Number theory 2 2024

Exercises 6

A number theoretical function f is a *multiplicative function*, if $f(1) = 1$ and

$$f(mn) = f(m)f(n)$$

for all $m, n \in \mathbb{N}$, with $\gcd(m, n) = 1$.

1. Let f be a multiplicative function. Let

$$n = p_1^{e_1} \cdots p_k^{e_k},$$

where $p_1 < \cdots < p_k$ are prime numbers and $e_1, \ldots e_k \in \mathbb{N} - \{0\}$ be the prime factorization of n. Prove that

$$f(n) = f(p_1^{e_1}) \cdots f(p_k^{e_k}).$$

Solution. As p_1 and p_2 are distinct primes, $\gcd(p_1^{e_1}, p_2^{e_2}) = 1$. By multiplicity, we have $f(p_1^{e_1} p_2^{e_2}) = f(p_1^{e_1}) f(p_2^{e_2})$. Assume that $f(p_1^{e_1} p_2^{e_2} \cdots p_n^{e_{n-1}}) = f(p_1^{e_1}) f(p_2^{e_2}) \cdots f(p_n^{e_{n-1}})$. As $p_1, p_2, \ldots, p_n$ are distinct primes, $\gcd(p_1^{e_1} p_2^{e_2} \cdots p_{n-1}^{e_{n-1}}, p_n^{e_n}) = 1$. By multiplicity and induction, we have

$$\begin{aligned} f(p_1^{e_1} p_2^{e_2} \cdots p_{n-1}^{e_{n-1}} p_n^{e_n}) &= f(p_1^{e_1} p_2^{e_2} \cdots p_{n-1}^{e_{n-1}}) f(p_n^{e_n}) \\ &= \Big(f(p_1^{e_1}) f(p_2^{e_2}) \cdots f(p_{n-1}^{e_{n-1}}) \Big) f(p_n^{e_n}). \end{aligned}$$

2. Let $r, m, n \in \mathbb{Z}$, $n \geq 2$, such that $\gcd(m, n) = 1$. Prove that any two different elements of

$$\{km + r : 0 \leq k \leq n - 1\}$$

are not congruent $\bmod n$.

Solution. If $1 \leq k < \ell \leq n - 1$, then $0 < \ell - k < n - 1$, and the difference

$$\ell m + r - (km + r) = (\ell - k)m$$

is not divisible by n.

3. Let p be a prime. Prove that $\phi(p^k) = p^{k-1}(p - 1)$ for all $k \in \mathbb{N}^*$.

Solution. The prime power p^k is only divisible by 1 and the prime powers p^m with $1 \leq m \leq k$. Thus, for numbers $1 \leq n \leq p$, $\gcd(n, p^k) \geq 1$ if and only if $n \equiv 0 \bmod p$. These are the numbers sp with $1 \leq s \leq p^{k-1}$. This implies that $\phi(p^k) = p^k - p^{k-1} = p^{k-1}(p - 1)$.

4. Determine $\phi(1000)$ and $\phi(2343)$.

Solution. Using multiplicativity, the prime decomposition of 1000 and by Exercise 3, we have

$$\phi(1000) = \phi(2^3 5^3) = \phi(2^3)\phi(5^3) = 2^2 5^2 4 = 400.$$

Using the usual divisibility tests, we get the prime factorization

$$2343 = 3 \cdot 781 = 3 \cdot 11 \cdot 71\,.$$

Again, by multiplicativity and Exercise 3, we have

$$\phi(2343) = \phi(3)\phi(11)\phi(71) = 2 \cdot 10 \cdot 70 = 1400\,.$$

5. What are the two last decimals of 3^{400}?

Solution. Using Euler's generalization of Fermat's little theorem, we compute

$$3^{400} = 3^{10\phi(100)} = \left(3^{\phi(100)}\right)^{10} \equiv 1^{10} = 1 \pmod{100}\,,$$

so the two last decimals are 01.

6. Let $m, n \in \mathbb{N}$ and let $s = \prod_{p|m \text{ and } p|n} p$. Prove that

$$\phi(mn) = s\,\frac{\phi(m)\phi(n)}{\phi(s)}\,.$$

Solution. By Theorem 12.10 of the lectures,

$$\phi(m) = m \prod_{p|m} \left(1 - \frac{1}{p}\right). \tag{1}$$

and

$$\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right). \tag{2}$$

If q is a prime factor of both m and n, then it appears in both products (1) and (2). Therefore,

$$\begin{aligned}
\phi(m)\phi(n) &= m \prod_{p|m} \left(1 - \frac{1}{p}\right) n \prod_{p|n} \left(1 - \frac{1}{p}\right) \\
&= mn \prod_{p|mn} \left(1 - \frac{1}{p}\right) \prod_{p|m \text{ and } p|n} \left(1 - \frac{1}{p}\right) \\
&= \phi(mn)\, \frac{s \prod_{p|s} \left(1 - \frac{1}{p}\right)}{s} = \phi(mn)\, \frac{\phi(s)}{s}\,.
\end{aligned}$$

Let $k \in \mathbb{N}$. The kth *divisor function* is $\sigma_k \colon \mathbb{N}^* \to \mathbb{N}$,

$$\sigma_k(n) = \sum_{d|n} d^k\,.$$

A positive natural number $n \in \mathbb{N}^*$ is *abundant*, if $\sigma_1(n) > 2n$.

7. Give an example of an odd abundant number.[1]

[1] See figures 0.1 and 0.2.

