GROUPS AND GEOMETRY

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1. Euclidean and spherical geometry

1.1. **Metric spaces.** A function $d\colon X \times X \to [0, +\infty[$ is a *metric* in the nonempty set X if it satisfies the following properties

(1) $d(x,x) = 0$ for all $x \in X$ and $d(x,y) > 0$ if $x \neq y$,
(2) $d(x,y) = d(y,x)$ for all $x, y \in X$, and
(3) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in X$ (the *triangle inequality*).

The pair (X,d) is a *metric space.* Open and closed balls in a metric space, continuity of maps between metric spaces and other "metric properties" are defined in the same way as in Euclidean space, using the metrics of X and Y instead of the Euclidean metric.

If (X_1, d_1) and (X_2, d_2) are metric spaces, then a map $i\colon X \to Y$ is an *isometric embedding*, if

$$d_2(i(x), i(y)) = d_1(x,y)$$

for all $x, y \in X_1$. If the isometric embedding i is a bijection, then it is called an *isometry* between X and Y. An isometry $i\colon X \to X$ is called an isometry of X.

The isometries of a metric space X form a group $\mathrm{Isom}(X)$, the *isometry group* of X, with the composition of mappings as the group law.

A map $i\colon X \to Y$ is a *locally isometric embedding* if each point $x \in X$ has a neighbourhood U such that the restriction of i to U is an isometric embedding. A (locally) isometric embedding $i\colon I \to X$ is

(1) a *(locally) geodesic segment*, if $I \subset \mathbb{R}$ is a (closed) bounded interval,
(2) a *(locally) geodesic ray*, if $I = [0, +\infty[$, and
(3) a *(locally) geodesic line*, if $I = \mathbb{R}$.

Note that in Riemannian geometry, the definition of a geodesic line is different from the above: in general a Riemannian geodesic line is only a locally geodesic line according to our definition.

1.2. **Euclidean space.** Let us denote the *Euclidean inner product* of $\mathbb{R}^n$ by

$$(x|y) = \sum_{i=1}^{n} x_i y_i \,.$$

The *Euclidean norm* $\|x\| = \sqrt{(x|x)}$ defines the *Euclidean metric* $d(x,y) = \|x - y\|$. The triple $\mathbb{E}^n = (\mathbb{R}^n, (\cdot|\cdot), \|\cdot\|)$ is n-dimensional *Euclidean space.*

Euclidean space is a *geodesic metric space*: For any two distinct points $x, y \in \mathbb{E}^n$, the map $j_{x,y}\colon \mathbb{R} \to \mathbb{E}^n$,

$$j_{x,y}(t) = x + t\frac{y-x}{\|y-x\|},$$

is a geodesic line that passes through the points x and y. The restriction $j_{x,y}|_{[0,\|x-y\|]}$ is a geodesic segment that *connects* x to y: $j(0) = x$ and $j(\|x-y\|) = y$. In fact, this is the only geodesic segment that connects x to y up to replacing the interval of

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definition $[0, \|x-y\|]$ of the geodesic by $[a, a+\|x-y\|]$ for some $a \in \mathbb{R}$. More precisely: A metric space (X, d) is *uniquely geodesic*, if for any $x, y \in X$ there is exactly one geodesic segment $j\colon [0, d(x,y)] \to X$ such that $j(0) = x$ and $j(d(x,y)) = y$.

Proposition 1.1. *Euclidean space is uniquely geodesic.*

Proof. If g is a geodesic segment that connects x to y and z is a point in the image of g, then, by definition, $\|x-z\| + \|z-y\| = \|x-y\|$. But, using the Cauchy inequality from linear algebra, it is easy to see that the Euclidean triangle inequality becomes an equality if and only if z is in the image of the linear segment $j|_{[0,\|x-y\|]}$. $\square$

If a metric space X is uniquely geodesic and $x, y \in X$, $x \neq y$, we denote the image of the unique geodesic segment connecting x to y by $[x, y]$.

A *triangle* in Euclidean space consists of three points $A, B, C \in \mathbb{E}^n$ (the *vertices*) and of the three *sides* $[A,B]$, $[B,C]$ and $[C,A]$. Let the lengths of the sides be, in the corresponding order, c, a and b, and let the angles between the sides at the vertices A, B and C be α, β and γ. These quantities are connected via the

Euclidean law of cosines.

$$c^2 = a^2 + b^2 - 2ab\cos\gamma\ .$$

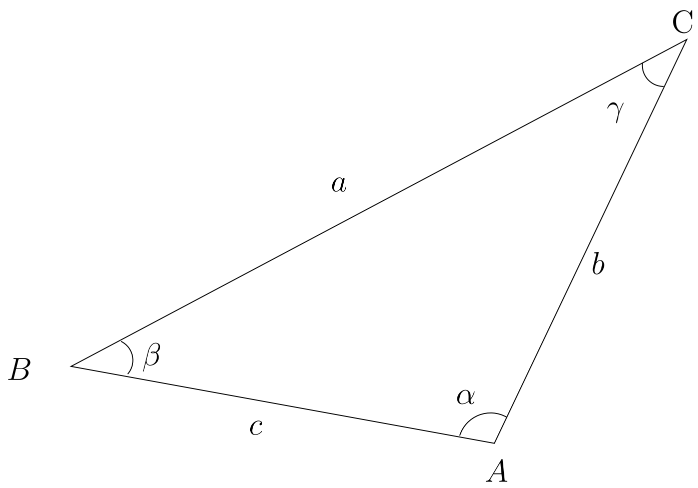

FIGURE 1

Proof. The proof is linear algebra:

$$\begin{aligned} c^2 = \|B-A\|^2 = \|B-C+C-A\|^2 &= b^2 + 2(B-C|C-A) + a^2 \\ = b^2 + 2(B-C|C-A) + a^2 &= b^2 - 2ab\cos\gamma + a^2\ . \end{aligned}$$ $\square$

The law of cosines can be proved without knowing that $\mathbb{E}^n$ is uniquely geodesic. In fact, using the law of cosines, it is easy to prove that Euclidean space is uniquely geodesic, compare with the case of the sphere treated in section 1.3 .

1.3. **The sphere.** The unit sphere in $(n-1)$-dimensional Euclidean space is

$$\mathbb{S}^n = \{x \in \mathbb{E}^{n+1} : \|x\| = 1\} .$$

Let us show that the *angle distance*

$$d(x, y) = \arccos(x|y) \in [0, \pi]$$

is a metric. In order to do this, we will use the analog of the Euclidean law of cosines, but first we have to define the objects that are studied in spherical geometry.

Each 2-dimensional linear subspace (plane) $T \subset \mathbb{R}^{n+1}$ intersects $\mathbb{S}^n$ in a *great circle*. If $A \in \mathbb{S}^n$ and $u \in \mathbb{S}^n$ is orthogonal to A ($u \in A^\perp$), then the path $j_{A,u} : \mathbb{R} \to \mathbb{S}^n$,

$$j_{A,u}(t) = A\cos t + u \sin t ,$$

parametrises the great circle $\langle A, u\rangle \cap \mathbb{S}^n$, where (A, u) is the linear span of A and u. The vectors A and u are linearly independent, so $\langle A, u\rangle$ is a 2- plane.

If $A, B \in \mathbb{S}^n$ such that $B \neq \pm A$, then there is a unique plane that contains both points. Thus, there is unique great circle that contains A and B, in the remaining cases, there are infinitely many such planes. The great circle is parametrised by the map $j_{A,u}$, with

$$u = \frac{B - (B|A)A}{\|B - (B|A)A\|} = \frac{B - (A|B)A}{\sqrt{1 - (A|B)^2}} . \tag{1}$$

Now $j(0) = A$ and $j(d(A, B)) = B$.

If $B = -A$, then there are infinitely many great circles through A and B: the map $j_{A,u}$ parametrises a great circle through A and B for any $u \in A^\perp$.

We call the restriction of any $j_{A,u}$ as above to any compact interval $[0, s]$ a *spherical segment*, and u is called the *direction* of $j_{A,u}$. Once we have proved that d is a metric, it is immediate that a spherical segment is a geodesic segment.

A triangle in $\mathbb{S}^n$ is defined as in the Euclidean case but now the sides of the triangle are the spherical segments connecting the vertices.

Let $j_{A,u}([0, d(C, A)])$ be the side between C and A, and let $j_{A,v}([0, d(C, B)])v$ be the side between C and B. The angle between $j_{A,u}([0, d(C, A)])$ and $j_{A,v}([0, d(C, B)])$ is $\arccos(u|v)$, which is the angle at A between the sides $j_{A,u}([0, d(A, B)])$ and $j_{A,v}([0, d(A, B)])$ in the ambient space $\mathbb{E}^{n+1}$. Now we can state and prove the

Spherical law of cosines.

$$\cos c = \cos a \cos b + \sin a \sin b \cos \gamma .$$

Proof. Let u and v be the initial tangent vectors of the spherical segments $j_{C,u}$ from C to A and $j_{C,v}$ from C to B. As u and v are orthogonal to C, we have

$$\begin{aligned}\cos c = (A|B) &= (\cos(b)C + \sin(b)u| \cos(a)C + \sin(a)v)\\ &= \cos(a)\cos(b) + \sin(b)\sin(a)(u|v) .\end{aligned}$$

□

Proposition 1.2. *The angle metric is a metric on* $\mathbb{S}^n$.

