Number theory 2 2024

Exercises 5

Rational numbers $\frac{a}{c}, \frac{b}{d} \in \mathbb{Q}$ are *adjacent*, if $|ad - bc| = 1$.

1. Determine the rational numbers that are adjacent to $\frac{2}{3}$.

Solution. It is easy to see that $\frac{1}{1}$ is adjacent to $\frac{2}{3}$. By Lemma 11.3 of the lectures, the set of all rational numbers adjacent to $\frac{2}{3}$ is

$$\left\{\frac{1+2k}{1+3k} : k \in \mathbb{Z}\right\}.$$

2. Determine the rational numbers that are adjacent to $\frac{23}{41}$.

Solution. Using the Euclidean algorithm, we find a solution $(x, y) = (9, 16)$ to the linear Diophantine equation

$$23y - 41x = 1\,,$$

and this means that $\frac{9}{16}$ is adjacent to $\frac{23}{41}$. By Lemma 11.3 of the lectures, the sert of all rational numbers adjacent to $\frac{23}{41}$ is

$$\left\{\frac{9+23k}{16+41k} : k \in \mathbb{Z}\right\}.$$

3. Let $\frac{a}{c}$ and $\frac{b}{d}$ be adjacent. Prove that $\frac{b+ka}{d+kc}$ is adjacent to $\frac{b+(k+1)a}{d+(k+1)c}$.

Solution. Just compute and use the assumption that $\frac{a}{c}$ and $\frac{b}{d}$ are adjacent:

$$(b+ka)(d+(k+1)c) - (d+kc)(b+(k+1)a) = bc - ad = 1\,.$$

If $\frac{a}{c}$ and $\frac{b}{d}$ are adjacent, the rational number $\frac{a+b}{c+d}$ is their *mediant*.

4. Let $r, s \in \mathbb{Q} \cup \{\frac{1}{0}\}$ be adjacent. Prove that there are exactly two elements of $\mathbb{Q} \cup \{\frac{1}{0}\}$ adjacent to both $\frac{a}{c}$ and $\frac{b}{d}$, and that one of these is the mediant of r and s.

Solution. Exercise 3 applied with $k = 0$ and $k = -1$ gives that $\frac{a+c}{b+d}$ and $\frac{a-c}{b-d}$ are adjacent to both $\frac{a}{c}$ and $\frac{b}{d}$. The elements that are adjacent to both must be in the set of rational numbers adjacent to $\frac{a}{c}$. Let $m \in \mathbb{Z}$, and compute

$$(b+ma)d - (d+mc)b = m(ad - bc) = m\,.$$

This shows that the only rational numbers adjacent to $\frac{a}{c}$ and $\frac{b}{d}$ are their mediant $\frac{a+c}{b+d}$ and $\frac{a-c}{b-d}$.

Let $\frac{p}{q} \in \mathbb{Q}$ with $\gcd(p,q)=1$. The *Ford disk* at $\frac{p}{q}$ is

$$\mathscr{H}_{\frac{p}{q}} = \left\{(x,y) \in \mathbb{R}^2 : \left\| w - \left(\frac{p}{q}, \frac{1}{2q^2}\right)\right\| \leq \frac{1}{2q^2}\right\}.$$

The boundary $\partial\mathscr{H}_{\frac{p}{q}}$ of $\mathscr{H}_{\frac{p}{q}}$ is the *Ford circle* at $\frac{p}{q}$.
The half-plane

$$\mathscr{H}_{\frac{1}{0}} = \{(x,y) \in \mathbb{R}^2 : y \geq 1\}$$

is the *Ford disk at infinity.*

5. Let $\frac{a}{c}, \frac{b}{d} \in \mathbb{Q} \cup \{\infty\}$, $\frac{a}{c} \neq \frac{b}{d}$. Prove that the Ford disks $\mathscr{H}_{\frac{a}{c}}$ and $\mathscr{H}_{\frac{b}{d}}$ are tangent if $\frac{a}{c}$ is adjacent to $\frac{b}{d}$, and that the disks are disjoint if $\frac{a}{c}$ is not adjacent to $\frac{b}{d}$.

Solution. The square of the distance between the centers of the Ford disks $\mathscr{H}_{\frac{a}{c}}$ and $\mathscr{H}_{\frac{b}{d}}$ satisfiess

$$\begin{aligned}
\left|\frac{a}{c} - \frac{b}{d}\right|^2 + \left|\frac{1}{2c^2} - \frac{1}{2d^2}\right|^2 &= \left|\frac{ad-bc}{cd}\right|^2 + \left|\frac{1}{2c^2} - \frac{1}{2d^2}\right|^2 \\
&\geq \left|\frac{1}{cd}\right|^2 + \left|\frac{1}{2c^2} - \frac{1}{2d^2}\right|^2 \\
&= \left(\frac{1}{2c^2} + \frac{1}{2d^2}\right)^2
\end{aligned}$$

with equality when $\frac{a}{c}$ and $\frac{b}{d}$ are adjacent. The claim follows because $\frac{1}{2c^2} + \frac{1}{2d^2}$ is the sum of the radii of $\mathscr{H}_{\frac{a}{c}}$ and $\mathscr{H}_{\frac{b}{d}}$.

