x, y \in \overline{\Omega}$.

We notice first that it is enough to prove that (2.4) holds whenever $x \in \Omega$, $y \in \partial\Omega$, and $|x - y| < \delta$ for some $0 < \delta < 1$. This can be deduced using the comparison principle and the translation invariance of p-harmonic functions, cf. Lemma 5.1 in [17], and by noting that if $|x - y| > \delta$, then

$$|u_p(x) - u_p(y)| \leq \frac{2\sup_{\partial\Omega}|f|}{\rho(\delta^{1/2})}\rho(|x-y|^{1/2}).$$

Thus it suffices to construct, for each $z_0 \in \partial\Omega$, barriers $\omega^+, \omega^- \in C^2(\Omega) \cap C(\overline{\Omega})$ such that

(i) $\omega^{\pm}(z_0) = f(z_0)$,
(ii) $\omega^- \leq u_p \leq \omega^+$ on $\partial(\Omega \cap B(z_0, \delta))$,
(iii) $-\Delta_p\omega^+ \geq 0$ and $-\Delta_p\omega^- \leq 0$ in $\Omega \cap B(z_0,\delta)$,
(iv) $|\omega^{\pm}(x) - f(z_0)| \leq C\rho(|x-z_0|^{1/2})$ for $x \in \Omega \cap B(z_0,\delta)$.

Indeed, (ii), (iii) and the comparison principle imply $\omega^- \leq u_p \leq \omega^+$ in $\Omega \cap B(z_0, \delta)$, which together with (i) and (iv) shows that

$$|u_p(x) - u_p(z_0)| \leq C\rho(|x-z_0|^{1/2}) \qquad \text{for } x \in \Omega \cap B(z_0,\delta).$$

In the construction of the required barriers, we utilize some well-known properties of the (signed) distance function $d(x)$, see e.g. [7], [9]. First, since Ω is a smooth domain, $d(x)$ is smooth in some small neighborhood U of the boundary and

$$-\Delta d(x) \geq (n-1)\min_{z\in\partial\Omega} H(z) > 0 \qquad \text{for } x \in U, \tag{2.5}$$

where $H(z)$ denotes the mean curvature of $\partial\Omega$ at z. Secondly, from the fact that $|\nabla d(x)| = 1$ for $x \in U$, it follows that $D^2d(x)\nabla d(x) = 0$. From now on, we assume that $\delta > 0$ is so small that $B(z, 2\delta) \subset U$ for every $z \in \partial\Omega$.

Following [17], we look for the barrier ω^+ in the form

$$\omega^+(x) = f(z_0) + g(v(x)),$$

where $g(t) = C\rho(t^{1/2})$,

$$v(x) = |x-z_0|^2 + \lambda d(x)$$

and $\lambda > 0$, $C > 0$ are to be determined. A direct computation gives

$$\Delta_p\omega^+ = g'(v)^{p-1}\Delta_p v + (p-1)g'(v)^{p-2}g''(v)|\nabla v|^p.$$

Using the fact that $D^2d(x)\nabla d(x) = 0$, we have

$$\Delta_p v = |\nabla v|^{p-2}\big(2n + \lambda\Delta d\big) + (p-2)|\nabla v|^{p-4}\big(8|x-z_0|^2 + 8\lambda\nabla d\cdot(x-z_0) + 2\lambda^2 + 4\lambda D^2 d(x-z_0)\cdot(x-z_0)\big).$$

Assuming $\lambda \geq 4\delta$, which implies $\lambda/2 \leq |\nabla v| \leq 2\lambda$, this leads to the estimate

$$\Delta_p v \leq C_1\lambda^{p-1}\Delta d + C_2\lambda^{p-2} \qquad \text{in } \Omega\cap B(z_0,\delta),$$

where both constants C_1, $C_2 > 0$ are independent of p for $1 < p < 2$. By (2.5) we therefore obtain $-\Delta_p v > 0$ in $\Omega \cap B(z_0,\delta)$ provided that $\lambda > 0$ is chosen to be sufficiently large. Since ρ is concave and increasing, we conclude