Proof. Clearly, the triangle inequality is the only property that needs to be checked to show that the angle metric is a metric. Let $A, B, C \in \mathbb{S}^n$ be three distinct points and use the notation introduced above for triangles. The function

$$\gamma \mapsto f(\gamma) = \cos(a)\cos(b) + \sin(a)\sin(b)\cos(\gamma)$$

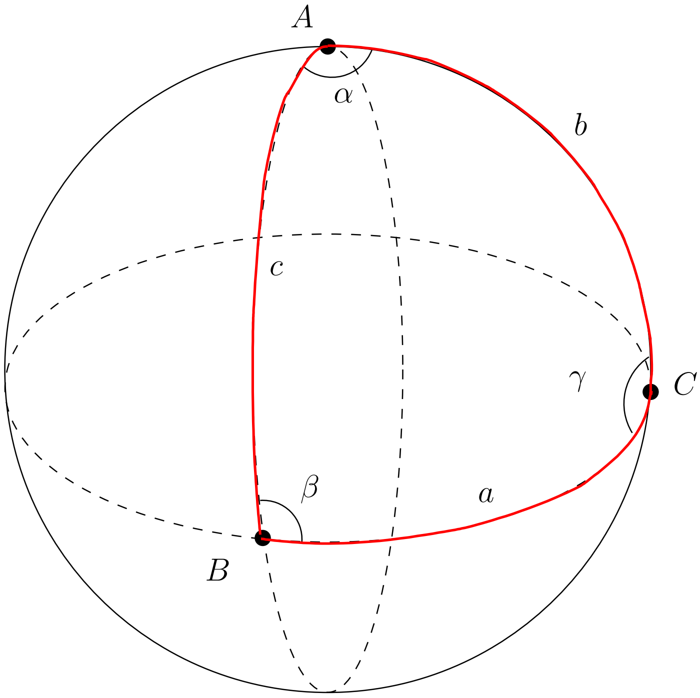

FIGURE 2

is strictly decreasing on the interval $[0, \pi]$, and

$$f(\pi) = \cos(a)\cos(b) - \sin(b)\sin(a) = \cos(a+b)\,.$$

Thus, the law of cosines implies that for all $\gamma \in [0, \pi]$, we have

$$\cos(c) = \cos(a)\cos(b) + \sin(a)\sin(b)\cos(\gamma) \geq \cos(a+b),$$

which implies $c \leq a + b$. Thus, the angle metric is a metric. □

Proposition 1.3. *$(\mathbb{S}^n, d)$ is a geodesic metric space. If $d(A, B) < \pi$, then there is a unique geodesic segment from A to B.*

Proof. If $x, y \in \mathbb{S}$ with $y \neq \pm x$, then the spherical segment with direction given by the equation (1) is a geodesic segment that connects x to y. If the points are antipodal, then it is immediate from the expression of the spherical segment that $j_{x,u}(\pi) = -x$. Thus, in this case there are infinitely many geodesic segments connecting x to y.

If j is a geodesic segment connecting A to B, then for C in $j([0, d(A, B)])$ satisfies

$$d(A, C) + d(C, B) = d(A, B)$$

by definition of a geodesic segment. Equality holds in the triangle inequality if and only if $\gamma = \pi$. In this case, all the points A, B and C lie on the same great circle and C is contained in the side connecting A to B. Thus, the spherical segments are the only geodesic segments connecting A and B. If $A \neq \pm B$, then there is exactly one 2-plane containing both points. This proves the second claim. □

Note that the sphere has no geodesic lines or rays because the diameter of the sphere is π.

1.4. **More on cosine and sine laws.** The law of cosines implies that a triangle in $\mathbb{E}^n$ or $\mathbb{S}^n$ is uniquely determined up to an isometry of the space, if the lengths of the three sides are known. In Euclidean space the angles are given by

$$\cos\gamma = \frac{a^2+b^2-c^2}{2ab}$$

and the corresponding equations for α and β obtained by permuting the sides and angles, and in the sphere we have

$$\cos\gamma = \frac{\cos c - \cos a\cos b}{\sin a \sin b}\,.$$

In Euclidean space, the three angles of a triangle do not determine the triangle uniquely but in $\mathbb{S}^n$ the angles determine a triangle uniquely. This is the content of

The second spherical law of cosines.

$$\cos c = \frac{\cos\alpha\cos\beta + \cos\gamma}{\sin\alpha\sin\beta}\,.$$

This formula follows from the first law of cosines by manipulation:

Proof. The first law of cosines implies

$$\sin^2\gamma = 1-\cos^2\gamma = \frac{1+2\cos a\cos b\cos c - (\cos^2 + \cos^2 b + \cos^2 c)}{\sin^2 a\sin^2 b} = \frac{D}{\sin^2 a\sin^2 b}\,,$$

and D is symmetric in a, b and c. Thus, using the law of cosines, we get

$$\frac{\cos\alpha\cos\beta+\cos\gamma}{\sin\alpha\sin\beta} = \frac{\frac{\cos a - \cos b\cos c}{\sin b\sin c}\frac{\cos b - \cos a\cos c}{\sin a\sin c} + \frac{\cos c - \cos a\cos b}{\sin a\sin b}}{\frac{D}{\sin a\sin b\sin^2 c}} = \cos c\,. \quad \square$$

The Euclidean law of sines. In Euclidean geometry, the relation

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

holds for any triangle. An analogous result holds for the sphere:

The spherical law of sines.

$$\frac{\sin a}{\sin\alpha} = \frac{\sin b}{\sin\beta} = \frac{\sin c}{\sin\gamma}\,.$$

Proof. We leave the Euclidean case as an exercise and prove the spherical sine law: In the proof of the second law of cosines we saw that he first law of cosines implies that

$$\left(\frac{\sin c}{\sin\gamma}\right)^2 = \frac{\sin^2 a\sin^2 b\sin^2 c}{D}\,.$$

The claim follows because this expression is symmetric in a, b and c. $\square$

1.5. **Isometries.** We will now study the isometries of Euclidean space and the sphere more closely. We begin by introducing some convenient terminology: A group G *acts* on a metric space X *by isometries* if there is a homomorphism $\Phi\colon G \to \mathrm{Isom}(X)$. Normally, one ignores the homomorphism Φ in notation, and writes $g(x)$ or $g\cdot x$ or something similar to mean $\Phi(g)(x)$. Similarly, one defines action by homomorphisms in a topological space, a linear action in a vector space etc.

The *(Euclidean) orthogonal group* of dimension n is

$$\begin{aligned}\mathrm{O}(n) &= \{A \in \mathrm{GL}_n(\mathbb{R}) : (Ax|Ay) = (x|x) \;\text{ for all } x,y\in\mathbb{E}^n\}\\ &= \{A \in \mathrm{GL}_n(\mathbb{R}) : {}^T\!AA = I_n\}\,.\end{aligned}$$

It is easy to check that $\mathrm{O}(n)$ acts by isometries on $\mathbb{E}^n$ and on $\mathbb{S}^{n-1}$ for any $n\in\mathbb{N}$. Recall the following basic result from linear algebra:

Lemma 1.4. *An $n \times n$-matrix $A = (a_1, \ldots, a_n)$ is in $\mathrm{O}(n)$ if and only if the vectors $a_1, \ldots, a_n$ form an orthonormal basis of $\mathbb{E}^n$.* □

For any $b \in \mathbb{R}^n$, let $t_b(x) = x + b$ be the translation by b. The *translation group*

$$\mathrm{T}(n) = \{t_b : b \in \mathbb{R}^n\}$$

acts by isometries on $\mathbb{E}^n$. Orthogonal maps and translations generate the *Euclidean group*

$$\mathrm{E}(n) = \{x \mapsto Ax + b : A \in \mathrm{O}(n), b \in \mathbb{R}^n\}$$

which acts by isometries on $\mathbb{E}^n$.

If a group G acts on a space X, and x is a point in X, the set

$$G(x) = \{g(x) : g \in g\}$$

is the *G-orbit* of x. The action of a group is said to be *transitive* if $G(x) = X$ for some (and therefore for any) $x \in X$.

Proposition 1.5. *(a)* $\mathrm{E}(n)$ *acts transitively by isometries on* $\mathbb{E}^n$*. In particular,* $\mathrm{Isom}(\mathbb{E}^n)$ *acts transitively on* $\mathbb{E}^n$*.*

(b) $\mathrm{O}(n+1)$ *acts transitively by isometries on* $\mathbb{S}^n$*. In particular,* $\mathrm{Isom}(\mathbb{S}^n)$ *acts transitively on* $\mathbb{S}^n$*.*

Proof. (a) The Euclidean group of $\mathbb{E}^n$ contains the group of translations $\mathrm{T}(n)$ as a subgroup. Clearly, this subgroup acts transitively.

(b) Transitivity follows from the fact that any element of $\mathbb{S}^n$ can be taken as the first element of an orthogonal basis of $\mathbb{E}^n$ or, equivalently, as the first column of an orthogonal matrix. □

An *affine hyperplane* of $\mathbb{E}^n$ is a subset of the form

$$H = H(P, u) = P + u^\perp ,$$

where $P, u \in \mathbb{E}^n$ and $\|u\| = 1$. The *reflection* in H is the map

$$r_H(x) = x - 2(x - P|u)u .$$

Reflections are very useful isometries, the following results give some of their basic properties.

Proposition 1.6. *Let H be an hyperplane in $\mathbb{E}^n$. Then*

(1) $r_H \circ r_H$ is the identity.

(2) $r_H \in \mathrm{E}(n)$. In particular, if $0 \in H$, then $r_H \in \mathrm{O}(n)$.

(3) $d(r_H(x), y) = d(x, y)$ for all $x \in \mathbb{E}^n$ and all $y \in H$.

(4) The fixed point set of r_H is H.

Proof. We will prove (3) and leave the rest as exercises. Let $x \in \mathbb{E}^n$ and $y \in H$. We have $r_H(x) = x - 2(x - y|u)u$, which implies

$$\begin{aligned} d(r_H(x), y)^2 &= (r_H(x) - y | r_H(x) - y) = (x - y - 2(x - y|u)u | x - y - 2(x - y|u)u) \\ &= (x - y|x - y) - 4(x - y|(x - y|u)u) + 4((x - y|u)u|(x - y|u)u) \\ &= (x - y|x - y) = d(x, y)^2 . \end{aligned}$$

□

The *bisector* of two distinct points p and q in $\mathbb{E}^n$ is the affine hyperplane

$$\mathrm{bis}(p, q) = \{x \in \mathbb{E}^n : d(x, p) = d(x, q)\} = \frac{p+q}{2} + (p - q)^\perp .$$

Proposition 1.7. *(1) If $r_H(x) = y$ and $x \notin H$, then $H = \operatorname{bis}(x, y)$.*
(2) If $p, q \in \mathbb{E}^n$, $p \neq q$, then $r_{\operatorname{bis}(p,q)}(p) = q$.
(3) Let $\phi \in \operatorname{Isom}(\mathbb{E}^n)$, $\phi \neq \operatorname{id}$. If $a \in \mathbb{E}^n$, $\phi(a) \neq a$, then the fixed points of ϕ are contained in $\operatorname{bis}(a, \phi(a))$.
(4) Let $\phi \in \operatorname{Isom}(\mathbb{E}^n)$, $\phi \neq \operatorname{id}$. If H is a hyperplane such that $\phi|_H$ is the identity, then $\phi = r_H$.

