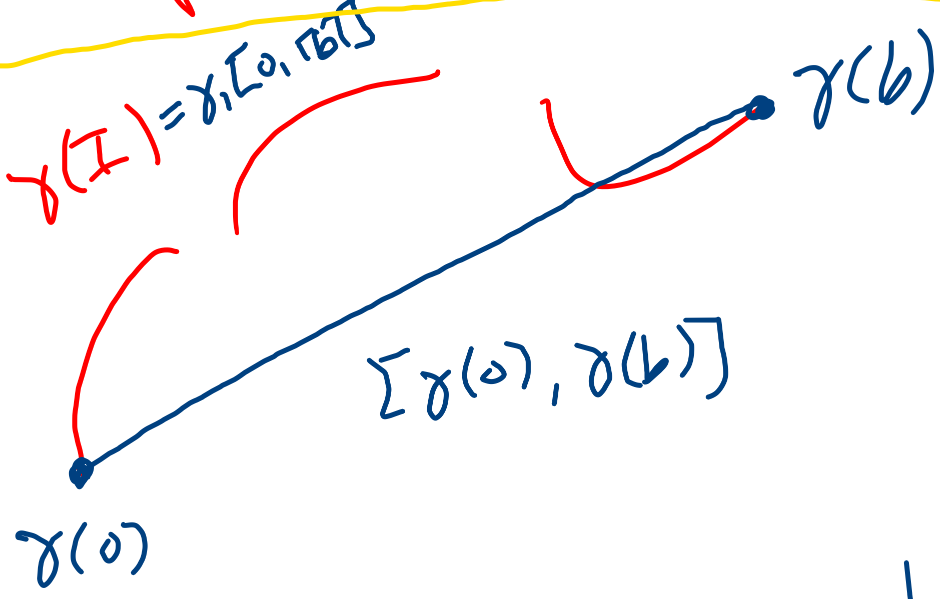
Neg. curved spaces 22.10.2020

Thm 7.8. X, Y quasiisometric geodesic spaces.
X Gromov-hyp $\Longleftrightarrow$ Y Gromov-hyp.