$$-\Delta_p\omega^+ \geq -g'(v)^{p-1}\Delta_p v > 0.$$

Hence (iii) holds for ω^+. Since properties (i) and (iv) are clearly valid, it remains to check (ii). By choosing $C > 1$ large enough, we have $\omega^+(x) \geq \sup_{\partial\Omega}|f| \geq u_p(x)$ for $x \in \Omega\cap\partial B(z_0,\delta)$. On the other hand,

$$u_p(x) = f(x) \leq f(z_0) + \rho(|x-z_0|) \leq \omega^+(x) \qquad \text{for } x\in\partial\Omega\cap B(z_0,\delta),$$

and hence also (ii) holds for ω^+. The barrier ω^- is constructed in an analogous manner. □

Remark 2.4. The assumption that Ω is a smooth domain is not really needed in Lemma 2.3, since it is not necessary that the barriers ω^+,ω^- are twice differentiable in Ω. Indeed, it is enough to have condition (iii) above valid in the viscosity sense, and this can be done, for example, if Ω is convex and $\partial\Omega$ everywhere has positive mean curvature "in the viscosity sense" : there exists $\mu > 0$ so that, whenever $x_0 \in \partial\Omega$ and $U \subset \Omega$ is a smooth domain such that $\partial U \cap \partial\Omega = \{x_0\}$, we have

$$H_U(x_0) \geq \mu > 0;$$

here $H_U(x_0)$ is the mean curvature of ∂U at x_0. Since this extension of Lemma 2.3 is not needed in what follows, we omit the details.

3. Global results

Our main result in this section is the following theorem:

Theorem 3.1. *Suppose that $\Omega \subset I\!\!R^n$ is a bounded smooth domain whose boundary has positive mean curvature and $f \in C(\partial\Omega)$, and let $h \in BV(\Omega)\cap C(\overline{\Omega})$ be the unique function of least gradient such that $h = f$ on $\partial\Omega$. Then if $u_p \in W^{1,p}_{loc}(\Omega)\cap C(\overline{\Omega})$ is the unique p-harmonic function satisfying $u_p = f$ on $\partial\Omega$, we have*

$$u_p \to h \qquad \textit{locally uniformly in } \Omega.$$

Proof. Suppose first that $f \in C^{1,1}(\Omega)$. Then h is Lipschitz continuous, see [17] Theorem 5.9. In particular, it belongs to the Sobolev space $W^{1,p}(\Omega)$ for every

$1 \le p \le \infty$. Since u_p minimizes the L^p norm of the gradient, we have by using Hölder's inequality that

$$\begin{aligned}\|\nabla u_p\|(\Omega) &= \int_\Omega |\nabla u_p|\,dx \le |\Omega|^{1-1/p}\left(\int_\Omega |\nabla u_p|^p\,dx\right)^{1/p} \\ &\le |\Omega|^{1-1/p}\left(\int_\Omega |\nabla h|^p\,dx\right)^{1/p} \le |\Omega|\operatorname{ess\,sup}_{x\in\Omega}|\nabla h(x)|.\end{aligned}$$

Thus $\{u_p\}_{1<p<2}$ is bounded in $BV(\Omega)$, and by the compactness properties of BV functions we find a subsequence, still denoted by (u_p), converging to a function $u_1 \in BV(\Omega)$ in L^1. By the lower semicontinuity of the total variation with respect to L^1 convergence, we obtain

$$\begin{aligned}\|\nabla u_1\|(\Omega) &\le \liminf_{p\to 1}\|\nabla u_p\|(\Omega) \le \liminf_{p\to 1}|\Omega|^{1-1/p}\left(\int_\Omega |\nabla h|^p\,dx\right)^{1/p} \\ &= \int_\Omega |\nabla h|\,dx = \|\nabla h\|(\Omega).\end{aligned} \tag{3.1}$$

Next we invoke Lemma 2.3 and the comparison principle to conclude that $\{u_p\}$ is equicontinuous and uniformly bounded in $\overline{\Omega}$. Thus, up to a subsequence, $u_p \to u_1$ uniformly. In particular, u_1 is continuous and $u_1 = f$ on $\partial\Omega$. Hence it follows from (3.1) and the uniqueness of h that $u_1 = h$ in Ω. This proves the theorem in the case $f \in C^{1,1}(\partial\Omega)$.

