Number theory 2 2024

Exercises 4

1. Find a solution to the inequality

$$\left|\sqrt{3} - \frac{p}{q}\right| < \frac{1}{q^2}$$

that satisfies $|q\sqrt{3} - p| < \frac{1}{9}$.[1]

Solution. As in Example 10.3 of the lectures, we compute the fractional parts of the numbers $k\sqrt{3}$ for $0 \leq k \leq 9$. We divide the unit interval in 9 segments of length $\frac{1}{9}$.

k	$k\sqrt{3} - \lfloor k\sqrt{3} \rfloor$	$\approx$	number of segment
0	0	0.	0
1	$\sqrt{3} - 1$	0.732051	6
2	$2\sqrt{3} - 3$	0.464102	4
3	$3\sqrt{3} - 5$	0.196152	1
4	$4\sqrt{3} - 6$	0.928203	8
5	$5\sqrt{3} - 8$	0.660254	5
6	$6\sqrt{3} - 10$	0.392305	3
7	$7\sqrt{3} - 12$	0.124356	1
8	$8\sqrt{3} - 13$	0.856406	7
9	$9\sqrt{3} - 15$	0.588457	5

We see that $\frac{1}{9} \leq 3\sqrt{3}-5\,,\ 7\sqrt{3}-12 < \frac{2}{9}$. Therefore, $|4\sqrt{3}-7| = |3\sqrt{3}-5-(7\sqrt{3}-12)| < \frac{1}{9}$, and $|\sqrt{3} - \frac{7}{4}| < \frac{1}{4\cdot 9} < \frac{1}{4^2}$.

2. Let $\frac{a}{b} \in \mathbb{Q}$. Prove that the inequality

$$\left|\frac{a}{b} - \frac{p}{q}\right| < \frac{1}{q^2} \tag{1}$$

has only finitely many solutions.

Solution. Assume $\frac{a}{b} \neq \frac{p}{q}$. Then $|aq-bp| > 0$ because it is a positive integer. The inequality (1) gives

$$\frac{1}{q^2} > \left|\frac{a}{b} - \frac{p}{q}\right| = \left|\frac{aq-bp}{bq}\right| \geq \frac{1}{|bq|}\,,$$

which implies $|q| < b$. If $q = 1$, then $\frac{p}{q}$ is an integer, and there are at most two integers at a distance less than 1 from any rational number. If $q \geq 2$, then for $n, m \in \mathbb{Z}$, we have

$$\left|\frac{n}{q} - \frac{m}{q}\right| \geq \frac{1}{q} \geq 2\,\frac{1}{q^2}\,.$$

[1] Example 10.3.

This implies that at most one value of p may give a solution to the inequality (1) for a fixed q.

The *discriminant* of a polynomial $P(X) = aX^2 + bX + c$ of degree 2 is

$$\mathrm{Disc}(P(X)) = b^2 - 4ac\,.$$

3. Let $a, b, c \in \mathbb{R}$ with $a \neq 0$, and let α and α' be the roots of the polynomial $P(X) = aX^2 + bX + c$. Prove that

$$\mathrm{Disc}(P(X)) = a^2(\alpha - \alpha')^2\,.$$

Solution. The roots of $P(X)$ are $\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\alpha' = \frac{-b-\sqrt{b^2-4ac}}{2a}$. Therefore,

$$a^2(\alpha - \alpha')^2 = a^2\Big(\frac{\sqrt{b^2-4ac}}{2a}\Big)^2 = b^2 - 4ac\,.$$

Let $F_0 = 0$ and $F_1 = 1$ and set for all $n \geq 2$

$$F_n = F_{n-1} + F_{n-2}\,.$$

The sequence $(F_n)_{n\in\mathbb{N}}$ is the *Fibonacci sequence*.

The roots of the polynomial $P(X) = X^2 - X - 1$ are the *golden ratio* $\varphi = \frac{1+\sqrt{5}}{2}$ and $\widehat{\varphi} = \frac{1-\sqrt{5}}{2}$.

4. Prove[2] that

$$F_n = \frac{\varphi^n - \widehat{\varphi}^n}{\sqrt{5}} \tag{2}$$

for all $n \in \mathbb{N}$.