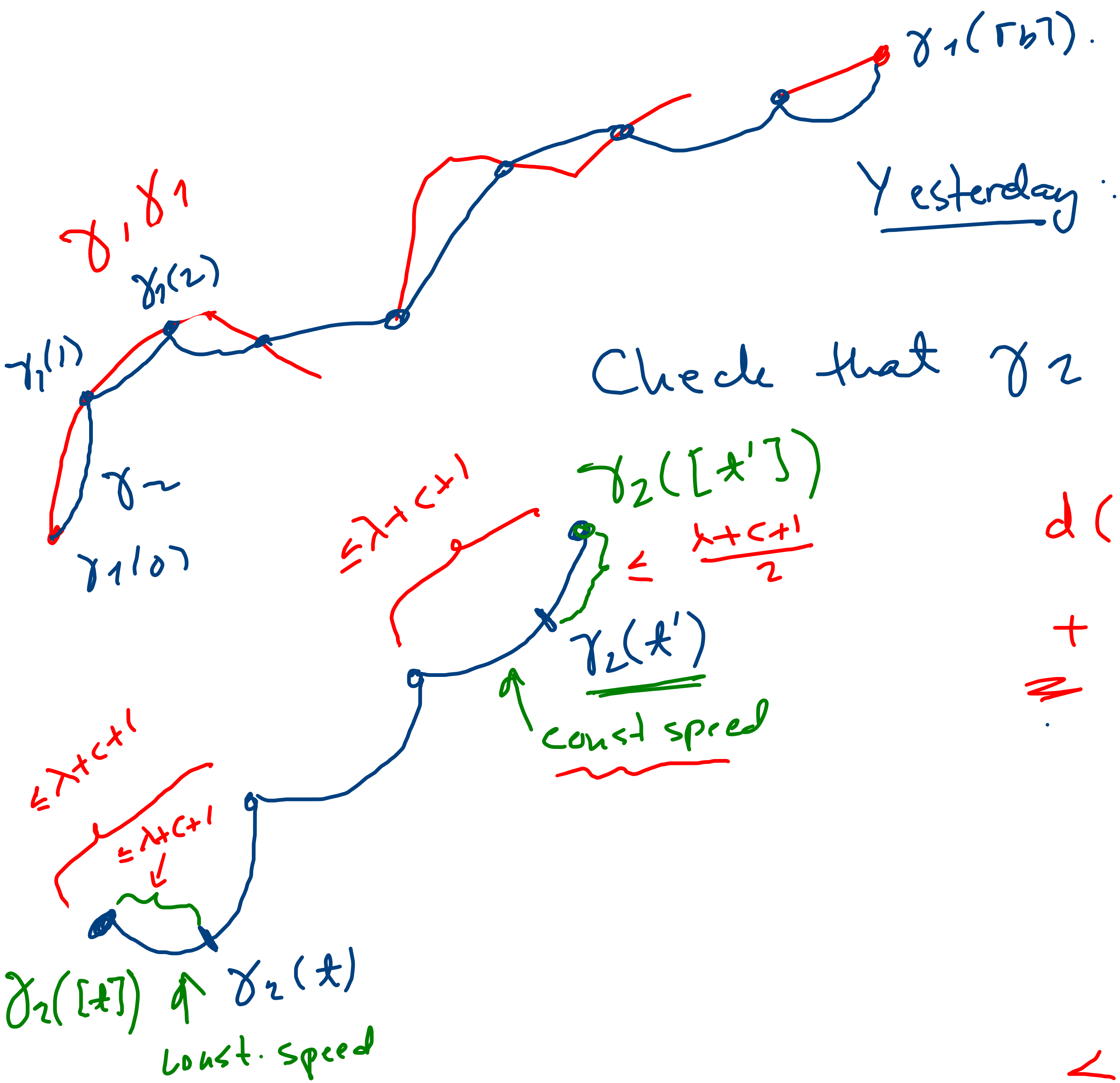
Thm 7.10 $\delta \geq 0$, $\lambda \geq 1$, $c \geq 0$. $\exists\, R = R(\delta, \lambda, c)$:
If $\gamma : I \to X$ (λ, c)-quasig. segment, X δ-hyp

stability of quasigeod. segments



$$d_{Haus}\left(\gamma(I), [\gamma(0), \gamma(b)]\right) \leq R.$$

Proof. $\gamma \rightsquigarrow \gamma_1$ defined on $[0, [b]]$
$\rightsquigarrow \gamma_2$ continuous path made of geod. segments and constant mappings: $\beta_1 * \beta_2 * \dots * \beta_{[b]}$



Yesterday: $\boxed{d_{Haus}(\gamma_1,\gamma_2) \le \lambda + \frac{3}{2}(c+1)}$

Check that γ_2 is quasigeodesic:

$$d(\gamma_2(t),\gamma_2(t')) \le \overbrace{d(\gamma_2(t),\gamma_2([t]))}^{\le \frac{\lambda+c+1}{2}}$$