The general case follows by using a rather standard approximation procedure. If f is merely continuous, we select a sequence (f_j) of smooth functions converging to f uniformly on $\partial\Omega$. For $1 < p < \infty$, let $u_p(f)$ and $u_p(f_j)$ denote the unique p-harmonic functions with boundary data f and f_j, respectively. The comparison principle for p-harmonic functions and (the proof of) Lemma 2.3 then imply that the set $\{u_p(f), u_p(f_j)_{j=1}^{\infty}\}$ is uniformly bounded and equicontinuous. Hence there exists a sequence $p_{k,0} \to 1$ and $u_1(f) \in BV(\Omega)$ such that

$$\sup_\Omega |u_{p_{k,0}}(f) - u_1(f)| < \frac{1}{k}. \tag{3.2}$$

Next we select inductively the sequences $(p_{k,j})$, $j = 1, 2, \ldots$, so that $(p_{k,j})$ is a subsequence of $(p_{k,j-1})$ and

$$\sup_\Omega |u_{p_{k,j}}(f_j) - h_j| < \frac{1}{k}, \tag{3.3}$$

where $h_j \in BV(\Omega)$ is the function of least gradient with boundary values f_j. Moreover, it follows from the comparison principle for the functions of least gradient, see [17], that

$$\sup_\Omega |h - h_j| \le \sup_{\partial\Omega} |f - f_j|, \tag{3.4}$$

where $h \in C(\overline{\Omega}) \cap BV(\Omega)$ is the unique function of least gradient with boundary values f.

In order to conclude the proof, it suffices to show that $u_1(f) = h$. To this end, we denote $p_j = p_{j,j}$ and compute, using (3.2), (3.3), (3.4) and the comparison principle for the p-harmonic functions,

$$\begin{aligned}\sup_\Omega |u_1(f) - h| \le\ & \sup_\Omega |u_1(f) - u_{p_j}(f)| + \sup_\Omega |u_{p_j}(f) - u_{p_j}(f_j)| \\ & + \sup_\Omega |u_{p_j}(f_j) - h_j| + \sup_\Omega |h_j - h| \\ \le\ & \tfrac{1}{j} + \sup_{\partial\Omega} |f - f_j| + \tfrac{1}{j} + \sup_{\partial\Omega} |f - f_j|.\end{aligned}$$

Since $f_j \to f$ uniformly on $\partial\Omega$, we are done. □

Remark 3.2. The approximation of the continuous boundary data f by a sequence of smooth functions (f_j) is needed in the proof above due to the fact that mere continuity of the boundary data is clearly not enough to guarantee any Sobolev regularity for the function of least gradient, not even locally. Indeed, let $C : [0, 1] \to [0, 1]$ be the usual Cantor function associated to the standard 1/3-Cantor set, cf. [16], and define

$$u(x, y) := C(x) \qquad \text{for } (x, y) \in B := B_{1/2}((1/2, 0)).$$

It is easy to see that $u \in BV(B) \cap C^\alpha(\overline{B})$ with $\alpha = \log 2/ \log 3$, but $u \notin W^{1,1}_{loc}(B)$ since C is not absolutely continuous. From the fact that superlevel sets of a function of least gradient are area-minimizing it readily follows that u is a function of least gradient in B. An analogous example can be constructed for any given $0 < \alpha < 1$.

Remark 3.3. It is proved in [12] that a function $u_p \in C(\Omega)$ is p-harmonic in Ω if and only if it is a viscosity solution of

$$-|\nabla u|^2 \Delta u - (p - 2)D^2 u \nabla u \cdot \nabla u = 0, \tag{3.5}$$

or, equivalently, of $-|\nabla u|^{4-p}\Delta_p u = 0$. Thus it follows from the stability results of viscosity solutions, see e.g. [3], and Theorem 3.1 above that if $h \in C(\Omega)$ is a function of least gradient, it satisfies

$$-|\nabla u|^2 \Delta u + D^2 u \nabla u \cdot \nabla u = 0 \tag{3.6}$$

in the viscosity sense. However, the converse is not true, as shown by the functions in [18], Example 3.6.

