Neg curved geometry 25.11.2020

CAT(-1)-spaces

$\kappa \in \mathbb{R}$

If (X,d) is a metric space, $k>0$,

$\rightsquigarrow (kd)(x,y) = k(d(x,y))$

(X, kd) is a metric space

- $(\mathbb{E}^n, d_{\mathbb{E}^n}) \rightsquigarrow (\mathbb{R}^n, k\, d_{\mathbb{E}^n})$.

$F(x) = \frac{1}{k} x$, $F : \mathbb{E}^n \to (\mathbb{R}^n, k d_{\mathbb{E}^n})$

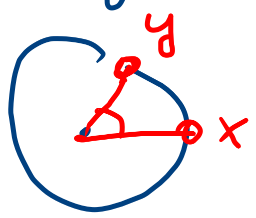
$$k \left\| \tfrac{1}{k} x - \tfrac{1}{k} y \right\| = \| x - y \| = d_{\mathbb{E}}(x,y)$$

$$\| $$

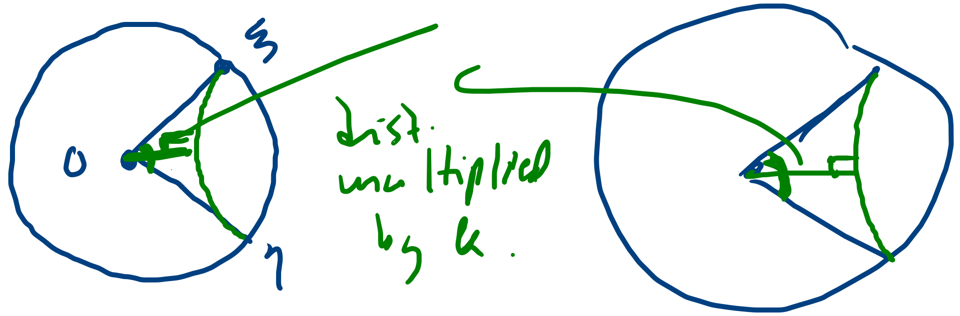
$$k\, d_{\mathbb{E}}(F(x), F(y)) \quad \therefore \ F \text{ isometry.}$$

- $(S^{n-1}, d_{S^{n-1}})$ $\rightsquigarrow$ $(S^{n-1}, k\, d_{S^{n-1}})$ isometric to the sphere with radius k with the angle metric, not isom. with $(S^{n-1}, d_{S^{n-1}})$ if $k \neq 1$.

unit sphere in $\mathbb{E}^n$ with angle metric



- $(\mathbb{H}^n, d_{\mathbb{H}^n}) \rightsquigarrow (\mathbb{H}^n, k\, d_{\mathbb{H}^n})$ not isom to $\mathbb{H}^n$ if $k \neq 1$.



Model space of curvature $\varkappa \in \mathbb{R}$

$$\overline{X}_\varkappa = \begin{cases} S^2_\varkappa = \left(S^2, \frac{1}{\sqrt{\varkappa}} d_{S^2}\right) & \varkappa > 0 \\ \mathbb{E}^2 & \varkappa = 0 \\ \mathbb{H}^2_\varkappa = \left(\mathbb{H}^2, \frac{1}{\sqrt{-\varkappa}} d_{\mathbb{H}^2}\right) & \varkappa < 0 \end{cases}$$

Note: Small spheres are more curved and big spheres are almost flat.

$\varkappa$ is the Riemannian / Gaussian curvature of $\overline{X}_\varkappa$ (curv. of $S^2 = 1$, $\mathbb{E}^2 = 0$, $\mathbb{H}^2 = -1$)

If X is a metric space and Δ a triangle in X / $x, y, z \in X$, there are triangles with the same side lengths as Δ / side lengths $d(x,y), d(y,z)$ and $d(z,x)$. in $\mathbb{E}^2$ and $\mathbb{H}^2_\varkappa$

