Riemannian geometry
Exercises 6, 8.12.2015

In problems 1 to 3, let g be the Riemannian metric

$$g = \frac{dx^2 + dy^2}{(x^2+y^2)\log^2\sqrt{x^2+y^2}}\,.$$

on the punctured disc $B(0,1) - \{0\}$.

1. Let $\mathbb{H}^2$ be the upper halfplane model of the hyperbolic plane. Let $J\colon \mathbb{C} \to \mathbb{C}$ be the mapping

$$J(z) = i\,z\,.$$

Show that the mapping $\operatorname{Exp} \circ \mathrm{J}|_{\mathbb{H}^2} \colon \mathbb{H}^2 \to \big(B(0,1) - \{0\}, g\big)$ is a local Riemannian isometry.

2. Show that the Riemannian manifold $\big(B(0,1) - \{0\}, g\big)$ is complete.

3. Describe the minimizing rays of the Riemannian manifold $(B(0,1) - \{0\}, g)$.

---

4. Show that the $(1,3)$-curvature tensor of a Riemannian manifold M is a $\mathscr{F}(M)$-multilinear mapping and that the $(0,4)$-curvature tensor is a $(0,4)$-tensor.

5. Let M be a Riemannian manifold and let $p \in M$. Show that

$$R(y,x,z,w) = R(x,y,w,z) = -R(x,y,z,w)$$

for all $x, y, z, w \in T_pM$.

6. Show that the cone

$$\{(x,y,z) \in \mathbb{E}^n : x^2 + y^2 = cz^2 : (x,y,z) \neq 0\}\,,$$

is a flat submanifold of the Riemannian manifold $\mathbb{E}^3$.

7. Let M be a Riemannian manifold and let $p \in M$. Let P be a 2-dimensional plane in T_pM. Let $x, y \in P$ be linearly independent. Show that the expression

$$K(P) = \frac{R(x,y,x,y)}{g(x,x)g(y,y) - g(x,y)^2}$$

is independent of the choice of x, y.

[6] Vihje: Spherical coordinates may turn out to be useful.