Remark 3.4. It easily follows from Theorem 3.1 and the comparison principle for p-harmonic functions that if Ω is bounded smooth domain whose boundary has positive mean curvature, $f_p \in C(\partial\Omega)$ for $1 < p < 2$,

$$\begin{cases} -\Delta_p u_p = 0 & \text{in } \Omega, \\ u_p = f_p & \text{on } \partial\Omega, \end{cases}$$

and $f_p \to f$ uniformly, then $u_p \to h$ uniformly and h is the unique function of least gradient with boundary data f.

4. Local results

In this last section, we investigate local convergence properties of p-harmonic functions as $p \to 1$. Since we do not assume anything about their boundary behavior nor about the domain Ω, it is clear that the limit functions, if there are any, need not be continuous. Hence it is reasonable to drop the requirement that a function of least gradient should be continuous. More precisely, we say that $h \in BV_{loc}(\Omega)$ is locally a function of least gradient, if

$$\|\nabla h\|(K) \leq \|\nabla v\|(K)$$

whenever $v \in BV_{loc}(\Omega)$ is such that $v = h$ outside a compact subset $K \subset \Omega$. This definition has been used for example in [1] and [15].

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^n$ be open and suppose that $u_p \in W^{1,p}_{loc}(\Omega) \cap C(\Omega)$ are p-harmonic functions so that for some $s > 1$, the sequence $\{u_p\}_{1<p<2}$ is bounded in $L^s(U)$ for every open $U \subset\subset \Omega$. Then there exists $p_j \to 1$ such that $u_{p_j} \to u_1 \in BV_{loc}(\Omega)$ in $L^q_{loc}(\Omega)$ for every $1 \leq q < n/(n-1)$ and u_1 is locally a (possibly discontinuous) function of least gradient.*

Remark 4.2. We prove in fact a slightly stronger result: any subsequential limit of $\{u_p\}_{1<p<2}$ in $L^q_{loc}(\Omega)$, $1 < q < n/(n-1)$, is locally a function of least gradient.

Before proving Theorem 4.1, let us point out a concrete situation where its assumptions are satisfied and Theorem 3.1 cannot be applied. Let $f : \partial\Omega \to \mathbb{R}$ be a bounded function, Ω any bounded domain, and denote by $\underline{h}_p$ the lower Perron solution (see e.g. [10], Chapter 9) to the Dirichlet problem

$$\begin{cases} -\Delta_p v = 0 & \text{in } \Omega, \\ v = f & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

Then $\underline{h}_p \in W^{1,p}_{loc}(\Omega) \cap C(\Omega)$ is p-harmonic in Ω, but, in general, it is not true that

$$\lim_{x \to z} \underline{h}_p(x) = f(z)$$

for $z \in \partial\Omega$. However, since f is bounded, it follows that

$$\sup_{x \in \Omega} |\underline{h}_p(x)| \leq \sup_{z \in \partial\Omega} |f(z)|.$$

Thus Theorem 4.1 applies to the sequence $\{\underline{h}_p\}_{1<p<2}$, and we conclude that any subsequential limit when $p \to 1$ is locally a function of least gradient. An analogous result holds for the sequence of the upper Perron solutions of (4.1).

We begin the proof of Theorem 4.1 by showing that the sequence $\{u_p\}_{1<p<s}$ contains a convergent subsequence.