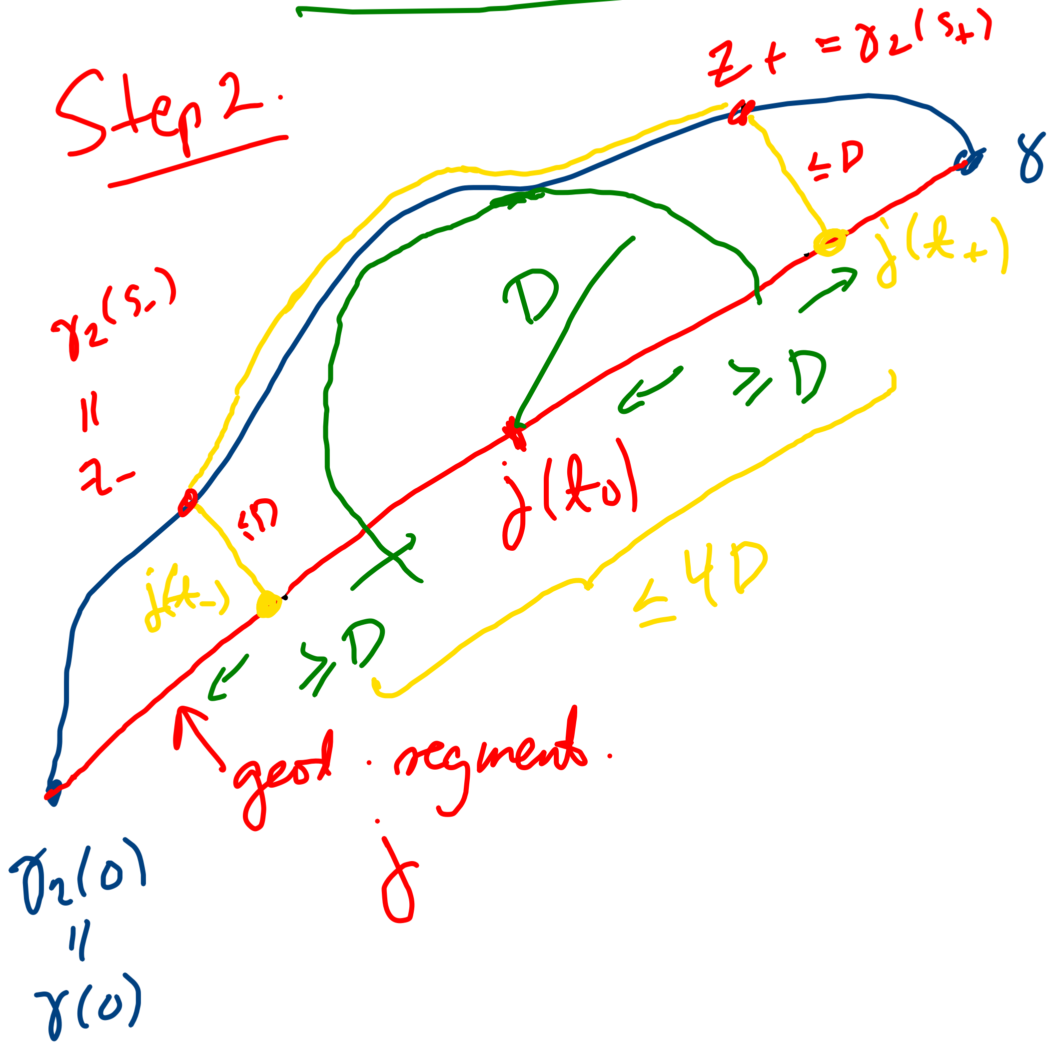
$$+ \underbrace{d(\gamma_{\underset{1}{2}}([t]),\gamma_{\underset{1}{2}}([t']))}_{\le \lambda \underbrace{|[t]-[t']|}_{|t-t'|+1} + c+1} + \underbrace{d(\gamma_2([t']),\gamma_2(t'))}_{\le \frac{\lambda+c+1}{2}}$$

$$\le \lambda|t-t'| + C$$

+ other direction

$\rightsquigarrow$ γ_2 is $(\lambda, 2(\lambda+c+1))$-quasigeodesic.

Step 2.