Proof. (1) follows from Proposition 1.6(3).
(2) From the definitions we get

$$r_{\operatorname{bis}(p,q)}(p) = p - 2(p - \frac{p+q}{2}|p-q)\frac{p-q}{\|p-q\|^2} = q\,.$$

(3) If $\phi(b) = b$, then $d(a, b) = d(\phi(a), \phi(b)) = d(\phi(a), b)$, so that $b \in \operatorname{bis}(a, \phi(a))$.
(4) Let $a \notin H$. We know that $H = \operatorname{bis}(a, \phi(a))$ and therefore that $r_H(a) = \phi(a)$. But this holds for all $a \notin H$. □

Next, we want to prove that all isometries of Euclidean space $\mathbb{E}^n$ are affine transformations with an orthogonal linear part.

Theorem 1.8. $\operatorname{Isom}(\mathbb{E}^n) = \mathrm{E}(n)$.

The idea of the proof is to show that each isometry of $\mathbb{E}^n$ is the composition of reflections in affine hyperplanes. In order to do this, we show that the isometry group has a stronger transitivity property than what was noted above.

Proposition 1.9. *Let $p_1, p_2, \ldots, p_k, q_1, q_2, \ldots, q_k \in \mathbb{E}^n$ be points that satisfy*

$$d(p_i, p_j) = d(q_i, q_j)$$

for all $i, j \in \{1, 2, \ldots, k\}$. Then, there is an isometry $\phi \in \operatorname{Isom}(\mathbb{E}^n)$ such that $\phi(p_i) = q_i$ for all $i \in \{1, 2, \ldots, k\}$. Furthermore, the isometry ϕ is the composition of at most k reflections in affine hyperplanes.

Proof. We construct the isometry by induction. If $p_1 = q_1$, let ϕ_1 be the identity, otherwise, let ϕ_1 be the reflection in the bisector of p_1 and q_1. Let $m > 1$ and assume that there is an isometry ϕ_m such that $\phi(p_i) = q_i$ for all $i \in \{1, 2, \ldots, m\}$, which is the composition of at most m reflections.

Assume that $\phi_m(p_{m+1}) \neq q_{m+1}$. Now, $q_1, \ldots q_m \in \operatorname{bis}(\phi_m(p_{m+1}), q_{m+1})$: For each $1 \leq i \leq m$, we have

$$d(q_i, \phi_m(p_{m+1})) = d(\phi_m(p_i), \phi_m(p_{m+1})) = d(p_i, p_{m+1}) = d(q_i, q_{m+1})\,.$$

The map

$$\phi_{m+1} = r_{\operatorname{bis}(\phi_m(p_{m+1}), q_{m+1})} \circ \phi_m$$

satisfies $\phi_{m+1}(p_i) = q_i$ for all $1 \leq i \leq m+1$. □

Note that Proposition 1.9 implies that if T and T' are two triangles in $\mathbb{E}^n$ with equal sides, then there is an isometry ϕ of $\mathbb{E}^n$ such that $\phi(T) = T'$.

Proof of Theorem 1.8. Consider the set $\{0, e_1, \ldots, e_n\}$ in $\mathbb{E}^n$. Note that this set is not contained in any affine hyperplane. $\mathbb{R}^n$.

Let $\phi \in \operatorname{Isom}(\mathbb{E}^n)$. Proposition 1.9 implies that there is an isometry $\phi_0 \in \mathrm{O}(n)$ such that $\phi_0(\phi(e_i)) = e_i$ for all $1 \leq i \leq m$ and $\phi_0(\phi(0)) = 0$. Since the set of fixed points of $\phi_0 \circ \phi$ contains the points $0, e_1, \ldots, e_n$, the fixed point set of ϕ_0 is not contained in any affine hyperplane. Proposition 1.7 implies that $\phi_0 \circ \phi$ is the identity map. Thus, $\phi = \phi_0^{-1}$. In particular, $\phi \in \mathrm{O}(n)$, which is all we needed to show. □

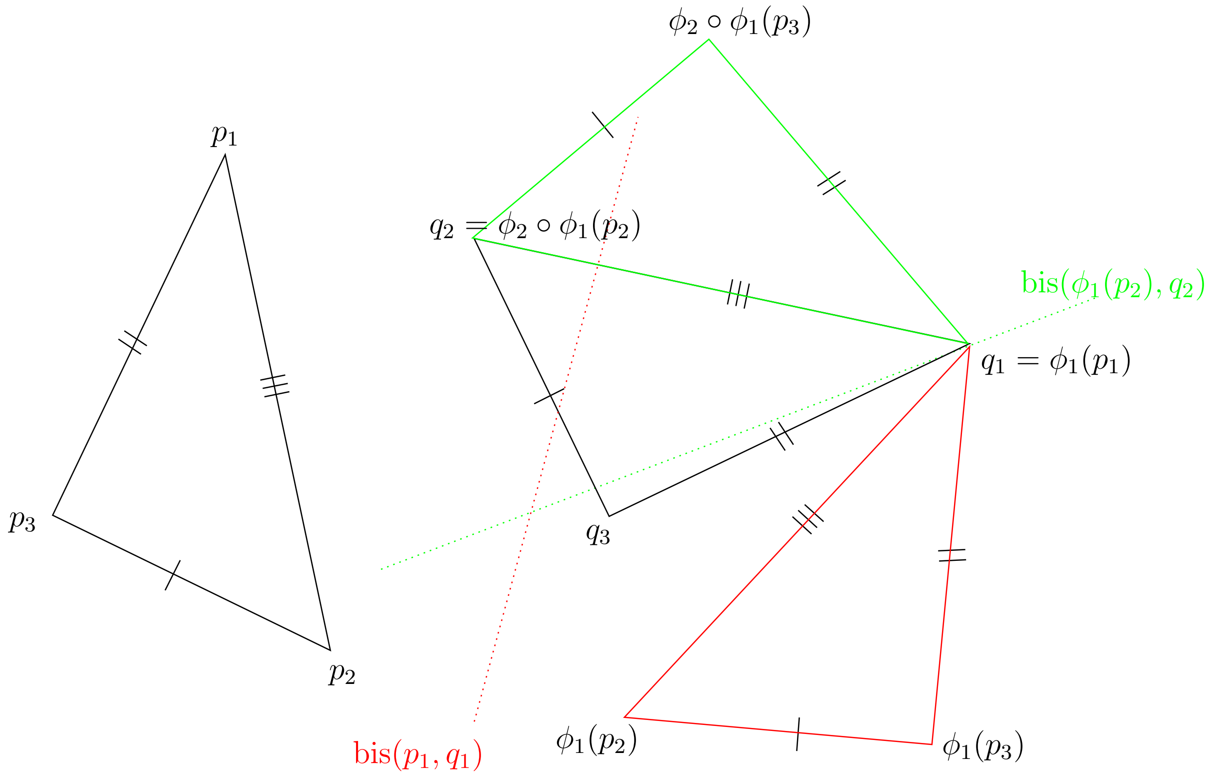

FIGURE 3

In the same way, we can show that the analogous result for the sphere. The proof works as in the Euclidean case once we have defined hyperplanes and bisectors in the appropriate, natural manner.

Let H_0 be a linear hyperplane in $\mathbb{E}^n$. The intersection $H = H_0 \cap \mathbb{S}^n$ is a *hyperplane* of $\mathbb{S}^n$. Note that each hyperplane of $\mathbb{S}^n$ is isometric with $\mathbb{S}^{n-1}$. The *reflection* r_H in H is the restriction of the reflection in H_0 to the sphere: $r_H = r_{H_0}|_{\mathbb{S}^n}$. The bisector of points $p, q \in \mathbb{S}^n$ is the hyperplane

$$\operatorname{bis}(p, q) = \{x \in \mathbb{S}^n : d(x, p) = d(x, q)\}\,.$$

Theorem 1.10. $\operatorname{Isom}(\mathbb{S}^n) = \mathrm{O}(n+1)$ □

Corollary 1.11. *Any isometry of $\mathbb{E}^n$ and $\mathbb{S}^n$ can be represented as the composition of at most $n+1$ reflections.* □

If a group G acts on a space X A is a nonempty subset of X, the *stabiliser* of A in G is

$$\operatorname{Stab}_G A = \{g \in G : gA = A\}\,.$$

Clearly, stabilisers are subgroups of G.

Proposition 1.12. *The stabiliser in* $\operatorname{Isom}(\mathbb{E}^n)$ *of any point $x \in \mathbb{E}^n$ is isomorphic to* $\mathrm{O}(n)$. *The stabiliser in* $\operatorname{Isom}(\mathbb{S}^n)$ *of any point $x \in \mathbb{S}^n$ is isomorphic to* $\mathrm{O}(n)$.

Proof. All point stabilisers are conjugate in $\operatorname{Isom}(\mathbb{E}^n)$ so it suffices to consider 0 for which the claim is obvious. We leave the spherical case to the reader. □

Proposition 1.13. *(1) Each affine k-plane of $\mathbb{E}^n$ is isometric with $\mathbb{E}^k$. For each k-plane P, there is an isometry $\phi \in \operatorname{Isom}(\mathbb{E}^n)$ such that*

$$\phi(P) = \{x \in \mathbb{E}^n : x^{k+1} = x^{k+2} = \cdots = x^n\}\,.$$

(2) Each k-plane of $\mathbb{S}^n$ is isometric with $\mathbb{S}^k$. For each k-plane P, there is an isometry $\phi \in \mathrm{Isom}(\mathbb{S}^n)$ such that

$$\phi(P) = \{x \in \mathbb{S}^n : x^{k+2} = x^{k+3} = \cdots = x^{n+1}\} .$$

Proof. This is a direct generalisation of Proposition 1.5. $\square$

2. Symmetries and finite subgroups

Proposition 2.1. *Let $G < \mathrm{Isom}(\mathbb{E}^n)$ be a finite subgroup. Then there is a point $p \in \mathbb{E}^n$ which is a common fixed point of all elements of G.*

Proof. Let $q \in \mathbb{E}^n$. The orbit $Gq = \{gq : g \in G\} = \{q_1, q_2, \dots, q_m\}$ is a finite set, on which G acts by permutations. Let

$$q_0 = \frac{q_1 + q_2 + \cdots + q_m}{m} .$$

Now, for any $g = A \circ T_b$ with $A \in \mathrm{O}(n)$ and $T_b \in \mathrm{T}(n)$, we have

$$g(q_0) = Aq_0 + b = \frac{Aq_1 + \cdots + Aq_m}{m} + b = \frac{Aq_1 + m + \cdots + Aq_m + m}{m} = q_0 . \quad \square$$

Proposition 2.1 implies that in order to understand the finite subgroups of the Euclidean group $\mathrm{E}(n)$, we can restrict to study finite subgroups of the orthogonal group $\mathrm{O}(n)$

In certain considerations it is convenient to restrict to the *special orthogonal group*

$$\mathrm{SO}(n) = \{A \in \mathrm{O}(n) : \det A = 1\} ,$$

which is a normal subgroup index 2 of $\mathrm{O}(n)$, being the kernel of the homomorphism $\det\colon \mathrm{O}(n) \to \{-1, 1\}$.