If $r,s,t \in \mathbb{Q} \cup \{\infty\}$ are pairwise adjacent, then the compact region in the complement of $\mathscr{H}_r$, $\mathscr{H}_s$ and $\mathscr{H}_t$ in the plane $\mathbb{R}^2$ is the *Ford triangle* $\Delta(r,s,t)$.

6. Let

$$\begin{aligned}
A &= (A_1, A_2) = \mathscr{H}_{\frac{p}{q}} \cap \mathscr{H}_{\frac{r}{s}}\,, \\
B &= (B_1, B_2) = \mathscr{H}_{\frac{r}{s}} \cap \mathscr{H}_{\frac{t}{u}} \quad \text{and} \\
C &= (C_1, C_2) = \mathscr{H}_{\frac{t}{u}} \cap \mathscr{H}_{\frac{p}{q}}
\end{aligned}$$

be the vertices of the Ford triangle $\Delta = \Delta\left(\frac{p}{q}, \frac{r}{s}, \frac{t}{u}\right)$ as in Figure 0.1. Prove that

$$A_1 = \frac{pq+rs}{q^2+s^2}\,, \quad B_1 = \frac{rs+tu}{s^2+u^2} \quad \text{and} \quad C_1 = \frac{tu+pq}{u^2+q^2}\,.$$

Solution. Assume that $q \geq s$ and $\frac{r}{s} < \frac{p}{q}$. Using similar triangles, we see that A_1 divides the segment $[\frac{r}{s}, \frac{p}{q}]$ in two pieces the ratio $\frac{1}{2s^2}$ to $\frac{1}{2q^2}$. This gives the equation

$$\frac{A_1 - \frac{r}{s}}{\frac{p}{q} - A_1} = \frac{q^2}{s^2}\,.$$

Solving for A_1, we get the claim.

The other cases follow by symmetry.

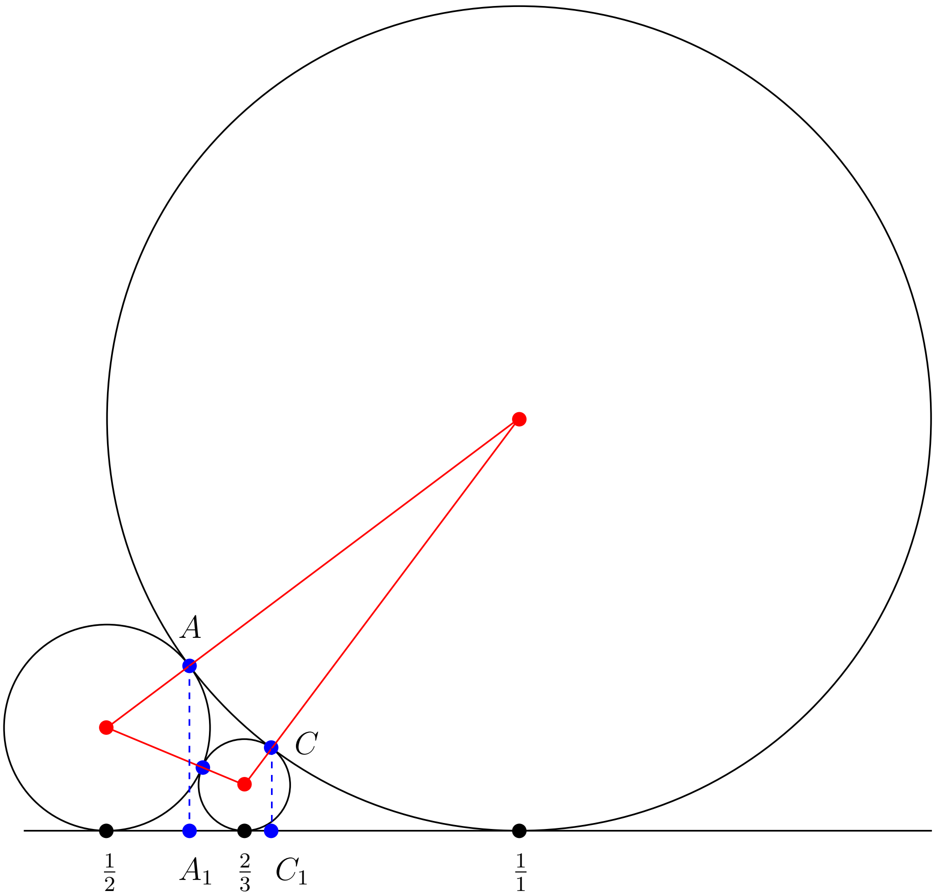

Figure 0.1: $\frac{p}{q} = \frac{1}{1}$, $\frac{r}{s} = \frac{1}{2}$ and $\frac{t}{u} = \frac{2}{3}$.

