Riemannian geometry

Projects

Study the topic of the project from the sources given in the statement of the project and/or other sources. Write a report that gives a detailed proof of the theorem or a detailed description of the example studied in the project.

The project should ideally be returned by the beginning of February. We will discuss the topics on February 2 at 16 in MaD381.

1 (Flat manifolds)**.** Show that a Riemannian manifold M is locally Euclidean if its curvature tensor is $R = 0$. The proof is presented in [Lee1, Thm. 7.3]. The proof refers to [Boo] but it is better to study the normal form of commuting vector fields in [Lee2, Thm. 18.6].

2 (First variation)**.** Show that geodesics are critical points of the functional defined by path length. See [Lee1, Thm. 6.7]. The *energy* of a path $\gamma\colon I \to M$ is

$$E(\gamma) = \frac{1}{2}\int_I |\dot{\gamma}|^2$$

It is shown in [GHL, 2.96 ja III.B] that γ is also a critical point of the energy functional.

3. (Semi-Riemannian metrics and hyperbolic space) Minkowski space [1] is a semi-Riemannian manifold: it is the pair $\mathbb{M}^{n,1} = (\mathbb{R}^{n+1}, h)$ where

$$h = -dt^2 + \sum_{i=1}^{n}(dx^i)^2$$

is a Lorentz metric which is not positive definite. Show that the Lorentz metric induces a Riemannian metric on the submanifold

$$\mathscr{H}^n = \left\{(t,x) \in \mathbb{M}^{n,1} : -t^2 + \sum_{i=1}^{n}(x^i)^2 = -1\right\}$$

that makes $\mathscr{H}^n$ isometric with the hyperbolic space $\mathbb{H}^n$. Describe isometries of $\mathscr{H}^n$ using the Lorentz group $\mathrm{O}^+(n,1)$ and describe the geodesics of $\mathscr{H}^n$. See [Lee1, Prop. 3.5, Prop. 3.6, Prop. 5.14]

References [Lee1] and [Lee2] are available as ebooks through the library. There are copies of the other books in the library.

Viitteet

[Boo] W. M. Boothby. *An introduction to differentiable manifolds and Riemannian geometry*, volume 120 of *Pure and Applied Mathematics*. Academic Press, Inc., Orlando, FL, second edition, 1986.

[GHL] S. Gallot, D. Hulin, and J. Lafontaine. *Riemannian geometry*. Universitext. Springer-Verlag, Berlin, second edition, 1990.

[Lee1] J. M. Lee. *Riemannian manifolds*, volume 176 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1997. An introduction to curvature.

[Lee2] J. M. Lee. *Introduction to smooth manifolds*, volume 218 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2013.

[1] the space of special relativity