Neg. curved geometry 21.10.2020

Quasi-isometric embeddings and quasi-isometries

Def$^{\underline{n}}$ X, Y metric spaces, $F: X \to Y$, $\lambda \geq 1$, $c \geq 0$

$$\frac{1}{\lambda} d(x_1, x_2) - c \leq d(F(x_1), F(x_2)) \leq \lambda d(x_1, x_2) + c \quad \forall x_1, x_2 \in X$$

F (λ, c)-quasi-isometric embedding.

$$\begin{tikzcd} X \arrow[r, bend left, "F"] & Y \arrow[l, bend left, "\bar{F}"] \end{tikzcd}$$

If $F: X \to Y$ is a q.i. embedding

$\bar{F}: Y \to X$

$$d(x, \bar{F}(F(x))) \leq K$$

$$d(y, F(\bar{F}(y))) \leq K$$

for some $K \geq 0$ $\forall x \in X$ $\forall y \in Y$.

$\bar{F}$ is a quasi-inverse of F

F is a quasi-isometry

Lemma 7.3 a) $X \xrightarrow{F} Y \xrightarrow{G} Z$ if F, G q.i. embeddings,

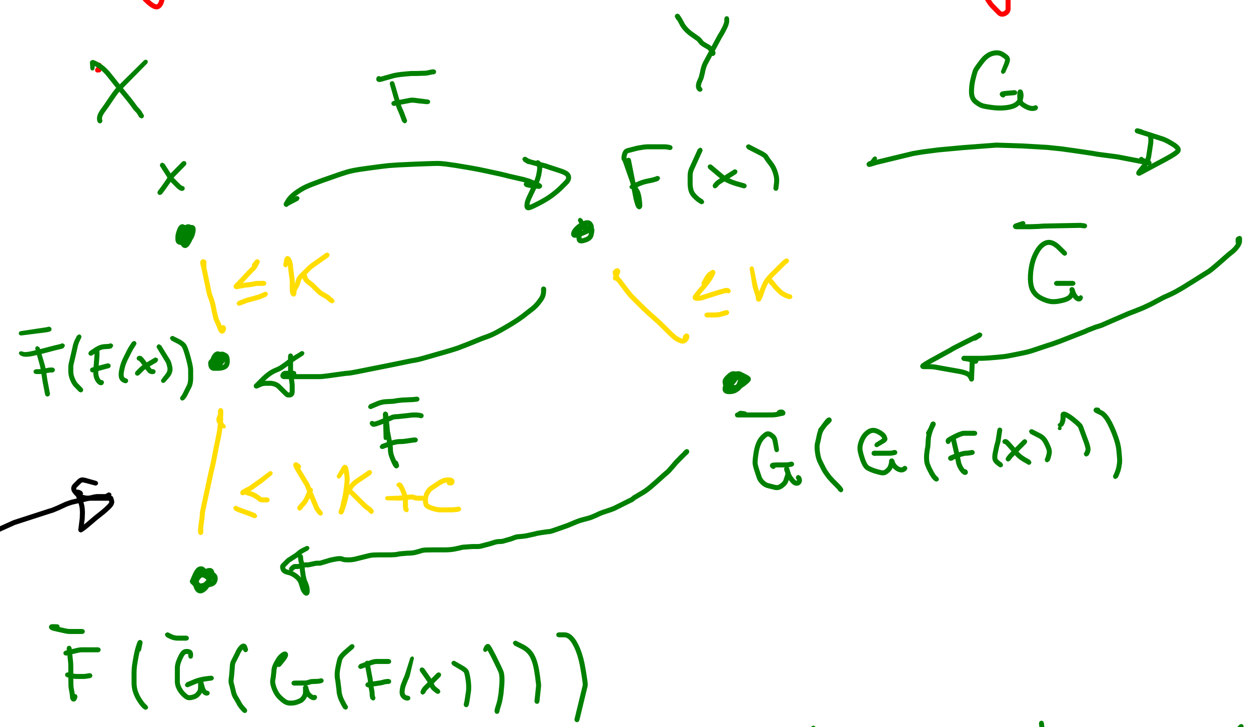
then $\boxed{G \circ F \text{ is a q.i. embedding}}$

Exercise

Lemma 7.4 $X \xrightarrow{F} Y \xrightarrow{G} Z$, $\bar{F}: Y \to X$, $\bar{G}: Z \to Y$. If F, G are q.i., then $G \circ F$ is a q.i.

Proof. Let $\bar{F}, \bar{G}$ be quasi-inverses of F and G.