Solution. Let us first check that

$$\frac{\varphi^0 - \widehat{\varphi}^0}{\sqrt{5}} = \frac{1-1}{\sqrt{5}} = 0 = F_0$$

and

$$\frac{\varphi^1 - \widehat{\varphi}^1}{\sqrt{5}} = \frac{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}}{\sqrt{5}} = 1 = F_1\,.$$

Assuming that (2) holds for all indices up to n, we have

$$\begin{aligned} F_{n+1} = F_n + F_{n-1} &= \frac{\varphi^n - \widehat{\varphi}^n + \varphi^{n-1} - \widehat{\varphi}^{n-1}}{\sqrt{5}} \\ &= \frac{\varphi^{n-1}(\varphi+1) - \widehat{\varphi}^{n-1}(\widehat{\varphi}+1)}{\sqrt{5}} = \frac{\varphi^{n-1}(\varphi^2) - \widehat{\varphi}^{n-1}(\widehat{\varphi}^2)}{\sqrt{5}} = \frac{\varphi^{n+1} - \widehat{\varphi}^{n+1}}{\sqrt{5}} \end{aligned}$$

using the defining equation of φ and $\widehat{\varphi}$.

[2]Induction.

5. Prove that
$$\lim_{n\to\infty}\frac{F_{n+1}}{F_n}=\varphi\,.$$

Solution. Using equation (2), we have
$$\lim_{n\to\infty}\frac{F_{n+1}}{F_n}=\lim_{n\to\infty}\frac{\varphi^{n+1}-\widehat{\varphi}^{n+1}}{\varphi^n-\widehat{\varphi}^n}=\varphi\lim_{n\to\infty}\frac{1-\left(\frac{\widehat{\varphi}}{\varphi}\right)^{n+1}}{1-\left(\frac{\widehat{\varphi}}{\varphi}\right)^{n}}=\varphi$$
because $\left|\frac{\widehat{\varphi}}{\varphi}\right|<1$.

6. Prove that
$$F_{n+2}F_n-F_{n+1}^2=(-1)^{n+1} \tag{3}$$
for all $n\in\mathbb{N}$.

Solution. Note that
$$\begin{pmatrix}1&1\\1&0\end{pmatrix}^n=\begin{pmatrix}F_{n+1}&F_n\\F_n&F_{n-1}\end{pmatrix}$$
for all $n\in\mathbb{N}^*$. This gives
$$F_{n+1}F_{n-1}-F_n^2=\det\begin{pmatrix}F_{n+1}&F_n\\F_n&F_{n-1}\end{pmatrix}=\det\begin{pmatrix}1&1\\1&0\end{pmatrix}^n=(-1)^n\,.$$

7. What do the previous exercises tell about how well the golden ratio φ is approximated by the sequence of rational numbers $(\frac{F_{n+1}}{F_n})_{n\in\mathbb{N}}$?

Solution. If n is odd, then $\left(\frac{\widehat{\varphi}}{\varphi}\right)^{n+1}>0$ and $\left(\frac{\widehat{\varphi}}{\varphi}\right)^{n}<0$, and we have
$$\frac{F_{n+1}}{F_n}=\varphi\frac{1-\left(\frac{\widehat{\varphi}}{\varphi}\right)^{n+1}}{1-\left(\frac{\widehat{\varphi}}{\varphi}\right)^{n}}<\varphi$$
and if n is even, we have
$$\frac{F_{n+1}}{F_n}=\varphi\frac{1-\left(\frac{\widehat{\varphi}}{\varphi}\right)^{n+1}}{1-\left(\frac{\widehat{\varphi}}{\varphi}\right)^{n}}>\varphi$$
Using equation (3) and the fact that the Fibonacci sequence is increasing, we get the estimate
$$\left|\varphi-\frac{F_{n+1}}{F_n}\right|<\left|\frac{F_{n+2}}{F_{n+1}}-\frac{F_{n+1}}{F_n}\right|=\left|\frac{F_{n+2}F_n-F_{n+1}^2}{F_{n+1}F_n}\right|=\frac{1}{|F_{n+1}F_n|}<\frac{1}{F_n^2}\,.$$