Show that $[\gamma(0), \gamma(b)]$ is close to γ_2.

$$D = \max\{ d(j(t), \gamma_2([0,\lceil b\rceil])) : t \in [0, d(\gamma(0),\gamma(b))]\} < \infty .$$

Let t_0 be such that $d(j(t_0), \gamma_2( )) = D$.

$$t_- = \max(0, t_0 - 2D)$$

$$t_+ = \min(d(\gamma(0),\gamma(b)), t_0 + 2D)$$

$$\boxed{d(z_-, z_+) \le 6D}$$

$$\ell(\gamma_2|_{[s_-,s_+]}) \le (\lambda + c + 1)([s_+] - [s_-] + 2)$$

$$\underbrace{d(\gamma_2(t), \gamma_2(t'))}_{} \geq \frac{1}{\lambda} \underbrace{|t-t'|}_{} - \underbrace{C'}_{}$$

$\rightsquigarrow$ bound on $\boxed{\ell(\eta) \le (K+2)D + K'}$ for K, K' depending on data. (λ, c)

Prop. 6.8 if $\tilde{\gamma}$ $: \tilde{I} \to X$ rectifiable and $\tilde{j}$ geod. joins its endpts, then $d(\tilde{j}(t), \tilde{\gamma}(\tilde{I})) \le \underset{A}{\delta} \log_2 \ell(\tilde{\gamma}) + 1$

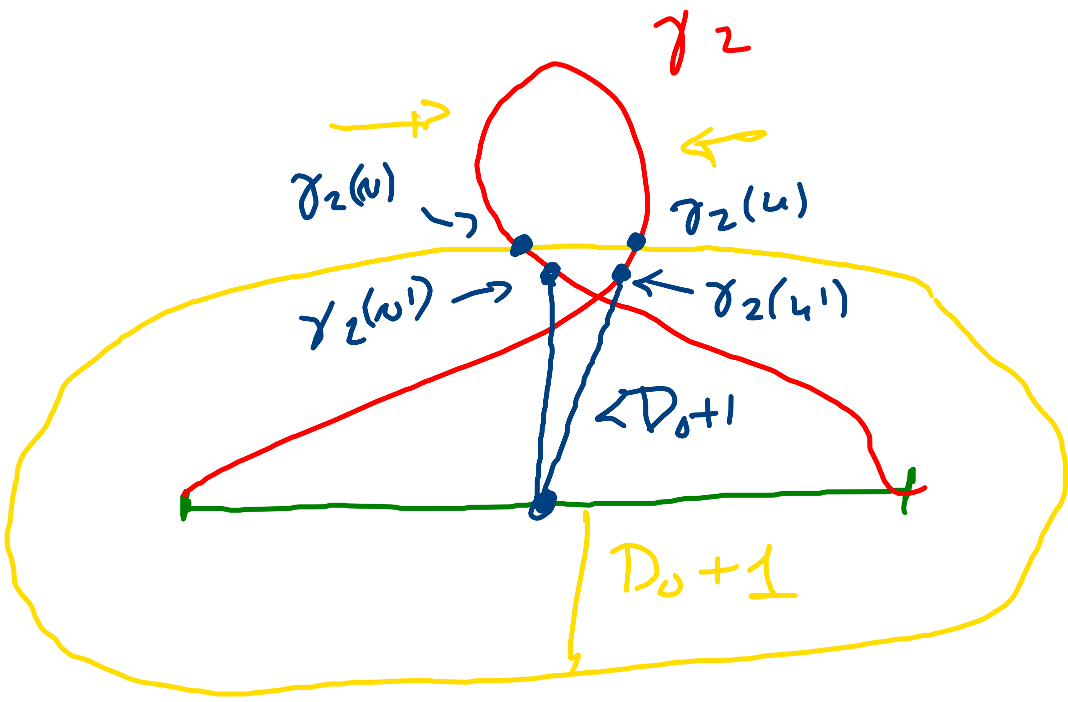
$$D = d(j(t_0), \eta) \le \delta \log_2 \ell(\eta) + 1 \le \delta \log_2\left(\underline{(K+2)D} + \underline{K'}\right) + 1$$

Cannot hold for big D $\rightsquigarrow$ get an upper bound on D

$\underline{D_0 = D_0(\lambda, c, \delta)}$

$$\Rightarrow \forall t \quad d(j(t), \gamma_2([0, \lceil b \rceil])) \le D_0$$

$$\rightsquigarrow [\gamma(0), \gamma(b)] \subset \underline{\overline{N}_{D_0}(\gamma_2([0, \lceil b \rceil]))}$$