$\boxed{\bar{F}\ (\lambda, c)\text{-q.i. embedding}}$



q.i. emb.

$$d(x, (\bar{F} \circ \bar{G}) \circ (G \circ F)(x))$$

$$\leq \underbrace{d(x, \bar{F}(F(x)))}_{\leq K} + \underbrace{d(\bar{F}(F(x)), \bar{F}(\bar{G}(G(F(x)))))}_{\leq \lambda K + C}$$

$$\leq (\lambda + 1) K + c \quad \forall x \in X.$$

Estimate for $d(y, (G \circ F) \circ (\bar{F} \circ \bar{G}))$ in the same way. $\square$

②

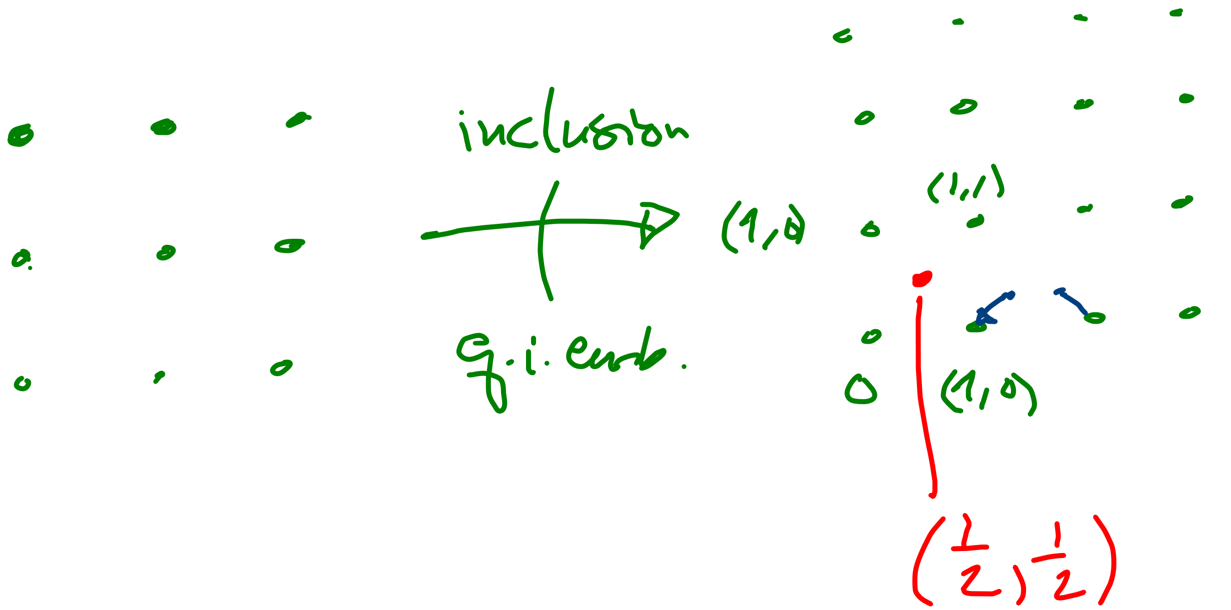
Prop. 7.6. If $F : X \to Y$ is a q.i. emb such that

$$\sup\{ d(y, F(X)) : y \in Y \} < \infty,$$

then F is a quasi-isometry.

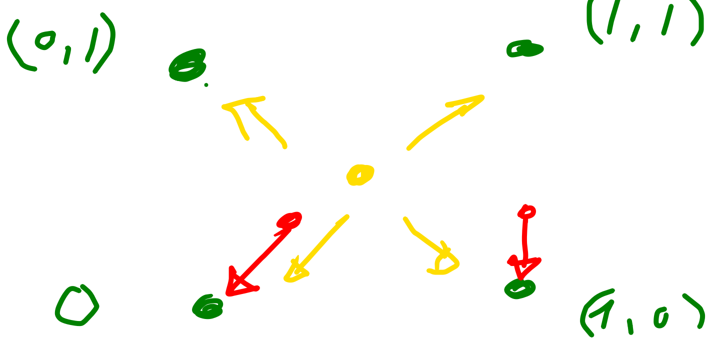
Proof. Exercise

Example $\mathbb{Z}^2 \xhookrightarrow{i} \mathbb{E}^2$



$$\sup\{ d(y, \mathbb{Z}^2) : y \in \mathbb{E}^2 \} = \frac{\sqrt{2}}{2}.$$

Prop. 7.6 : i is a q.i.



Thm. 7.8 If X, Y are geodesic quasi-isometric, then X Gromov-hyperbolic $\iff Y$ —"— ——— .