Solution. In Figure 0.2, we see that there is some odd abundant number between 940 and 950. We find the prime factorization $945 = 3^3 \cdot 5 \cdot 7$, and Proposition 12.14 of the lectures gives

$$\sigma_1(945) = \frac{3^4 - 1}{3 - 1}\frac{5^2 - 1}{5 - 1}\frac{7^2 - 1}{7 - 1} = 40 \cdot 6 \cdot 8 = 1920 > 1890 = 2 \cdot 945\,.$$

This shows that 945 is abundant.

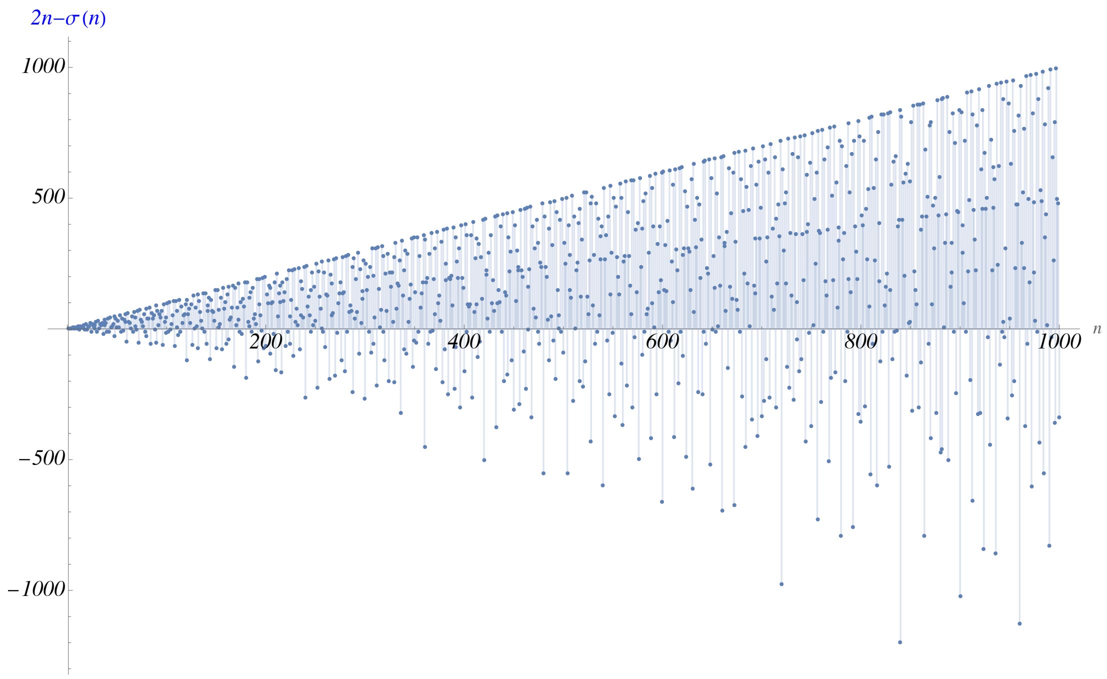

Figure 0.1: The values of $2n - \sigma_1(n)$ for $1 \leq n \leq 999$.

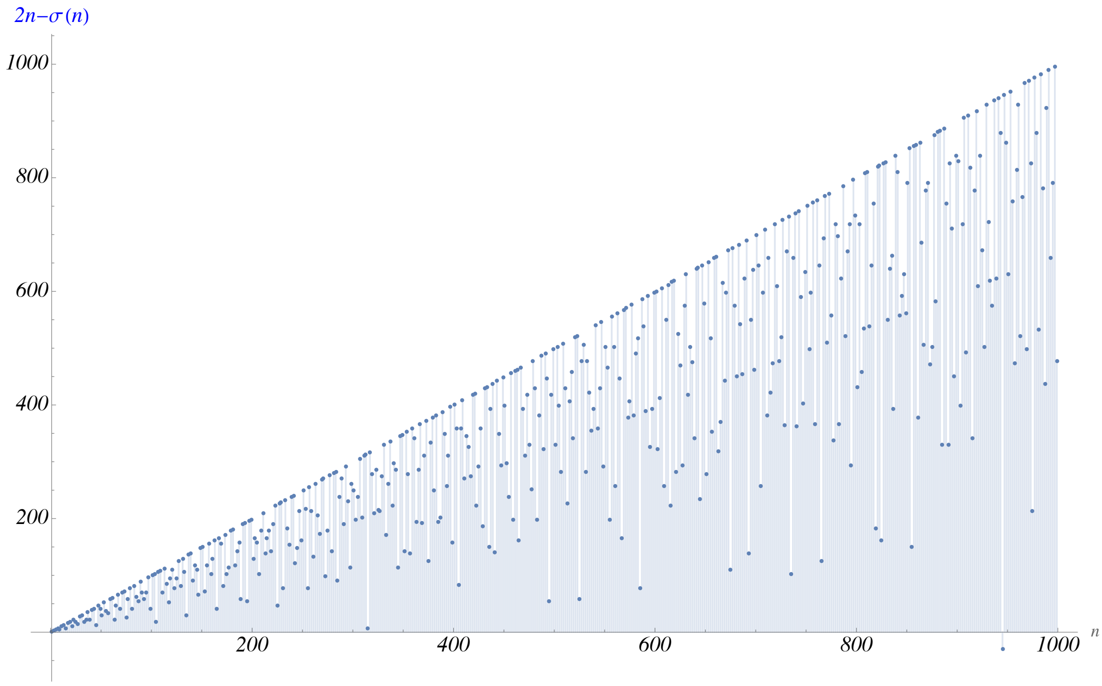

Figure 0.2: The values of $2n - \sigma_1(n)$ for odd numbers $1 \leq n \leq 999$.