2.1. Symmetries of regular polygons.

Let $A \in \mathrm{O}(2)$. The columns of A form an orthonormal basis of $\mathbb{E}^2$. If we write the first column as $\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$, then orthogonality implies that the second column is either $\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}$ or $\begin{pmatrix}\sin\theta\\ -\cos\theta\end{pmatrix}$. Therefore, there are exactly two kinds of orthogonal maps of the plane: the rotation by θ,

$$R_\theta = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix} \in \mathrm{SO}(2)$$

and the reflection in the line $L = (-\cos\frac{\theta}{2}, \sin\frac{\theta}{2})^\perp$,

$$S_\theta = \begin{pmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{pmatrix} \in \mathrm{O}(2) - \mathrm{SO}(2) .$$

The rotations form the normal subgroup $\mathrm{SO}(2)$of $\mathrm{O}(2)$.

Remark 2.2. (1) It is easy to check that $S_\theta = R_\theta S_0$.
(2) Using complex numbers, R_θ is multiplication by $e^{i\theta}$ and the reflection S_θ is the mapping $S_\theta(z) = e^{i\theta}\bar{z}$.

Any hyperplane P divides $\mathbb{E}^n$ (or $\mathbb{S}^n$) into two closed *halfspaces* which are the closures of the two components of the complement of P. A (spherical) polytope is a compact intersection with nonempty interior of a finite collection of (spherical) halfspaces. In dimension $n = 2$, polytopes are called *polygons* and in dimension $n = 3$, *polyhedra.*

Three halfplanes in $\mathbb{E}^2$, whose pairwise intersections are nonempty and not halfplanes, define a polygon with three *sides* that are geodesic segments, rays or lines.

On the other hand, any nondegenerate triangle in $\mathbb{E}^2$ defines a polygon: The side $[A, B]$ is contained in a geodesic line L_C. Let H_C be the closed halfplane determided by L_C that contains C, and define the halfplanes H_A and H_B in the same way. Now, $P = H_A \cap H_B \cap H_C$ is a polygon, which we also refer to as a triangle.

A polygon is *regular* if all its edges have equal lengths and all the interior angles are equal. It is easy to construct a regular polygon: For any $n \geq 3$ take the vertices to be the nth roots of unity $e^{k\frac{2pi}{n}i}$ and connect them in succession in the order they appear on the unit circle to form the sides of the regular polygon P_n. More precisely, for each $k \in \{0, 1, \dots, n-1\}$, let H_k be the half plane containing 0 determined by the affine line through $e^{k\frac{2pi}{n}i}$ and $e^{(k+1)\frac{2pi}{n}i}$.

Now, $R_{\frac{2\pi}{n}}$ generates a cyclic group of order n of transformations that rotate P_n to itself. Each reflection $S_{\frac{k2\pi}{n}}$ maps P_n to itself, reversing the orientation. The group of symmetries of the polygon P_n is the *dihedral group* D_n. The cyclic group $\langle R_{\frac{2\pi}{n}} \rangle = D_n \cap \mathrm{SO}(2)$ is again a normal subgroup of index 2 in D_n. The dihedral group D_n has $2n$ elements and therefore some authors prefer the notation D_{2n}.

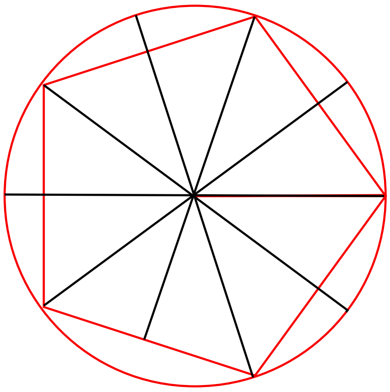

FIGURE 4

Proposition 2.3. *A finite subgroup of* $\mathrm{O}(2)$ *is either a cyclic group, the Klein four-group or the symmetry group of a regular polygon.*

Proof. There are infinitely many cyclic subgroups of order 2 and all groups of order 2 are cyclic. Assume now that the group G has more than 3 elements. Consider first the case $G < \mathrm{SO}(2)$. Let

$$\phi_G = \min \left\{ \phi \in]0, 2\pi] : R_\phi \in G \right\}.$$

The finiteness of G implies that $\phi_G > 0$. For any $R_\theta \in G$, $\theta = k\phi_G + \psi$ for some $0 \leq \psi < \phi_G$. But minimality of ϕ_G implies $\psi = 0$. Thus $R_\theta = R_{\phi_G}^k$. As this holds for any element in G, $G = \langle R_{\phi_G} \rangle$ is a cyclic group.

If G is not contained in $\mathrm{SO}(2)$, the first part of the proof implies that $H = G \cap \mathrm{SO}(2)$ is a cyclic group. Furthermore, H is a normal subgroup of index 2. The mappings $R_\theta \circ S_0$ are all different for different $\theta = k\frac{2\pi}{n}$, $n \in \{0, 1, \dots, n-1\}$. Thus, $G = D_n$ if $n \geq 2$ and the Klein four-group if $n = 2$. $\square$

2.2. **Actions of finite groups on finite sets.** Let us consider finite group actions in a more general situation. For any nonempty set X, let $\mathscr{S}(X)$ be the *permutation group* of X, that is, the group of bijections of X to itself. A group G *acts* on a set X if there is a homomorphism $\Phi \colon G \to \mathscr{S}(X)$. We denote the *quotient* of the set X by the action of G by

$$G \backslash X = \{Gx : x \in X\}.$$

For each $a \in Gx$, the stabilizer $\operatorname{Stab}_G a$ is isomorphic with $\operatorname{Stab}_G x$. In particular, the cardinality of stabilizers is constant along any orbit.

The set of *fixed points* of $g \in G$ is

$$\operatorname{fix}(g) = \{x \in X : gx = x\}.$$

The following are basic results on group actions on finite sets.

Proposition 2.4. *Let G be a finite group that acts on a finite nonempty set X.*
(1) (The class formula) Let A be a subset of X that intersects each G-orbit exactly once.

$$\#X = \sum_{a \in A} \frac{\#G}{\# \operatorname{Stab}_G a}.$$

(2) (The orbit stabilizer lemma) For each $x \in X$, the map $gx \mapsto g \operatorname{Stab}_G(x)$ is a bijection between the sets Gx and $G/\operatorname{Stab}_G(x)$. In particular,

$$\#G = \# \operatorname{Stab}_G(x) \#(Gx)$$

for all $x \in X$. □
(3) (The orbit counting lemma) The number of distinct orbits equals the average number of fixed points of the elements of G:

$$\#G \backslash X = \frac{\sum_{g \in G} \# \operatorname{fix}(g)}{\#G}.$$

Proof. We prove (1) and (3) and leave (2) as an exercise.
(1) The number of points in each orbit Gx is $\#G / \# \operatorname{Stab}_G(x)$. Now

$$\#X = \sum_{a \in A} \#Ga,$$

and the claim follows.
(3) Let us count the elements of

$$Y = \{(g, x) \in G \times X : gx = x\}.$$

in two ways. On the one hand, adding the cardinalities of fixed point sets of the elements of G, we get,

$$\#Y = \sum_{g \in G} \# \operatorname{fix}(g).$$

On the other hand, adding there cardinalities of the stabilisers of points of X, we get

$$\begin{aligned} \#Y &= \sum_{x \in X} \# \operatorname{Stab}_G(x) = \sum_{Gx \in G \backslash X} \sum_{z \in Gx} \# \operatorname{Stab}_G(z) \\ &= \sum_{Gx \in G \backslash X} \#(Gx) \# \operatorname{Stab}_G(x) = \sum_{Gx \in G \backslash X} \#G = \#(G \backslash X) \#G \end{aligned}$$

by the orbit stabilizer lemma. The claim follows by equating the results of the two counts. □

Proposition 2.4 is also known as Burnside's lemma.

2.3. **Symmetries of Platonic solids.** As in the 2-dimensional case, $\mathrm{SO}(n)$ is a normal subgroup of $\mathrm{O}(n)$ of index 2. The transformations $A \in \mathrm{O}(3) - \mathrm{SO}(3)$ are all of the form $A_0 J$ with $J = \mathrm{diag}(1, 1, -1)$. The following property simplifies the treatment of $\mathrm{SO}(3)$:

Proposition 2.5. *The nonidentity elements of* $\mathrm{SO}(3)$ *are rotations* $R_{v,\theta}$ *by* $\theta \in]0, 2\pi[$ *about an axis given by a unit vector* $v \in \mathbb{S}^2$. □

The classification of finite subgroups of $\mathrm{SO}(3)$ is more complicated than for $\mathrm{SO}(2)$. To begin with, note that $\mathrm{SO}(2)$ can be seen as a subgroup of $\mathrm{SO}(3)$, it is the image of the injective homomorphism $A \mapsto \mathrm{diag}(A, 1)$. Thus, we see that the cyclic groups of order n for any n appear as finite subgroups of $\mathrm{SO}(3)$.

Furthermore, it is easy to check that if a regular polygon is isometrically embedded in $\mathbb{E}^3$, then the reflections of its dihedral group can be extended to half-turns in $\mathbb{E}^n$ preserving the embedded polygon. Thus, dihedral groups also appear as finite subgroups of $\mathrm{SO}(3)$.

Let us show that there are only finitely many further possibilities up to conjugation.

Let $G < \mathrm{SO}(3)$ be finite. Let X be the union of the fixed point sets in $\mathbb{S}^2$ of the nonidentity elements of G. Each such element seen is a rotation of $\mathbb{E}^3$ and it fixes 2 points in $\mathbb{S}^2$. The group G acts on X by permutations, and we can apply the results of Subsection 2.2.

Noting that the identity fixes all points of X, the Orbit counting lemma and the Class formula imply

$$\begin{aligned}(2)\qquad \#(G\backslash X) &= \frac{2(\#G-1)+\#X}{\#G} = \frac{2(\#G-1)+\#G\sum_{Gx\in G\backslash X}\frac{1}{\#\,\mathrm{Stab}_G\,x}}{\#G}\\ &= 2(1-\frac{1}{\#G}) + \sum_{Gx\in G\backslash X}\frac{1}{\#\,\mathrm{Stab}_G\,x}\,.\end{aligned}$$

Thus,

$$(3)\quad 2(1-\frac{1}{\#G}) = \#(G\backslash X) - \sum_{Gx\in G\backslash X}(1-\frac{1}{\#\,\mathrm{Stab}_G\,x}) = \sum_{Gx\in G\backslash X}(1-\frac{1}{\#\,\mathrm{Stab}_G\,x})$$

Since G has more than 2 elements, we have $1 \le 2(1-\frac{1}{\#G}) < 2$ and since each point-stabiliser has at least 2 elements, we have $\frac{1}{2} \le 1 - \frac{1}{\#\,\mathrm{Stab}_G\,x} < 1$. Thus, the only possible values of $\#\,\mathrm{Stab}_G\,x$ are 2 and 3.