Thm 7.10 Let $\delta \geq 0$, $\lambda \geq 1$, $c \geq 0$. $\exists R = R(\delta, \lambda, c)$ such that:

If $\gamma : I \to X$ ($I = [0,b]$) is a (λ, c)-quasigeodesic segment in a δ-hyp space X,

(λ, c)-q.i.

then $d_{Haus}(\gamma(I), [\gamma(0), \gamma(b)]) \leq R$

any geod. segment w/ endpts $\gamma(0), \gamma(b)$

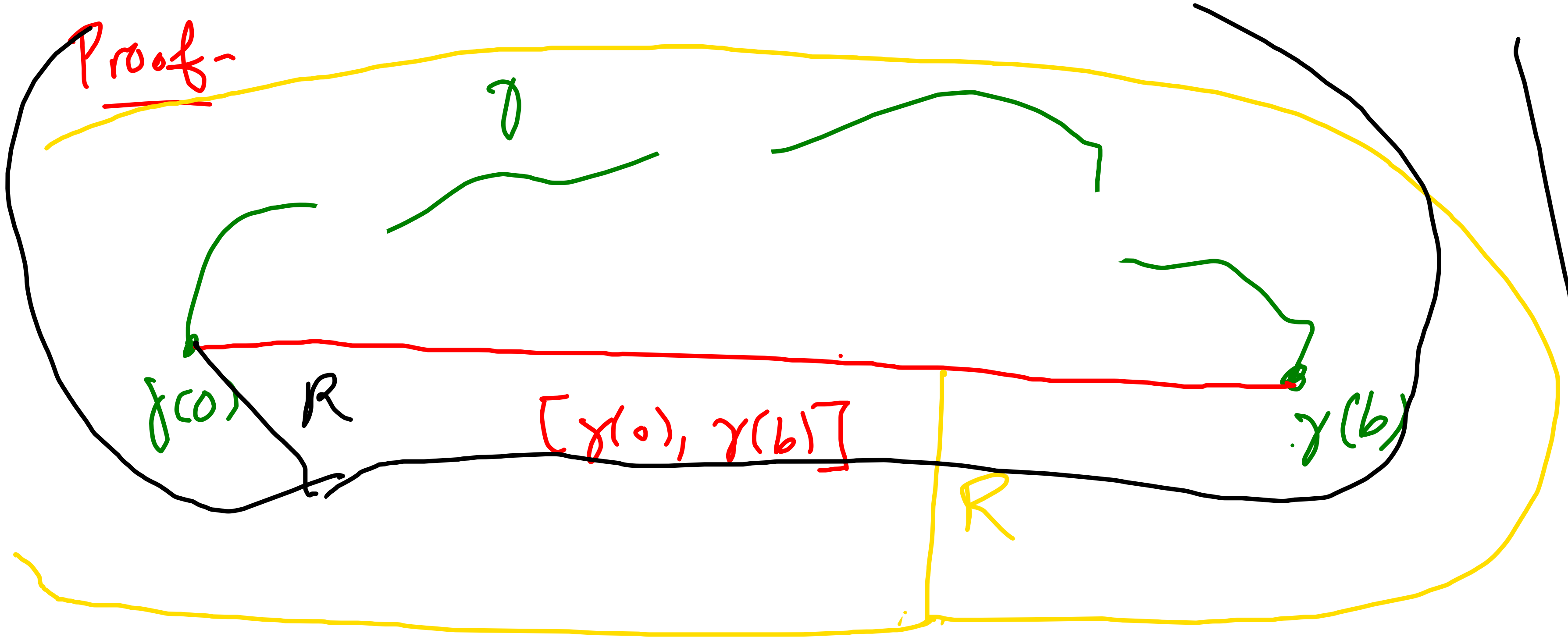
Def^{n} X metric space, $A, B \subset X$, $A \neq \emptyset \neq B$.

$$d_{Haus}(A,B) = \inf\{\varepsilon > 0 : A \subset N_\varepsilon(B) \ \& \ B \subset N_\varepsilon(A)\}$$

Hausdorff distance of A and B.

- $d_{Haus}(A,A) = 0$
- symm.
- triangle ineq OK
- may be ∞
- $d_{Haus}(A, \bar{A}) = 0$

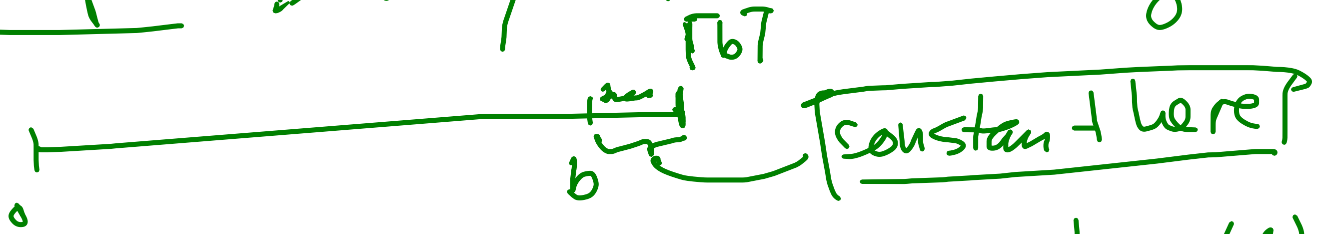
Proof-



Strategy 1) Replace γ by a continuous quasigeod that is close to γ

2) Use continuity & compactness to show this new quasigeod is "close" to any geod. segment with the same endpoints.