Prop. 2.4 Prop. 5.10

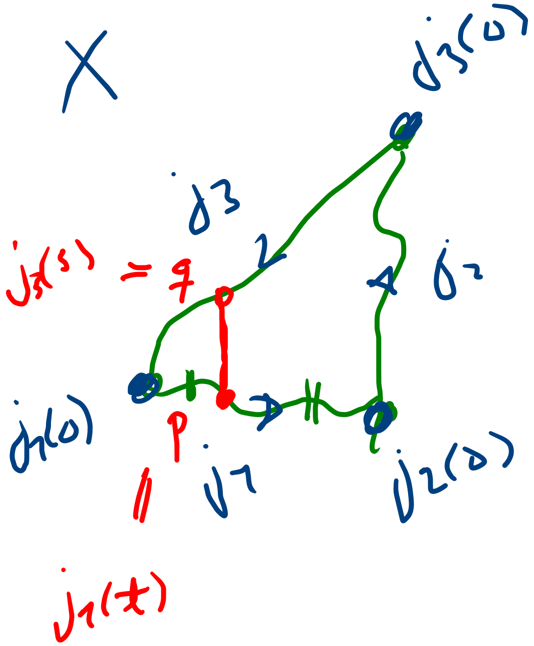
Comparison triangle of x, y, z.

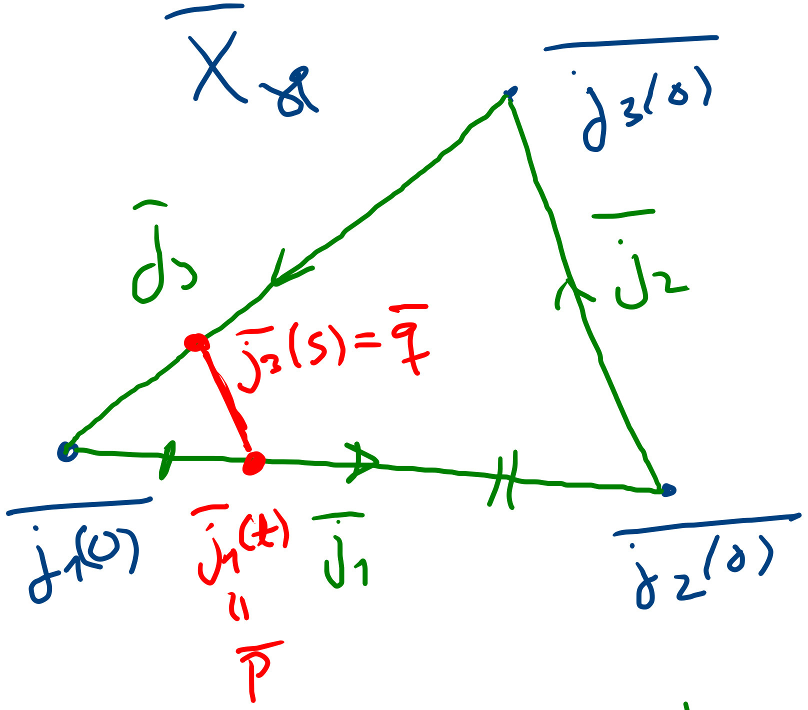
—— " —— of Δ.

②

X geodesic metric space, $\{\gamma_1, \gamma_2, \gamma_3\}$ a triangle in X. $\kappa \leq 0$

geodesic segments $\gamma_i : I_i \to X$, $I_i = [0, b_i]$, $\gamma_1(b_1) = \gamma_2(0)$, $\gamma_2(b_2) = \gamma_3(0)$, $\gamma_3(b_3) = \gamma_1(0)$.





$\exists\ \overline{\gamma_1(0)}, \overline{\gamma_2(0)}, \overline{\gamma_3(0)} \in \bar X_\kappa$ s.t. $d(\overline{\gamma_i(0)}, \overline{\gamma_k(0)}) = d(\gamma_i(0), \gamma_k(0))$

Let $\bar\gamma_1, \bar\gamma_2, \bar\gamma_3$ be the sides of the comp. triangle with vertices $\overline{\gamma_i(0)}$

s.t. $\bar\gamma_i(0) = \overline{\gamma_i(0)}$ for $i \in \{1, 2, 3\}$.

Any point $\gamma_i(t) = p$ has a comparison point $\bar\gamma_i(t) = \bar p$

$d(\gamma_1(0), p) = d(\bar\gamma_i(0), \bar p)$

X is a CAT(κ)-space if $d(p,q) \leq d(\bar p, \bar q)$ for all pts all triangles.