If $\#G\backslash X = 2$, then the first equality of equation (2) implies $\#X = 2$. Thus, all group elements fix the same axis, and the group G is cyclic.

If $\#G\backslash X = 3$, then equation (3) gives

$$(4)\qquad 1+\frac{2}{\#G} = \frac{1}{\#\,\mathrm{Stab}_G\,x_1} + \frac{1}{\#\,\mathrm{Stab}_G\,x_2} + \frac{1}{\#\,\mathrm{Stab}_G\,x_3}\,.$$

The only possibilities for the cardinalities of the stabilisers such that the sum of their inverses is greater than 1 are $(2, 2, n)$, $(2, 3, 3)$, $(2, 3, 4)$ and $(2, 3, 5)$. We call these triples the *signatures* of the finite groups of symmetries.

The above considerations are summarised in

Proposition 2.6. *Let G be a finite subgroup of* $\mathrm{SO}(3)$. *Then G is cyclic or a the signature of G is one of the following* $(2, 2, n)$ *with* $n \ge 2$, $(2, 3, 3)$, $(2, 3, 4)$ *or* $(2, 3, 5)$.

We will show that the remaining finite subgroups of O(3) are symmetry groups of regular polyhedra. In order to do this, we introduce some convenient terminology.

Let P be an n-dimensional polytope. An n-tuple $(F_0, F_1, \dots, F_{n-1})$, where F_0 is a vertex of P and for any $k \geq 1$, F_k is a k-cell containing F_{k-1}, is a *flag* of P. The isometries of $\mathbb{E}^n$ which map P to itself for m the *group of symmetries* of P. The intersection of the group of symmetries of P with SO(3) is called the *group of rotational symmetries* of P. The group of symmetries of P acts naturally on the set of flags $\mathscr{F}(P)$ of P. A polytope Q is *regular* if the group of symmetries of Q acts transitively on $\mathscr{F}(Q)$. A regular polyhedron is often called a Platonic solid or a regular solid.

Proposition 2.7. *If Q is a regular polytope, then the group of symmetries of Q acts simply transitively on $\mathscr{F}(Q)$.*

Proof. Assume that 0 is contained in the interior of the regular polytope P and that ϕ is a symmetry of P that fixes a flag. But then ϕ fixes an affine hyperplane that contains the $(n-1)$-cell of the flag and it does not reflect 0 across this hyperplane. Thus, Proposition 1.7 (4) implies that ϕ is the identity. □

The standard 3-dimensional simplex

$$\{x \in \mathbb{E}^4 : \sum_{i=1}^{4} x_i = 1, x_1, x_2, x_3, x_4 \geq 0\}$$

is a regular tetrahedron. It is easy to realize a regular tetrahedron in $\mathbb{E}^3$: Take an equilateral triangle in $\mathbb{E}^2 \times \{0\} \cong \mathbb{C}$ with vertices at the third roots of unity and place the third vertex at the correct height in $\{(0,0)\} \times \mathbb{E}^1$ such that its distance from the three vertices of the triangle is equal to the edge length $\sqrt{3}$ of the triangle. Another natural position to place the tetrahedron is to take as vertices the points $(1,1,1)$, $(1,-1,-1)$, $(-1,1,-1)$ and $(-1,-1,1)$.

A regular tetrahedron has two kinds of rotational symmetries :

- Three half-turns about an axis through the centers of opposite edges.
- Four cyclic groups of order 3 about an axis through a vertex and the opposite edge.

The fixed points of the half-turns are all in the same orbit whose point-stabilizers have order 2 and the fixed points of the rotations of order three form two orbits; the direction of the axis cannot be reversed by a symmetry because the points of intersection of the axis with the surface of the tetrahedron are a vertex and the centre of the opposing triangle. In addition to the rotational symmetry group, there are the 12 reflection through the planes that pass through two vertices and the midpoint of the unique edge that contains neither of the above vertices. Thus, using Proposition 2.7 and the fact that a tetrahedron has $4 \times 3 \times 2 = 24$ flags, we see that a tetrahedron is a regular polygon with a symmetry group of signature $(2,3,3)$.

In a similar way, one can check that the *cube*

$$K_3 = \{x \in \mathbb{E}^3 : |x_1|, |x_2|, |x_3| \leq 1\}\,.$$

is a regular polyhedron with a symmetry group of signature $(2,3,4)$.

The *dual* of a polytope Q in $\mathbb{E}^n$ is

$$Q^* = \{x \in \mathbb{E}^n : (x|y) \leq 1 \text{ for all } x \in Q\}\,.$$

The dual of the cube is the octahedron, also called the 3-dimensional *cocube*

$$K_3^* = \{x \in \mathbb{E}^3 : |x_1| + |x_2| + |x_3| \leq 1\}.$$

Proposition 2.8. *If Q is a polytope, then $Q^{**} = Q$.*

Proof. It is immediate from the definition that Q^* contains Q. Assume then that $a \in \mathbb{E}^n - Q$. By definition of Q, there is some closed half space H such that $Q \subset H$ and $a \notin H$. Let $h \in \mathbb{E}^n - \{0\}$ be such that $H = \{x \in \mathbb{E}^n : (x|h) = 1\}$. (It is a straightforward exercise to prove that such a vector exists.) Now, for all $x \in Q$, we have $(x|h) \leq 1$, which implies $h \in Q^*$. On the other hand, $(a|h) > 1$, which implies $a \notin Q^{**}$. $\square$

Corollary 2.9. *The group of symmetries of a polytope equals that of its dual.* $\square$

In particular, the symmetry groups of the cube and the octahedron are the same.

We will now consider the remaining regular polytopes: Let $\phi = \frac{1+\sqrt{5}}{2}$ be the golden ratio. In particular, $\phi^2 = \phi + 1$.

Proposition 2.10. *The* 12 *points* $(0, \pm\phi, 1)$, $(\pm 1, 0, \pm\phi)$ *and* $(\pm\phi, \pm 1, 0)$ *are the vertices of an icosahedron that all lie on the edges of an octahedron given by the inequality* $|x_1| + |x_2| + |x_3| \leq \frac{3+\sqrt{5}}{2}$. *The signature of the symmetry group of an icosahedron has signature* $(2, 3, 5)$.

Proof. The points in the statement are all on a sphere and 2 is the minimal distance. Note that the points $(\phi, -1, 0)$, $(0, \phi, \pm 1)$ and $(1, 0, \pm\phi)$ are at distance 2 from $(\phi, 1, 0)$. The group generated by half-turns around the coordinate axes and the cyclic permutation of the coordinates (which is a rotation of order 3 acts transitively on the set of vertices. Connect all points at distance 2 from each other by edges and fill all the equilateral triangles of edge length 2 to form the surface of a polytope. One can check by a direct count or using Euler's formula that the surface of the polytope consists of 20 triangles.

It can be checked that the neighbouring vertices of $(\phi, 1, 0)$ lie on the affine plane $\{x \in \mathbb{E}^3 : \phi x_1 + x_2 = \phi\}$. A rotation of order 5 around the line throughout 0 and $(\phi, 1, 0)$ preserves the set of the five triangles that meet at $(\phi, 1, 0)$. As this is a rigid configuration and it is repeated at each vertex of the polyhedron, the polyhedron is mapped to itself by this rotation.

A reflection in any of the three coordinate hyperplanes also preserves the polytope. As one of the edges abutting $(\phi, 1, 0)$ lies in the plane $e_3^\perp$, we see that the group of symmetries acts transitively on flags and the polyhedron is indeed regular. $\square$

The remaining regular solid, the *dodecahedron* is the dual of the icosahedron. In particular, it has the same group of symmetries.

In the above proof, we used the following result that restricts the combinatorics of polyhedra.

Proposition 2.11 (Euler's formula)**.** *Let P be a polyhedron. Let V be the number of vertices of P, E the number of edges of P and F the number of faces of P. Then*

$$V - E + F = 2\,.$$

Proof. This proof is due to W. Thurston, see [Thu, 1.3]. Turn the polyhedron in such a way that none of its edges is parallel to the plane $\mathbb{E}^2 \times \{0\}$ or to the line $\{0\} \times \mathbb{E}^1$. Place a positive charge to each vertex and inside each face. Place a negative charge to the midpoint of each edge. Move each charge a small distance on the surface of the polyhedron along a plane parallel to $\mathbb{E}^2 \times \{0\}$, counterclockwise seen from above. Now the unique highest and lowest points have positive charges that remain fixed. Each face receives charges from an open interval that contains one edge more than its number of vertices. Thus, the charges inside each face cancel, leaving a total charge $+2$. $\square$

If each face of a polyhedron P is an a-gon and each vertex belongs to b faces (equivalently, to b edges), then we say that the *Schläfli symbol*of P is $\{a, b\}$.

Corollary 2.12. *The Schläfli symbol of a polyhedron is either* $\{3,3\}$*,* $\{3,4\}$*,* $\{4,3\}$*,* $\{3,5\}$ *or* $\{5,3\}$*.*

Corollary 2.12 implies that the only Platonic solids are the tetrahedron, the octahedron, the cube, the icosahedron and the dodecahedron.

We are now in a position to state the main result on finite subgroups of SO(3):

Theorem 2.13. *Any finite subgroup of* SO(3) *is either*

- *cyclic group of order n*
- *the Klein* 4*-group*
- *the dihedral group D_n*
- *the symmetry group of the tetrahedron, the cube or the icosahedron.*

Furthermore, all the above possibilities are realised.

Proof. We took care of the case of cyclic groups above.

If the cardinalities of the three orbits are $(2, 2, 2)$, then equation (4) implies that G has 4 elements. As the stabilizers have 2 elements each, the nonidentity elements have order 2, and the only possible group with this structure is Klein's four-group. Such a group is realized by taking the rotations by π (half-turns) around the coordinate axes.