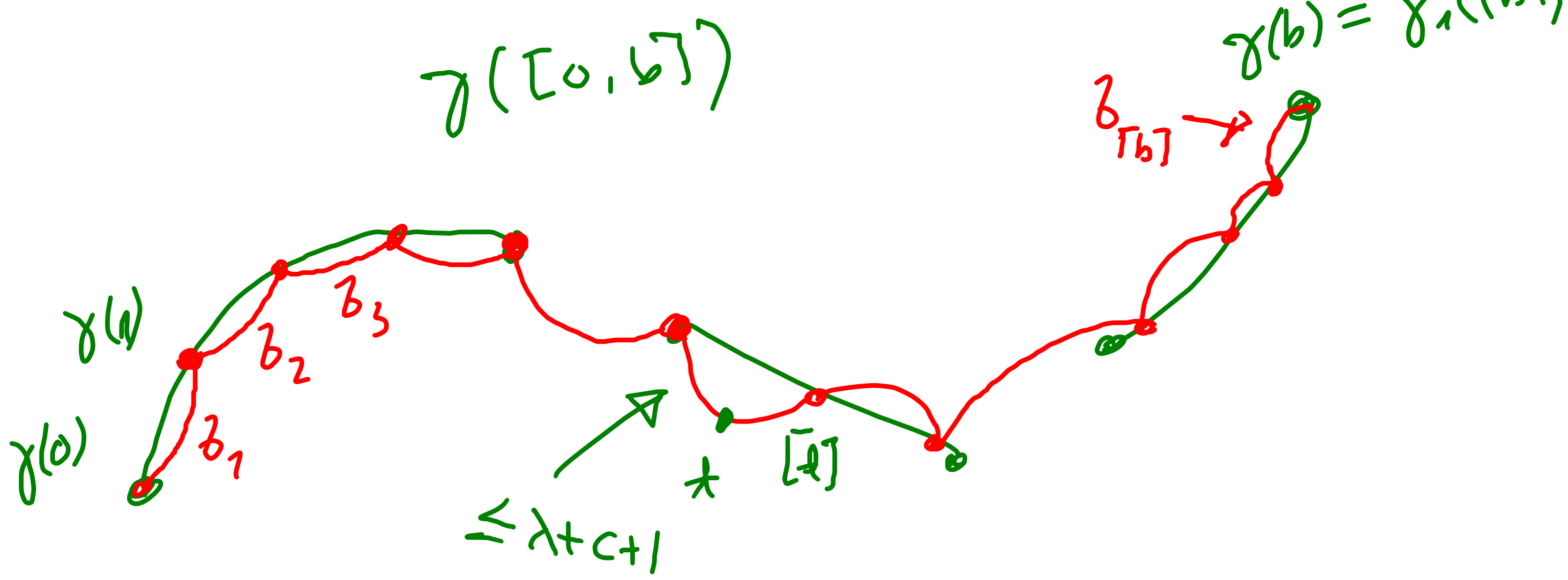
Step 0 b may not be an integer.



$\gamma_1 : [0, \lceil b \rceil] \to X$, $\gamma_1(t) = \begin{cases} \gamma(t) & \text{if } t \leq b \\ \gamma(b) & \text{if } t > b. \end{cases}$

γ_1 is a $(\lambda, C+1)$-quasigeod. segment.

Step 1



$\beta_i : [i-1, i] \longrightarrow X$

affinely reparametrized geod. segment or constant map

constant speed that may not be 1

$d(\beta_i(s), \beta_i(t)) = k_i |s-t|$

const. speed

s.t. $\beta_i(i-1) = \gamma_1(i-1)$

$\beta_i(i) = \gamma_1(i)$

$\gamma_2 = \beta_1 * \beta_2 * \cdots * \beta_{[b]}$

continuous path

γ_1 $(\lambda, c+1)$ q.geod segment $\Rightarrow d(\gamma_1(i-1), \gamma_1(i)) \leq \lambda \cdot 1 + c + 1$.

$d_{Haus}(\gamma_1, \gamma_2) \leq \lambda + \frac{3}{2}(c+1)$

$d(\gamma_2(t), \gamma_2([t])) \leq \frac{\lambda + c + 1}{2}$.

$|t - [t]| \leq \frac{1}{2}$

$d(\gamma_1(t), \gamma_2([t])) \leq \frac{\lambda}{2} + c + 1$

$\Rightarrow d(\gamma_2(t), \gamma_1(t)) \leq \lambda + \frac{3}{2}(c+1)$