7. Find four rational solutions to the inequality

$$\left|\frac{1}{\sqrt{2}} - \frac{p}{q}\right| < \frac{1}{2q^2}$$

with the help of Figures 0.2 0.3.

Solution. The two large disks in Figure 0.2 have equal sizes so they must be $\mathscr{H}_{\frac{0}{1}}$ and $\mathscr{H}_{\frac{1}{1}}$. We can then move down following the red vertical line by constructing the successive mediants. The solutions to the inequality are those fractions whose Ford disk the res line intersects. These fractions $\frac{1}{1}$, $\frac{2}{3}$, $\frac{5}{7}$ and $\frac{12}{17}$ are indicated in blue in the figure.

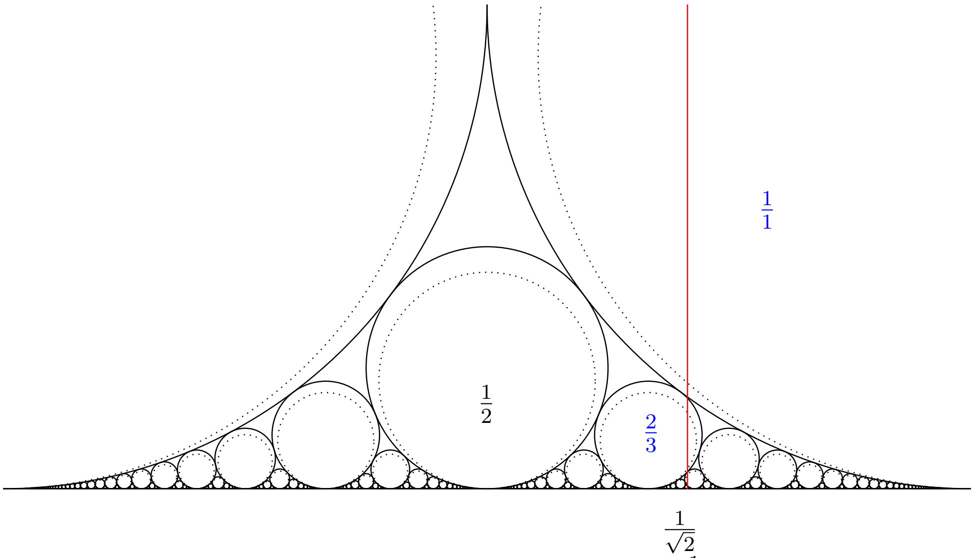

Figure 0.2: Approximation of $\frac{1}{\sqrt{2}}$.

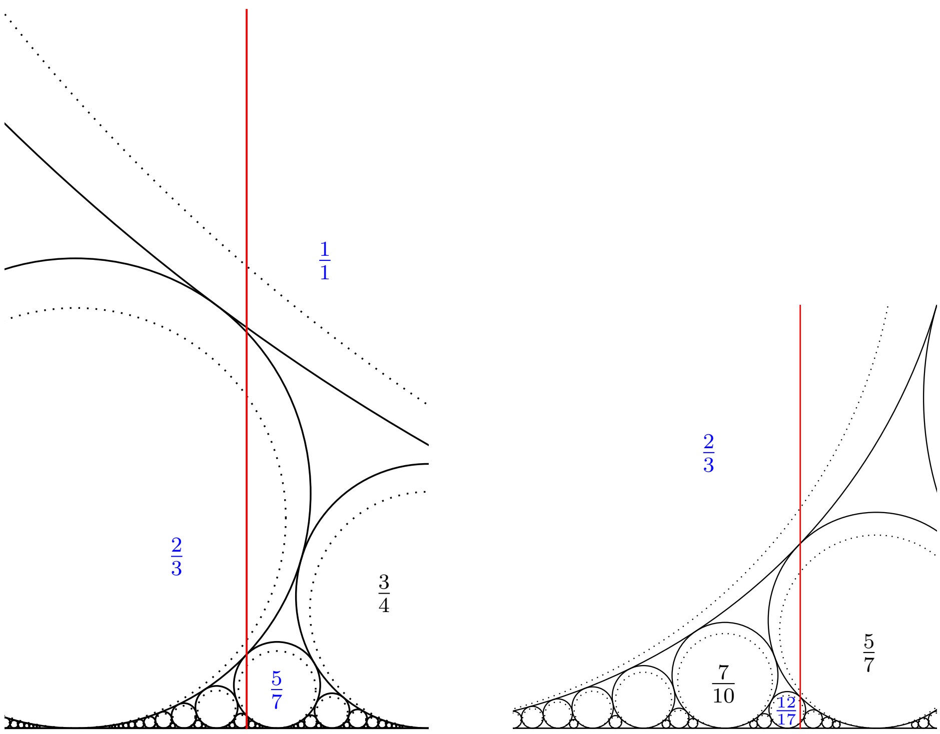

Figure 0.3: Approximation of $\frac{1}{\sqrt{2}}$, close-ups.