It is an exercise to check that the dihedral group D_n can be realized as a subgroup of SO(3) as the rotational symmetry group of a regular n-gon isometrically embedded in $\mathbb{E}^2 \times \{0\} \subset \mathbb{E}^3$. The two points $(0, 0, \pm 1)$, which form the intersection of the axis of the rotation of order n with $\mathbb{S}^2$, are stabilized by the cyclic subgroup of order n. The $2n$ points of intersection of the axes of the half-turns form two orbits of size n, where each point has a stabilizer of order 2 consisting of a half-turn and the identity. Assume then that G is a group with signature $(2, 2, n)$ with $n \geq 3$. Equation (4) implies that G has $2n$ elements. A point x_3 fixed by n elements has to be an intersection point with $\mathbb{S}^2$ of the axis of a rotation of order n, and any other nonidentity element of the group has to interchange the two fixed points $\{\pm z\}$ of the rotation group of order n. Consider the orbit of x_1 which is a fixed point of one of the half- turns. The orbit of x_1 consists of n distinct points that can be taken as the vertices of a regular n-gon. The $2n$ elements of G are thus seen to form the rotational symmetry group of this pentagon.

It is an exercise to check that the rotational symmetry group of the tetrahedron has signature $(2, 3, 3)$. Assume that G has signature $(2, 3, 3)$. Equation (4) implies that G has 12 elements. Consider the orbit of any fixed point x_3 with stabilizer of order 3. The orbit Gx_3 has four points by the Orbit stabiliser lemma. Let w be one of the points in $Gx_3 - \{\pm x_3\}$. The point w is not fixed by any element of $\operatorname{Stab}_G x_3$. Thus, $Gx_3 = x_3 \cup (\operatorname{Stab}_G x_3)w$. In particular, the points $(\operatorname{Stab}_g x_3)w$ are the vertices of an equilateral triangle. Repeating the argument for the points of $Gx_3 - \{x_3\}$, we see that Gx_3 are the vertices of a regular tetrahedron. All rotations of G preserve this tetrahedron so G, having 12 elements, must be its rotational symmetry group.

A group with orbit stabilizers of type $(2,3,4)$ has 24 elements. In a similar manner, it can be checked that the rotational symmetry group of any cube is of type $(2,3,4)$ and that any group with this orbit structure is conjugate to the symmetry group of the cube K_3.

In the remaining case of signature $(2,3,5)$, Equation (4) implies that G has 60 elements. We can use similar techniques as above to show that such a group is conjugate with the symmetry group of the icosahedron. See for example [Arm] for the details. $\square$

In any dimension, there are at least three regular polytopes, the simplex Σ_d which is self-dual and the cube K_3 and its dual the cocube K_3^*. In dimension 4, there are three in addition to the above:

- the 24-cell which is self-dual whose boundary is made up of 24 octahedra
- the 120-cell whose boundary is made up of 120 dodecahedra, and its dual
- the 600-cell whose boundary is made up of 600 tetrahedra.

In higher dimensions, there are only the three obvious regular polytopes. For more, see Chapter 12 of [Ber] or [Cox].

If we place a regular polyhedron in $\mathbb{E}^3$ so that the fixed point of its group of rotational symmetries is 0 and the polyhedron is contained in the unit ball, we can consider the radial projection of the surface of the polyhedron to $\mathbb{S}^2$. Each 2-cell in the boundary of the regular polytope is mapped to a regular polygon on the sphere. Thus, we get a *tiling* of $\mathbb{S}^2$ by

- 4 triangles with angles $2\pi/3$ (tetrahedron)
- 8 triangles with angles $\pi/2$ (octahedron)
- 6 quadrilaterals with angles $2\pi/3$ (cube)
- 12 pentagons with angles $2\pi/3$ (dodecahedron)
- 20 triangles with angles $2\pi/5$ (icosahedron).

3. Discrete subgroups and tilings

We saw that finite subgroups of $\operatorname{Isom}\mathbb{E}^n$ are finite subgroups of $\operatorname{Isom}\mathbb{S}^{n-1}$ and they lead to tilings of the sphere. Analogously, in $\mathbb{E}^n$, one considers *discrete or discontinuous groups*. In this section, we study quotient spaces of Euclidean space under the action of discontinuous groups of isometries. If the elements of the groups in question have no fixed points in Euclidean space, then the quotient spaces are metric spaces which are locally isometric with Euclidean space.

A group $G < \operatorname{Isom}\mathbb{E}^n$ acts *discontinuously* on $\mathbb{E}^n$, if for any compact subset $K \subset \mathbb{E}^n$, the set $\{g \in G : g(K) \cap K \neq \emptyset\}$ is finite. If G acts discontinuously, we say that G is a *discontinuous group* of isometries.

Example 3.1. A basic example is the action of $\mathbb{Z}^n < \mathbb{R}^n < E(n) = \operatorname{Isom}\mathbb{E}^n$ acting by translations. If $K \subset \mathbb{E}^n$ is compact, then for any $k \in \mathbb{Z}^n$ with $|k_i| > \operatorname{diam} K$, we have $\|(x+k)-x\| > \operatorname{diam} K$, and thus, $x+k \notin K$. This implies that the cardinality of $\{g \in \mathbb{Z}^n : g(K) \cap K \neq \emptyset\}$ is at most $(2 \operatorname{diam} K)^n$. Thus, $\mathbb{Z}^n$ acts discontinuously.

The images of the closed unit cube $[0,1]^n$ under $\mathbb{Z}^n$ cover all of $\mathbb{E}^n$ and the images of the open unit cube $]0,1[^n$ are pairwise disjoint. Thus, the action of $\mathbb{Z}^n$ by translations gives a tiling of $\mathbb{E}^n$ by translates of the unit cube.

Generelizing the above example, let $v_1, v_2, \ldots, v_n$ be a basis of $\mathbb{E}^n$ as a vector space. The subgroup $\Gamma_{v_1,v_2,\ldots,v_n}$ of $\operatorname{Isom}\mathbb{E}^n$ generated by the translations T_{v_i} is easily seen to be discontinuous. In this way, we can get uncountably many non-conjugate discontinuous subgroups of $\operatorname{Isom}\mathbb{E}^n$ which are all isomorphic as abstract groups.

Theorem 3.2. *A group $G < \operatorname{Isom}(\mathbb{E}^n)$ is discontinuous if and only if all G-orbits are discrete and closed, and the stabilizers of points are finite.*

Proof. Assume first that G is discontinuous. Let $x \in \mathbb{E}^n$, and assume that y is an accumulation point of the orbit Gx. There is a sequence $(g_i)_{i\in\mathbb{N}} \in G$, such that $g_i(x) \to y$ as $i \to \infty$ with $g_i(x) \neq g_j(x)$ if $i \neq j$. The set $\{g_i(x) : i \in \mathbb{N} \cup \{0\}\} \cup \{y\}$ is compact. But now for all $k \in \mathbb{N}$, we have $g_k(x) \in g_k K \cap K$. Thus, the set $\{g \in G : gK \cap K \neq \emptyset\}$ is infinite, which is a contradiction as we assumed that G is discontinuous. Thus, the orbit Gx is discrete and closed. The fact that the stabilizer of any point is finite is clear from discontinuity because points are compact sets.

Assume now that each G-orbit is discrete and closed and that stabilizers of points are finite. Assume that the group G is not discontinuous. Then there is a compact set $K \subset \mathbb{E}^n$ such that the set $\{g \in G : gK \cap K \neq \emptyset\}$ is infinite. Thus, there is a sequence $(g_i)_{i\in\mathbb{N}} \in G$ such that $g_i \neq g_j^{\pm 1}$ when $i \neq j$ and such that $g_i K \cap K \neq \emptyset$ for all $i \in \mathbb{N}$. For each $i \in \mathbb{N}$, there is some $x_i \in K$ such that $g_i(x_i) \in K$. As K is compact, by passing to a subsequence, there are points $x, y \in K$ such that $x_i \to x$ and $g_i(x_i) \to y$ as $i \to \infty$. All the maps g_i are isometries, so we have

$$d(g_j(x), y) \leq d(g_j(x), g_j(x_j)) + d(g_j(x_j), y) = d(x, x_j) + d(g_j(x_j), y) \to 0$$

as $j \to \infty$. The sequence $(g_j(x))_{j\in\mathbb{N}}$ is infinite because otherwise the stabilizer of x is infinite. Thus, y is an accumulation point of Gx, which is a contradiction. □

In fact, it is enough to check the condition of Theorem 3.2 for just one point to conclude discontinuity of the group:

Proposition 3.3. *Let $G < \operatorname{Isom}(\mathbb{E}^n)$. If the G-orbit of $x \in \mathbb{E}^n$ is discrete and closed and $\operatorname{Stab}_G x$ is finite, then the same holds for any point in $\mathbb{E}^n$. In particular, G is discontinuous.* □

Corollary 3.4. *A group $G < \operatorname{Isom}(\mathbb{E}^n)$ is discontinuous if for some $x \in \mathbb{E}^n$, Gx is discrete and closed and the stabilizer of x is finite.* □

A subset $F \subset \mathbb{E}^n$ is a *fundamental set* of $G < \operatorname{Isom}(\mathbb{E}^n)$ if it contains exactly one point from each G-orbit. A nonempty open subset $D \subset \mathbb{E}^n$ is a *fundamental domain* of G if

- the closure of D contains a fundamental set of G,
- $gD \cap D = \emptyset$ for all $g \in G - \{\mathrm{id}\}$, and
- the boundary of D has measure 0.

Proposition 3.5. *If $G < \operatorname{Isom}(\mathbb{E}^n)$ has a fundamental domain, then G is discontinuous.* □

Let $G < \operatorname{Isom}(\mathbb{E}^n)$ be a discontinuous group, and let $x \in \mathbb{E}^n$ be a point that is not fixed by any element of g. The set

$$D(x) = \{y \in \mathbb{E}^n : d(x, y) < d(g(x), y) \text{ for all } g \in G - \{\mathrm{id}\}\}$$

is a *Dirichlet domain* of G *centered* at x.

Proposition 3.6. *Let $G < \operatorname{Isom}(\mathbb{E}^n)$ be a discontinuous group and let $w \in \mathbb{E}^n$. Then G has a Dirichlet domain centered at w. The Dirichlet domain $D(w)$ is a convex fundamental domain of G.*

Proof. Note that if we set for any $g \in G$,

$$H_g(w) = \{x \in \mathbb{E}^n : d(x, w) < d(x, g(w))\},$$

then each H_g is a halfspace and

$$D(w) = \bigcap_{g \in G - \{\mathrm{id}\}} H_g(w).$$

The collection $(\partial H_g)_{g \in G - \{\mathrm{id}\}}$ is locally finite because $d(w, \partial H_g(w)) = \frac{1}{2} d(w, g(w))$ and therefore, by discontinuity, any ball centered at w intersects only a finite number of the collection of hyperplanes $(\partial H_g)_{g \in G}$. Local finiteness implies that $D(w)$ is open and the boundary of $D(w)$ is contained in the measure zero subset $\bigcup_{g \in G} \partial H_g$ of $\mathbb{E}^n$.