③

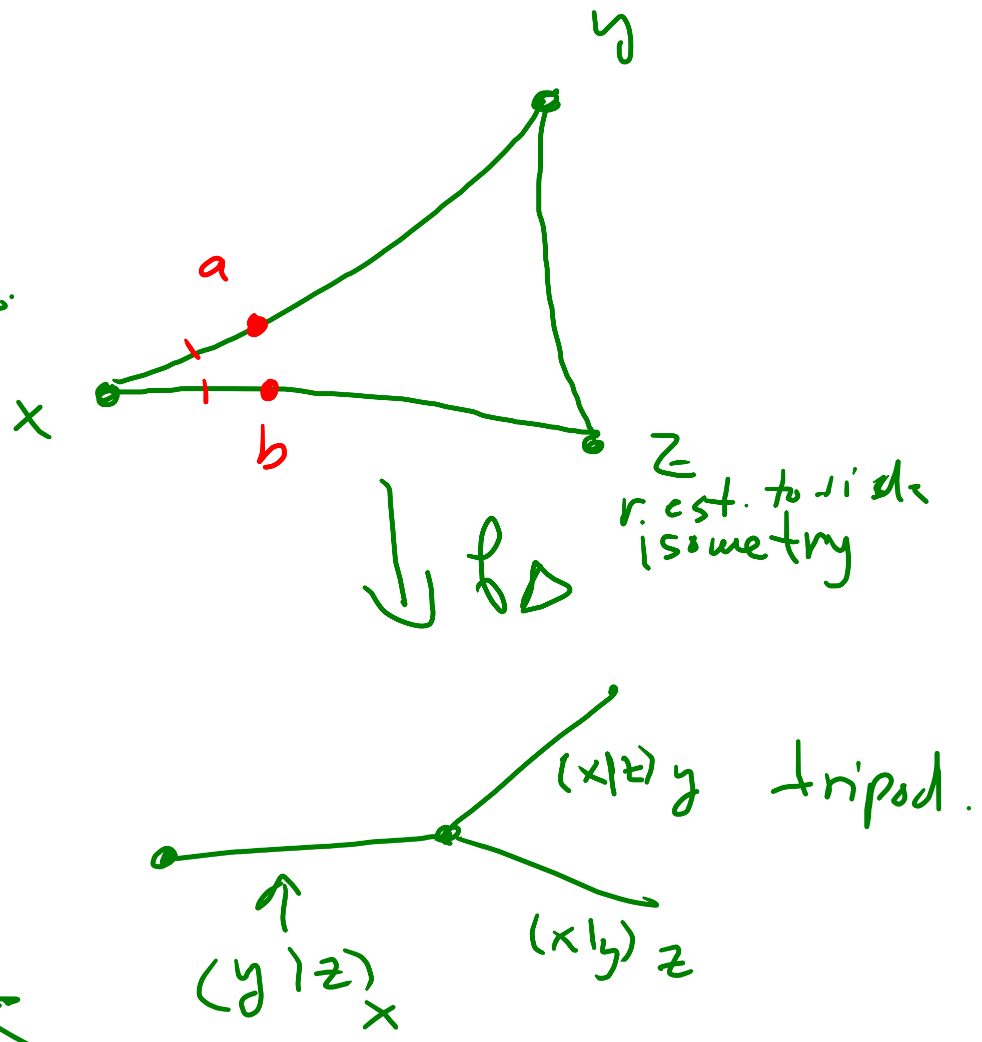
Let $\kappa = -1 \rightsquigarrow \overline{X}_{-1} = \mathbb{H}^2_{-1} = \mathbb{H}^2$.

If X is CAT(-1), then X is Gromov-hyperbolic.

E. Cartan, Aleksandrov, Toponogov

Recall that X is Gromov-hyperbolic if $\exists\, \delta > 0$ s.t. for all triangles $\Delta \subset X$ $d(a,b) \leq \delta$ for a, b in Δ with $\underline{f_\Delta(a) = f_\Delta(b)}$

$\Leftrightarrow$ dist. of a and b from a vertex is the same.



If X is CAT(-1), then $d(a,b) \leq \underline{d(\bar{a},\bar{b}) \leq \delta}$

where $\bar{a}, \bar{b}$ are comparison points in a comparison triangle $\overline{\Delta}$ of Δ in $\mathbb{H}^2$. Note: the tripod of $\overline{\Delta}$ is the same as for Δ $\Rightarrow$ $\boxed{f_{\overline{\Delta}}(\bar{a}) = f_{\overline{\Delta}}(\bar{b})}$

because $\mathbb{H}^2$ is Gromov hyp.

Prop. 10.8 If $\kappa < \kappa' \le 0$ and X is a CAT(κ)-space, then X is a CAT(κ')-space.