Let $[u,v] \subset [0,\lceil b\rceil]$ be maximal such that $\gamma_2([u,v])$ is not in $\mathcal{N}_{D_0+1}([\gamma(0),\gamma(b)])$.

$$\underset{J}{\underset{\uparrow}{[\gamma(0),\gamma(b)]}} \subset \underbrace{\mathcal{N}_{D_0+1}(\gamma_2|_{[0,u[})}_{U \text{ open}} \cup \underbrace{\mathcal{N}_{D_0+1}(\gamma_2(]v,\lceil b\rceil]))}_{V \text{ open}}$$

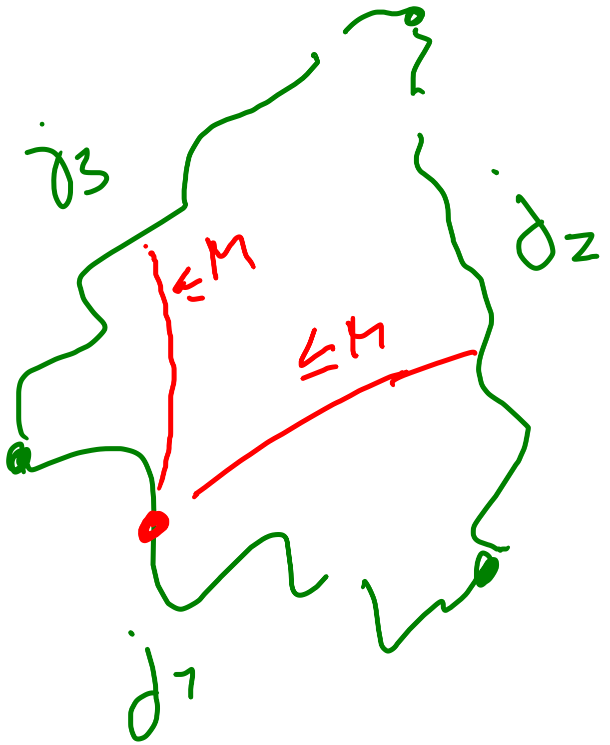
$\underline{\underline{U \cap J \neq \emptyset}}$, $\underline{\underline{V \cap J = \emptyset}}$, J connected

$\Rightarrow (U \cap J) \cap (V \cap J) \neq \emptyset$.

$$d(\gamma_2(u'),\gamma_2(v')) \leq 2(D_0+1)$$

As in step 2 $\rightsquigarrow$ $\ell(\gamma_2|_{[u',v']}) \leq 2K(D_0+1)+K' \Rightarrow$ all pts in $\underline{\underline{\gamma_2([u',v'])}}$ are at dist $\leq \boxed{K(D_0+1)+\frac{K'}{2}}$ of $\underline{\mathcal{N}_{D_0+1}([\gamma(0),\gamma(b)])}$. $\square$

⑤



$\lambda \geq 1$, $c \geq 0$, $\gamma_k : \underset{=[0,b_k]}{I_k} \to X$ (λ, c)-quasigeod segm.

$k \in \{1, 2, 3\}$.

$\gamma_1(b_1) = \gamma_2(0)$

$\gamma_2(b_2) = \gamma_3(0)$

$\gamma_3(b_3) = \gamma_1(0)$

$\longrightarrow$ (λ, c)-quasi triangle.

Extend Rips condition to quasitriangles.

Corollary 7.11 X δ-hyp, $\lambda \geq 1$, $c \geq 0$. $\exists$ $M = M(\delta, \lambda, c)$ s.t. all (λ, c)-quasitriangles in X satisfy the Rips condition with constant M.

Proof. Ex.