Let us show that G has a fundamental set F that satisfies $D(w) \subset F \subset \overline{D(w)}$. Discreteness and closedness of the orbit Gz imply that there is some $z^* \in Gz$ for which $d(z^*, w) = d(Gz, w)$. Let us choose one such z^* for each orbit, and let

$$F = \{z^* : Gz \in G \backslash \mathbb{E}^n\}.$$

Clearly, F is a fundamental set that contains $D(w)$. Furthermore, if the open interval $]w, z^*[$ intersects some H_g, $g \in G - \{\mathrm{id}\}$, then, by definition of the halfspace H_g, we have

$$d(g^{-1}(z^*), w) = d(z^*, g(w)) < d(z^*, w),$$

which is a contradiction. Thus, $F \subset \overline{D(w)}$. □

It is easy to check that for the action of $\mathbb{Z}^2$ on $\mathbb{E}^2$ as in Example 3.1, we have

$$D(0) = \left] -\frac{1}{2}, \frac{1}{2} \right[^2 = H_{(0,1)} \cap H_{(0,-1)} \cap H_{(1,0)} \cap H_{(-1,0)}.$$

The maximal distance from 0 to the boundary of the above cube is $\frac{1}{\sqrt{2}}$ and the open ball of radius $\sqrt{2}$ contains only the five points 0, $(0,1)$, $(0,-1)$, $(1,0)$ and $(-1,0)$.

The action of the group G on $\mathbb{E}^n$ defines an equivalence relation $\sim$ in $\mathbb{E}^n$ by setting $x \sim y$ if and only if $Gx = Gy$. In other words, $x \sim y$ if and only if there is some $g \in G$ such that $g(x) = y$. The quotient space which is the set of G-orbits is denoted by $G \backslash \mathbb{E}^n$ (or very often $\mathbb{E}^n / G$).

Example 3.7. When $n = 2$, then the quotient spaces $X_{v_1,v_2} = \Gamma_{v_1,v_2} \backslash \mathbb{E}^2$ in Example 3.1 are homeomorphic to a torus $\mathbb{S}^1 \times \mathbb{S}^1$. Geometrically, the quotient spaces for different bases v_1, v_2 can be very different If $v_1 = te_1$ and $v_2 = \frac{1}{t} e_2$, then it appears that the torus becomes "long and thin" as $t \to \infty$. We will make this more precise after Proposition 3.8.

Let us define a function $d \colon G \backslash \mathbb{E}^n \times G \backslash \mathbb{E}^n \to \mathbb{R}$ by

$$d(Gx, Gy) = \min_{g,h \in G} d(g(x), h(y))$$

for any $Gx, Gy \in G \backslash \mathbb{E}^n$. The fact that orbits are discrete ad closed implies that this minimum exists. It is easy to check that

$$d(Gx, Gy) = d(x, Gy) = d(Gx, y).$$

The proof of the following basic result uses the facts that the G-orbits of a discontinuous group are closed and that all the maps under consideration are isometries.

Proposition 3.8. *If G is a discontinuous group of isometries of $\mathbb{E}^n$, then $(G \backslash \mathbb{E}^n, d)$ is a metric space.*

Proof. Let $Gx \neq Gy$. Then, $d(Gx, Gy) = d(x, Gy) > 0$ because Gy is closed and $x \notin Gy$. It is clear that d is symmetric, so it remains to check the triangle inequality. Let $Gx, Gy, Gz \in G\backslash\mathbb{E}^n$. Now, for any $g, h \in G$, the triangle inequality gives

$$d(x, g(y)) + d(y, h(z)) = d(x, g(y)) + d(g(y), gh(z)) \geq d(x, gh(z)) .$$

Thus,

$$d(Gx, Gz) = d(x, Gz) \leq d(x, Gy) + d(y, Gz) = d(Gx, Gy) + d(Gy, Gz) . \quad \square$$

After this, we can make Example 3.7 more precise: For $0 < s < t$, the quotient spaces of the groups $\Gamma_{se_1, \frac{1}{s}e_2}$ and $\Gamma_{te_1, \frac{1}{t}e_2}$ are not isometric for example because balls of radius $s < r < t$ are contractible in the latter space but not in the former.

If none of the elements of $G - \{\mathrm{id}\}$ have fixed points in $\mathbb{E}^n$, then we say that G acts *without fixed points* and the action of G on $\mathbb{E}^n$ is said to be *free*.

Proposition 3.9. *If G is a discontinuous group of isometries of $\mathbb{E}^n$ that acts on $\mathbb{E}^n$ without fixed points, then the quotient map $\pi\colon \mathbb{E}^n \to G\backslash\mathbb{E}^n$, $\pi(x) = Gx$, is a locally isometric covering map.*

Proof. Let $x \in \mathbb{E}^n$, and let $s = \frac{1}{4}d(x, Gx - \{x\})$. Let us show that the restriction of π to $B(x, s)$ is an isometry. Let $y, z \in B(x, s)$. If $g \in G - \{\mathrm{id}\}$, then

$$d(y, g(z)) \geq d(x, g(x)) - d(x, y) - d(g(x), g(z)) \geq 4s - 2s = 2s .$$

Thus, $d(Gx, Gy) = d(x, y)$.

The quotient map is surjective by definition, and the preimage of the open set $B(Gx, s) = \pi(B(x, s))$ is the disjoint union $\bigsqcup_{g\in G} B(g(x), s) = \bigsqcup_{g\in G} gB(x, s)$, which implies that π is a covering map. $\square$

If $G < \mathrm{Isom}(\mathbb{E}^n)$ acts on $\mathbb{E}^n$ freely and discontinuously, then the quotient space $G\backslash\mathbb{E}^n$ is a *Euclidean manifold*: Each point Gx has a small neighbourhood given by the proof of Proposition 3.9 which is isometric with a ball in $\mathbb{E}^n$.

A discontinuous subgroup of $\mathrm{Isom}\,\mathbb{E}^n$ that consists entirely of translations and has a compact quotient is called a *lattice*.

Theorem 3.10 (Bieberbach)**.** *Any discontinuous subgroup of* $\mathrm{Isom}\,\mathbb{E}^n$ *with compact quotient contains a lattice.*

Proof. See [Ber, 1.7.5.2] for the case of dimension 2 and [Bus] for the general case. $\square$

4. Minkowski space and conic sections

4.1. **Bilinear forms and Minkowski space.** Let V and W be real vector spaces. A map $\Phi\colon V \times W \to \mathbb{R}$ is a *bilinear form*, if the maps $v \mapsto \Phi(v, w_0)$ and $v \mapsto \Phi(v_0, w)$ are linear for all $w_0 \in W$ and all $v_0 \in V$. A bilinear form Φ is *nondegenerate* if

- $\Phi(x, y) = 0$ for all $y \in W$ only if $x = 0$, and
- $\Phi(x, y) = 0$ for all $x \in V$ only if $y = 0$.

If $W = V$, then Φ is *symmetric* if $\Phi(x, y) = \Phi(y, x)$ for all $x, y \in V$. It is

- *positive semidefinite* if $\Phi(x, x) \geq 0$ for all $x \in V$,
- *positive definite* if $\Phi(x, x) > 0$ for all $x \in V - \{0\}$,
- *negative (semi)definite* if $-\Phi$ is positive (semi)definite, and
- *indefinite* otherwise.

A positive definite symmetric bilinear form is often called an inner product or a scalar product.

If V is a vector space with a symmetric bilinear form Φ, we say that two vectors $u, v \in V$ are *orthogonal* if $\Phi(u,v) = 0$, and this is denoted as usual by $u \perp v$. The *orthogonal complement* of $u \in V$ is

$$u^{\perp} = \{v \in V : u \perp v\}.$$

Let us consider the indefinite nondegenerate symmetric bilinear form $\langle \cdot | \cdot \rangle$ on $\mathbb{R}^{n+1}$ given by

$$\langle x|y\rangle = -x_0 y_0 + \sum_{i=1}^{n} x_i y_i = -x_0 y_0 + (\bar{x}|\bar{y}),$$

where $x = (x_0, x_1, \ldots, x_n) = (x_0, \bar{x})$. We call $\langle \cdot | \cdot \rangle$ the *Minkowski bilinear form*, and the pair

$$\mathbb{M}^{1,n} = \left(\mathbb{R}^{n+1}, \langle \cdot | \cdot \rangle\right)$$

is the $n+1$-dimensional *Minkowski space*. A basis $\{v_0, v_1, \ldots, v_n\}$ of $\mathbb{M}^{1,n}$ is *orthonormal* if the basis elements are pairwise orthogonal and if $\langle v_0|v_0\rangle = -1$ and $\langle v_i|v_i\rangle = 1$ for all $i \in \{1, 2, \ldots, n\}$. The following observation, which is a special case of *Sylvester's law of inertia* confirms that this is a sensible definition:

Proposition 4.1. *Let H be a maximal vector subspace of $\mathbb{M}^{1,n}$ such that the restriction of the Minkowski bilinear form to H is positive definite. Then* $\dim H = n$.

Proof. As H cannot be all of $\mathbb{M}^{1,n}$, we have $\dim H \leq n-1$. On the other hand, $H^{\perp}$ contains no vectors v with $\Phi(v,v) > 0$. Thus, $H^{\perp} \cap e_0^{\perp} = \emptyset$. This implies $\dim H^{\perp} \leq 1$. $\square$

Minkowski space has a number of geometrically significant subsets: (1) The subset of *null-vectors* is the *lightcone*

$$\mathscr{L}^n = \left\{x \in \mathbb{M}^{1,n} - \{0\} : \langle x|x\rangle = 0\right\}.$$

The name lightcone comes from Einstein's special theory of relativity, which lives in $\mathbb{M}^{1,3}$. Furthermore, we will occasionally use the physical terminology and say that a nonzero vector $x \in \mathbb{M}^{1,n}$ is

- *lightlike* if $\langle x|x\rangle = 0$,
- *timelike* if $\langle x|x\rangle < 0$, and
- *spacelike* if $\langle x|x\rangle > 0$.