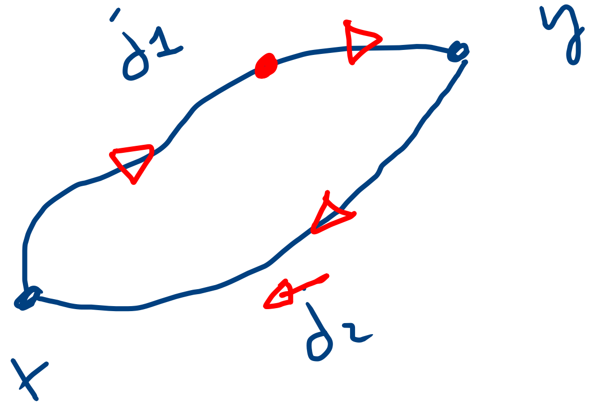
Proof. Tomorrow.

$\leadsto$ CAT(-1)-spaces are CAT(0)-spaces.

Prop. 10.9 CAT(0)-spaces are uniquely geodesic.

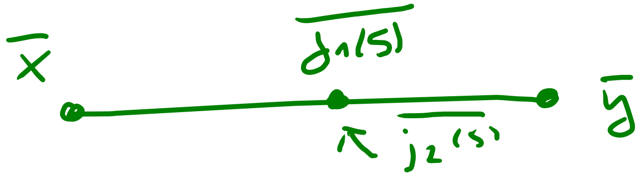
(P.10.8 $\Rightarrow$ CAT(κ)-spaces are uniquely geodesic for $\kappa \le 0$)

Proof. Let X be a CAT(0)-space. Let $x, y \in X$. Let $\gamma_1, \gamma_2 : [0, d(x,y)] \to X$ be geod. segments s.t. $\gamma_1(0) = \gamma_2(0) = x$ and $\gamma_1(d(x,y)) = \gamma_2(d(x,y)) = y$.



Let $0 < s < d(x,y)$ and consider the triangle Δ with sides $\gamma_1|[0,s]$, $\gamma_1|[s, d(x,y)]$ and γ_2 (in the opposite direction).

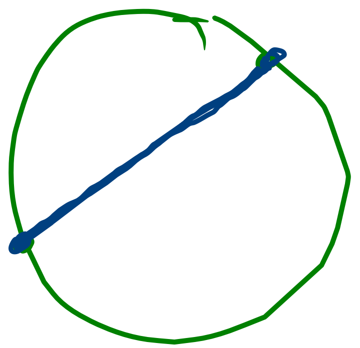
The comparison triangle of Δ in $\mathbb{E}^2$ has side lengths s, $d(x,y) - s$, $d(x,y)$ so $\bar{\Delta}$ is degenerate:



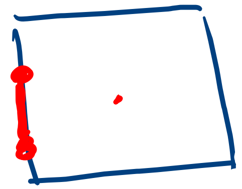
$\Rightarrow d(\gamma_1(s), \gamma_2(s)) \le d(\overline{\gamma_1(s)}, \overline{\gamma_2(s)}) = 0$

$\Rightarrow \gamma_1 = \gamma_2$. $\square$

Prop. 10.10 Ball are <u>strictly convex</u> in CAT(0)-spaces.



($(\mathbb{R}^2, d_\infty)$

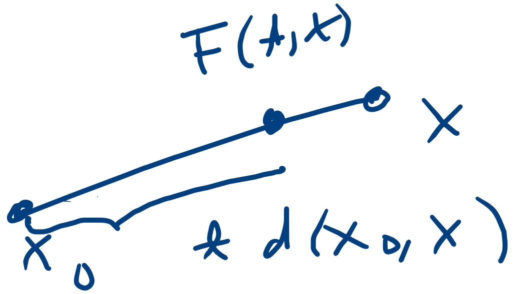


ball not strictly convex!)

Proof. Exercise.

Prop. 10.12. CAT(0)-spaces are <u>contractible</u>: If X is CAT(0) and $x_0 \in X$ $\exists$ $F : [0,1] \times X \to X$ continuous s.t. $F(0,x) = x_0$ $\forall x \in X$ and $F(1,x) = x$ $\forall x \in X$.

Proof. $x_0 \in X$. $\forall x \in X - \{x_0\}$ $\exists_1$ geod. segment $g_x : [0, d(x_0,x)] \to X$ (Prop. 10.9)
s.t. $g_x(0) = x_0$, $g_x(d(x_0,x)) = x$.



$F(t,x) = g_x(t\,d(x_0,x))$.

$F(1,x) = g_x(d(x_0,x)) = x$, $F(0,x) = g_x(0) = x_0$. to be continued...