Lemma 4.3. *Let $\Omega \subset \mathbb{R}^n$ be any open set and let $\{u_p\}_{1<p<s}$ be a sequence of p-harmonic functions that is locally bounded in $L^s(\Omega)$. Then there exists a subsequence $p_j \searrow 1$ and $u_1 \in BV_{loc}(\Omega)$ such that $u_{p_j} \to u_1$ in $L^q_{loc}(\Omega)$ for any $1 \leq q < n/(n-1)$.*

Proof. Using $u_p\psi^p \in W_0^{1,p}(\Omega)$, where $\psi \in C_0^\infty(\Omega)$, as a test-function in the weak formulation of the p-Laplace equation, we obtain a Caccioppoli-type inequality

$$\int_\Omega |\nabla u_p|^p \psi^p \, dx \le p^p \int_\Omega |u_p|^p |\nabla \psi|^p \, dx. \tag{4.2}$$

See e.g. [10], Chapter 3 for details. Now we have for $1 < p < s$ and for each ball $B = B(x,r)$ for which $2B \subset\subset \Omega$ that

$$\begin{aligned}\int_B |\nabla u_p| \, dx &\le |B|^{(p-1)/p} \left(\int_B |\nabla u_p|^p \, dx \right)^{1/p} \\ &\le C(n)|B|^{(p-1)/p} p r^{-1} \left(\int_{2B} |u_p|^p \, dx \right)^{1/p} \\ &\le C(n)|B|^{(p-1)/p} p r^{-1} \left(\int_{2B} |u_p|^s \, dx \right)^{1/s} |B|^{(s-p)/sp} \\ &\le C(n) p \, \|u_p\|_{L^s(\Omega)} \, r^{n-1-n/s}.\end{aligned}$$

Thus $\{u_p\}_{1<p<s}$ is bounded in $W^{1,1}_{loc}(\Omega)$, and we obtain, by using the Rellich-Kondrachov compactness theorem, that there exists a subsequence $p_j \to 1$ and $u_1 \in BV_{loc}(\Omega)$ so that $u_{p_j} \to u_1$ in $L^q_{loc}(\Omega)$ for $1 \le q < n/(n-1)$ as claimed. □

Remark 4.4. Via the Moser iteration technique it follows from a refined version of (4.2) that the sequence $\{u_p\}_{1<p<s}$ is actually locally bounded in $L^\infty(\Omega)$. Indeed, for a ball B such that $2B \subset\subset \Omega$ one obtains

$$\operatorname{ess\,sup}_B |u_p| \le C|B|^{-1/s} \left(\int_{2B} |u|^s \, dx \right)^{1/s}.$$

The novelty is that this holds with a constant C that is independent of p for $1 < p < s$.

Theorem 4.1 follows from Lemma 4.3 and

Proposition 4.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and suppose that $u_p \in W^{1,p}(\Omega) \cap C(\Omega)$ are p-harmonic functions so that $u_p \to u_1 \in BV(\Omega)$ in $L^q(\Omega)$ for some $1 < q < n/(n-1)$. Then u_1 is a (possibly discontinuous) function of least gradient in Ω.*

Proof. Let $v \in BV(\Omega)$ be such that $v = u_1$ outside a compact subset $K \subset \Omega$. Our aim is to show that

$$\|\nabla u_1\|(K) \le \|\nabla v\|(K). \tag{4.3}$$

For a given $\varepsilon > 0$, let us choose open sets Ω_0, U_i, V_i, $i = 1, 2$ with Lipschitz boundary so that

$$K \subset V_1 \subset\subset U_1 \subset\subset \Omega_0 \subset\subset U_2 \subset\subset V_2 \subset\subset \Omega,$$

and

$$\|\nabla u_1\|(V_2 \setminus \overline{V}_1) < \varepsilon;$$

this is possible because the measure ∇u_1 has finite total mass. Let us also choose a smooth cut-off function $\eta : \Omega \to \mathbb{R}$ satisfying

$$0 \leq \eta \leq 1, \qquad \eta \equiv 1 \text{ in } \Omega_0, \qquad \eta \equiv 0 \text{ in } \Omega \setminus U_2.$$

For $\delta > 0$ we let v_δ be the smooth function obtained by mollifying v. Here we assume that δ is so small that outside V_1 we have $v_\delta = (u_1)_\delta$. In other words, $\delta > 0$ is taken to be smaller than $\operatorname{dist}(K, \Omega \setminus V_1)$.