(2) The variety

$$\mathscr{H}_{-}^{n} = \{x \in \mathbb{M}^{1,n} : \langle x|x\rangle = -1\}$$

is a two-sheeted hyperboloid, and its upper sheet is

$$\mathbb{H}^n = \{x \in \mathbb{M}^{1,n} : \langle x|x\rangle = -1,\ v_0 > 0\}.$$

The set $\mathbb{H}^n$ can be given the structure of a Riemannian manifold induced by the restriction of the Minkowski bilinear form to its tangent spaces at each point, analogously to the Riemannian metric of $\mathbb{S}^n$ induced from the ambient Euclidean space. Endowed with this Riemannian metric, the upper sheet of the hyperboloid becomes *hyperbolic space*, see [BH, s. 18-21].

(3) The variety

$$\mathscr{H}_{+}^{n} = \{x \in \mathbb{M}^{1,n} : \langle x|x\rangle = 1\}$$

is a one-sheeted hyperboloid.

4.2. **The orthogonal group.** Let $J_{1,n} = \operatorname{diag}(-1, 1, \dots, 1)$, and note that

$$\langle x|y\rangle = {}^{T}\!xJy$$

for all $x, y \in \mathbb{H}^n$. The *orthogonal group* of the Minkowski bilinear form is

$$\begin{aligned} \mathrm{O}(1,n) &= \{A \in \mathrm{GL}_n(\mathbb{R}) : \langle Ax|Ay\rangle = \langle x|y\rangle \text{ for all } x, y \in \mathbb{M}^{1,n}\} \\ &= \{A \in \mathrm{GL}_n(\mathbb{R}) : {}^{T}\!AJ_{1,n}A = J_{1,n}\}\,. \end{aligned}$$

Clearly, the linear action of $\mathrm{O}(1,n)$ on $\mathbb{M}^{1,n}$ preserves the lightcone and the two-sheeted hyperboloid $\mathscr{H}^n$.

Let us write an $(n+1) \times (n+1)$-matrix A in terms of its column vectors $A = (a_0, a_1, \dots, a_n)$. If $A \in \mathrm{O}(1,n)$, then $a_0 = A(x_0)$ for $(x_0 = 1, 0, \dots, 0) \in \mathbb{H}^n$. Thus $A(x_0) \in \mathbb{H}^n$ if and only if $A_{00} > 0$, and therefore the stabiliser in $\mathrm{O}(1,n)$ of the upper sheet $\mathbb{H}^n$ is

$$\begin{aligned} \mathrm{O}^+(1,n) &= \{A \in O(1,n) : A\mathbb{H}^n = \mathbb{H}^n\} \\ &= \{A \in \mathrm{GL}_n(\mathbb{R}) : A_{00} > 0,\ \langle Ax|Ay\rangle = \langle x|y\rangle \text{ for all } x, y \in \mathbb{M}^{1,n}\} \\ &= \{A \in \mathrm{GL}_n(\mathbb{R}) : A_{00} > 0,\ {}^{T}\!AJ_{1,n}A = J_{1,n}\}\,, \end{aligned}$$

which is the identity component of $\mathrm{O}(1,n)$.

The following observation is proved in the same way as its Euclidean analog:

Lemma 4.2. *An $(n+1) \times (n+1)$-matrix $A = (a_0, a_1, \dots, a_n)$ is in $\mathrm{O}(1,n)$ if and only if the vectors $a_0, a_1, \dots, a_n$ form an orthonormal basis of $\mathbb{M}^{1,n}$. Furthermore, $A \in \mathrm{O}^+(1,n)$ if and only if $A \in \mathrm{O}(1,n)$ and $a_0 \in \mathbb{H}^n$.* □

Example 4.3. (1) Let $t \in \mathbb{R}$. The matrix

$$L_t = \begin{pmatrix} \cosh t & \sinh t & 0 \\ \sinh t & \cosh t & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{SO}(1,2)$$

stabilises any affine hyperplane

$$H_c = \{x \in \mathbb{M}^{1,2} : x_2 = c\}\,.$$

(2) For any $\theta \in \mathbb{R}$, let $\widehat{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in \mathrm{O}(2)$, and let

$$R_\theta = \operatorname{diag}(1, \widehat{R}_\theta) = \begin{pmatrix} 1 & 0 \\ 0 & \widehat{R}(\theta) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \in \mathrm{O}(1,2)\,.$$

This mapping is a Euclidean rotation around the vertical axis by the angle θ. The rotation R_θ stabilizes each affine hyperplane

$$E_r = \{x \in \mathbb{M}^{1,2} : x_0 = r\}\,.$$

Another important mapping that comes by extension from $\mathrm{O}(2)$ is given by the matrix $\operatorname{diag}(1, 1, -1)$, which is a reflection in the hyperplane $H_0 = e_2^\perp$ defined above.
(3) For each $v \in \mathscr{L}^2$ and $t \in \mathbb{R}$, let

$$P_{v,t} = \{x \in \mathbb{H}^2 : \langle v|x\rangle = t\}\,.$$

The vector hyperplane $P_{v,0}$ intersects the light cone in the line spanned by v. The mapping given by the matrix

$$N_s = \begin{pmatrix} 1 + \frac{s^2}{2} & -\frac{s^2}{2} & s \\ \frac{s^2}{2} & 1 - \frac{s^2}{2} & s \\ s & -s & 1 \end{pmatrix} \in \mathrm{O}(1,2)$$

stabilizes the affine hyperplanes $P_{(1,1,0),t}$ for all $t \in \mathbb{R}$.

All of these examples can be generalised to higher dimensions:

- L_t is extended as the identity on the last coordinates to $\mathrm{diag}(L_t, I_{n-2}) \in \mathrm{O}(1, n)$.
- Any Euclidean orthogonal matrix $A \in \mathrm{O}(n)$ gives an isometry $\mathrm{diag}(1, A) \in \mathrm{O}(1, n)$.
- N_s is extended as the identity on the last coordinates to $\mathrm{diag}(N_s, I_{n-2}) \in \mathrm{O}(1, n)$.

We will concentrate on the three-dimensional case with the classical theory of conic sections in mind. We say that a plane L in $\mathbb{M}^{1,2}$ is

(1) *elliptic* if $\langle\cdot|\cdot\rangle|_L$ is positive definite
(2) *parabolic* if $\langle\cdot|\cdot\rangle|_L$ is degenerate
(3) *hyperbolic* if $\langle\cdot|\cdot\rangle|_L$ is indefinite

An affine hyperplane A is called elliptic/hyperbolic/parabolic if the unique vector hyperplane parallel with A is elliptic/hyperbolic/parabolic.

A modification of the proof of Proposition 1.5 gives the following result.

Proposition 4.4. *The orthogonal group of Minkowski space acts transitively on $\mathscr{H}^n_+$, $\mathscr{H}^n_-$ and $\mathscr{L}^n$.*

Proof. Clearly, each of the sets $\mathscr{H}^n_+$, $\mathscr{H}^n_-$ and $\mathscr{L}^n$ is invariant. If $x \in \mathbb{H}^n$, then $x = (\sqrt{\|\bar{x}\|^2 - 1}, \bar{x})$. There is some $\widehat{R}_\theta \in \mathrm{O}(n)$ such that $Ax = \|x\|e_1$, and thus, $R_\theta(x) = (\sqrt{\|\bar{x}\|^2 - 1}, \|x\|e_1)$. Furthermore, $L_{\operatorname{arsinh}\|x\|}e_0 = (\sqrt{\|\bar{x}\|^2 - 1}, \|x\|e_1)$. This implies that $\mathbb{H}^n$ is contained in the $\mathrm{O}(1,n)$-orbit of e_0. If $x \in \mathscr{H}^n_- - \mathbb{H}^n$, then $\mathrm{diag}(-1, \mathrm{id}_n) \in \mathbb{H}^n$. All the maps used here are in $\mathrm{O}(1, n)$.

The remaining cases are proved in a similar way. □

Proposition 4.5. *A hyperplane $W \subset \mathbb{M}^{1,n}$ is*

(1) *hyperbolic if and only if $W = u^\perp$ for some $u \in \mathscr{H}^n_+$.*
(2) *parabolic if and only if $W = u^\perp$ for some $u \in \mathscr{L}^n$.*
(3) *elliptic if and only if $W = u^\perp$ for some $u \in \mathscr{H}^n_-$.*

Proof. All hyperplanes are orthogonal complements of vectors because the Minkowski bilinear form is nondegenerate. If $u \in \mathscr{H}^n_+ \cup \mathscr{H}^n_-$, then the law of inertia, Proposition 4.1 implies that the orthogonal complement is as in the claim. If $u \in \mathscr{L}$, then $u \in u^\perp$. Thus, $u^\perp$ is a parabolic hyperplane. □

The following result on (vector) hyperplanes is an immediate corollary.

Corollary 4.6. *The orthogonal group of Minkowski space acts transitively on the sets of elliptic, parabolic and hyperbolic hyperplanes.*

Consider now an affine hyperplane

$$H = \{x \in \mathbb{M}^{1,2} : \langle x|u\rangle = -r\}$$

for some $u \in \mathscr{L}^2 \cup \mathscr{H}^2_+ \cup \mathscr{H}^2_-$. If $c \neq 0$, then this hyperplane intersects $\mathscr{L}^2$. Using orthogonal mappings, we can bring it to standard position: If H is elliptic, then it is in the orbit of the standard elliptic plane $\{x \in \mathbb{M}^{1,2} : \langle x|u\rangle = -r\}$ and the intersection $H \cap \mathscr{L}^2$ is the image of a circle by a linear transformation. In the same way, the conical sections by hyperbolic and parabolic hyperplanes are seen to be hyperbola and parabola.

The isometries introduced in Example 4.3 are classified according to the conic sections they correspond to. The mapping L_t and any of its conjugates in $\mathrm{Isom}(\mathbb{H}^n)$ is called *hyperbolic* because L_t maps each affine plane parallel to the (x_0, x_1)-plane in $\mathbb{M}^{1,2}$ to itself, and these planes intersect the lightcone in hyperbola or, in the case of the (x_0, x_1)-plane itself, a degenerate hyperbola that consists of two lines.

The mapping $R(\theta)$ and any of its conjugates is called *elliptic* because it preserves all horizontal hyperplanes in $\mathbb{M}^{1,2}$ and their intersections with $\mathscr{L}^2$, which are circles with centers in the line spanned by e_0 or a point in the case of the horizontal coordinate hyperplane.

The mapping N_s and any of its conjugates is called *parabolic* because it preserves all affine hyperplanes $\{x \in \mathbb{M}^{1,2} : \langle v|x\rangle = c\}$, which intersect $\mathscr{L}^2$ in a parabola when $c \neq 0$ and in a line if $c = 0$.

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