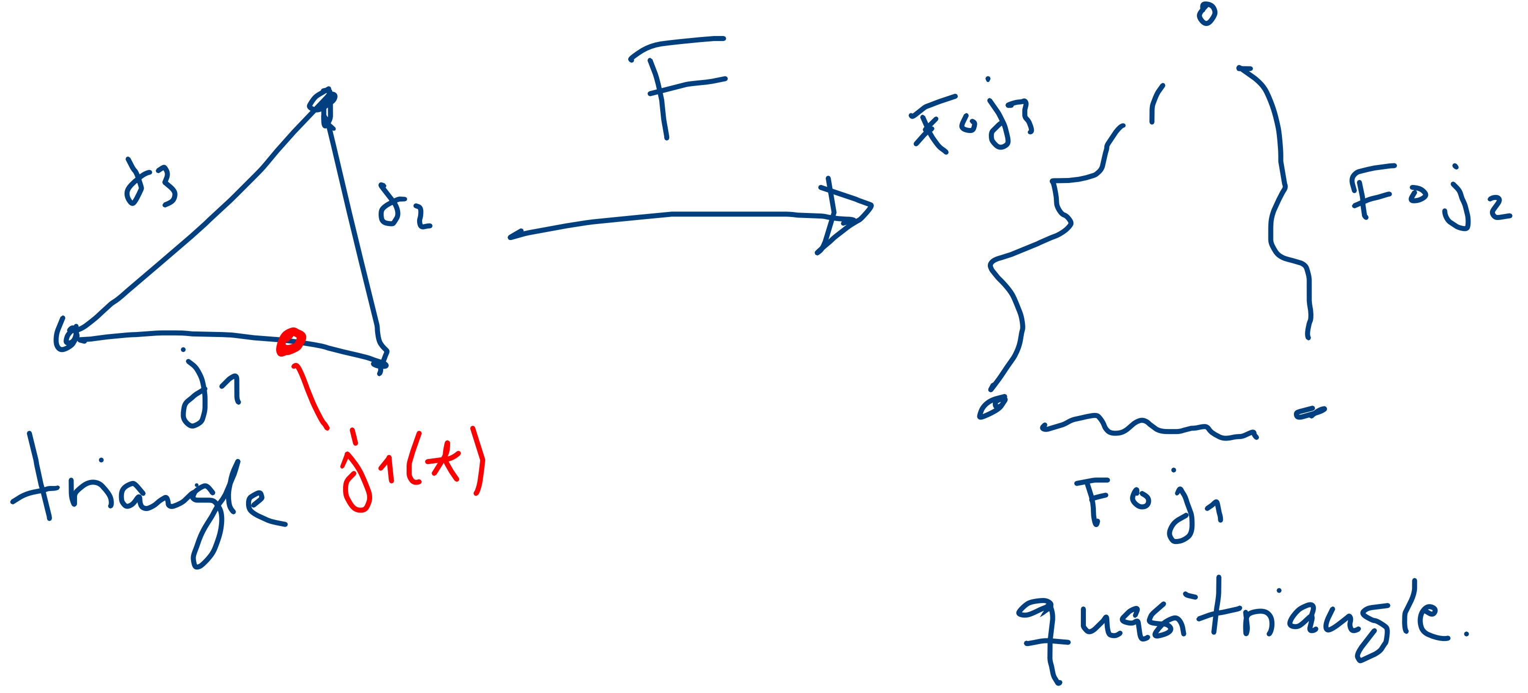
Proof of Thm 7.8 is an immediate corollary of thm 7.12.

Thm 7.12 X geod. metric space, Y δ-hyp, $F: X \to Y$ (λ, c)-q.i./emb.

Then X is δ'-hyperbolic for some δ'.

Proof. $j_i : I_i \to X$

X



Cor 7.11: $\exists\ s \in I_2$ (or I_3 :)

$$d(F \circ j_1(*), F \circ j_2(s)) \le M$$

$$\Rightarrow d(j_1(*), j_2(s)) \le \underbrace{\lambda M + c}_{\delta'}$$

∴ triangle satisfies Rips condition with δ'.

⇒ X is δ'-hyp. ∎