Now let us use the function

$$w_p = \eta^p (v_\delta - u_p) \in W_0^{1,p}(U_2)$$

as a test-function for the p-harmonic function u_p. By standard calculations, this yields

$$\int_{U_2} \eta^p |\nabla u_p|^p \, dx \leq \int_{U_2} \eta^p |\nabla u_p|^{p-1} |\nabla v_\delta| \, dx + p \int_{U_2} \eta^{p-1} |v_\delta - u_p| \, |\nabla u_p|^{p-1} |\nabla \eta| \, dx.$$

Using Hölder's inequality and the properties of the cut-off function η, we thus obtain

$$\left(\int_{U_2} \eta^p |\nabla u_p|^p \, dx \right)^{1/p} \leq \left(\int_{U_2} \eta^p |\nabla v_\delta|^p \, dx \right)^{1/p} + p \left(\int_{U_2 \setminus \Omega_0} |(u_1)_\delta - u_p|^p \, |\nabla \eta|^p \, dx \right)^{1/p}.$$

The second integral on the right hand side can be estimated as follows:

$$\int_{U_2 \setminus \Omega_0} |(u_1)_\delta - u_p|^p \, |\nabla \eta|^p \, dx \leq \left(\int_{U_2 \setminus \Omega_0} |(u_1)_\delta - u_1|^q \, dx \right)^{\frac{p}{q}} \left(\int_{U_2 \setminus \Omega_0} |\nabla \eta|^t \, dx \right)^{\frac{p}{t}} + \left(\int_{U_2 \setminus \Omega_0} |u_1 - u_p|^q \, dx \right)^{\frac{p}{q}} \left(\int_{U_2 \setminus \Omega_0} |\nabla \eta|^t \, dx \right)^{\frac{p}{t}}.$$

Here $t = qp/(q-p)$.

Let us now finish the proof. Since $u_p \to u_1$ in L^q, we obtain

(4.4)

$$\begin{aligned}\|\nabla u_1\|(\Omega_0) &\le \liminf_{p\to 1} \|\nabla u_p\|(\Omega_0) \le \liminf_{p\to 1}\left(\int_{U_2} \eta^p |\nabla u_p|^p\, dx\right)^{1/p} \\ &\le \int_{U_2} |\nabla v_\delta|\, dx + \left(\int_{U_2\setminus\Omega_0} |(u_1)_\delta - u_1|^q\, dx\right)^{\frac{1}{q}} \left(\int_{U_2\setminus\Omega_0} |\nabla \eta|^{\frac{q}{q-1}}\, dx\right)^{\frac{q-1}{q}} \\ &= \int_{U_1} |\nabla v_\delta|\, dx + \int_{U_2\setminus\overline{U}_1} |\nabla (u_1)_\delta|\, dx + C\left(\int_{U_2\setminus\Omega_0} |(u_1)_\delta - u_1|^q\, dx\right)^{\frac{1}{q}},\end{aligned}$$

where the constant $C > 0$ is independent of δ. Next we use the fact that regularization does not increase the total variation. More precisely, it follows from [19], Theorem 5.3.1, that for δ small enough

$$\|\nabla v_\delta\|(U_1) \le \|\nabla v\|(\Omega_0)$$

and

$$\int_{U_2\setminus\overline{U}_1} |\nabla (u_1)_\delta|\, dx \le \|\nabla u_1\|(V_2 \setminus \overline{V}_1) < \varepsilon.$$

Combining these estimates with (4.4) yields

$$\|\nabla u_1\|(\Omega_0) \le \|\nabla v\|(\Omega_0) + \varepsilon + C\left(\int_{U_2\setminus\Omega_0} |(u_1)_\delta - u_1|^q\, dx\right)^{\frac{1}{q}}.$$

Since $u_1 \in L^{n/(n-1)}_{loc}(\Omega)$ by the Sobolev embedding theorem and $1 < q < n/(n-1)$, we have that $(u_1)_\delta \to u_1$ in L^q as $\delta \to 0$. Hence

$$\|\nabla u_1\|(\Omega_0) \le \|\nabla v\|(\Omega_0).$$

Since $v = u_1$ in $\Omega_0 \setminus K$, (4.3) now follows. □

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