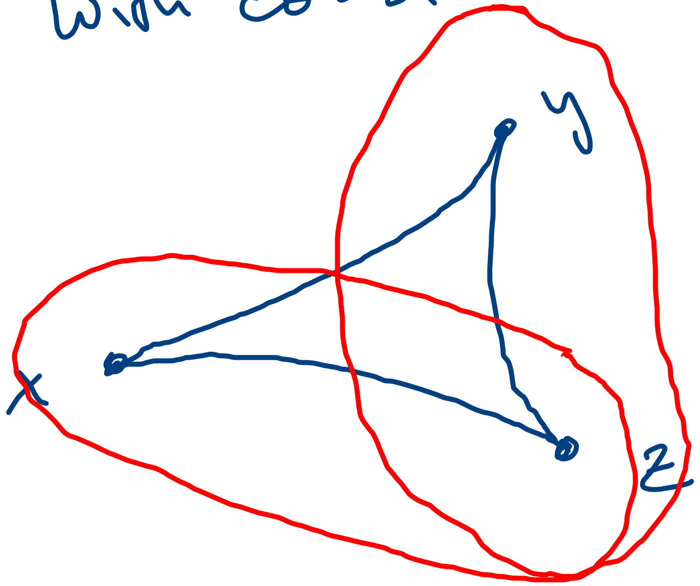
Neg. curved spaces 14.10.2020

X geod. metric space, $\delta > 0$, $\Delta \subset X$ a triangle satisfies the Rips cond. with const. δ if any side of Δ is contained in $\overline{N_\delta}$(other sides)



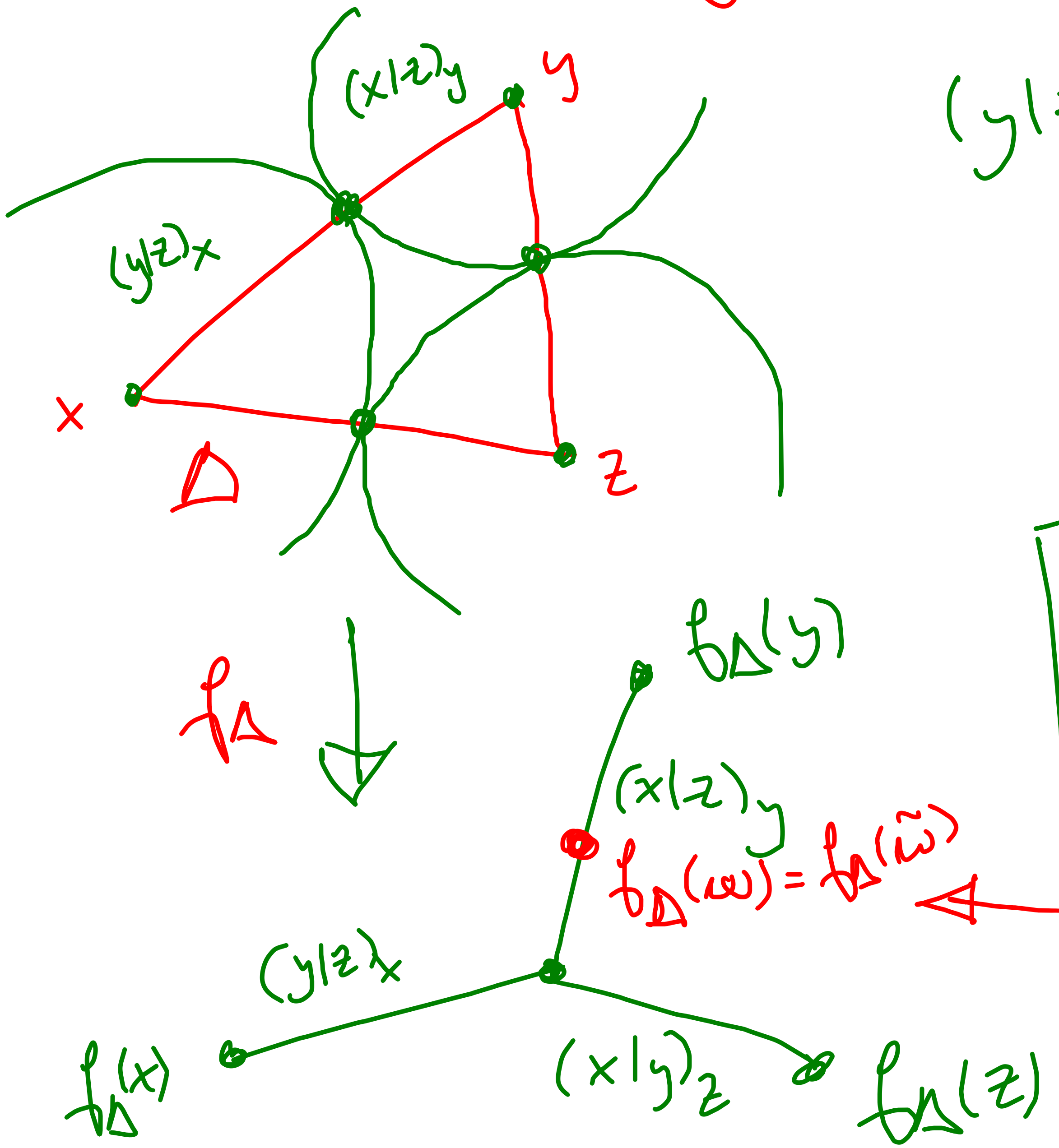
fix δ.

X is δ-hyperbolic if all triangles in X satisfy the Rips condition for δ

Ex:
- $\mathbb{H}^n$ is $\log(1+\sqrt{2})$-hyperbolic
- trees are δ-hyp $\forall \delta > 0 \rightsquigarrow 0$-hyperbolic
- $\mathbb{E}^n, n \geq 2$, is not δ-hyp for any $\delta > 0$.
- If X is bounded, X is diam X-hyp.

X is Gromov-hyperbolic if it is δ-hyp. for some $\delta > 0$.

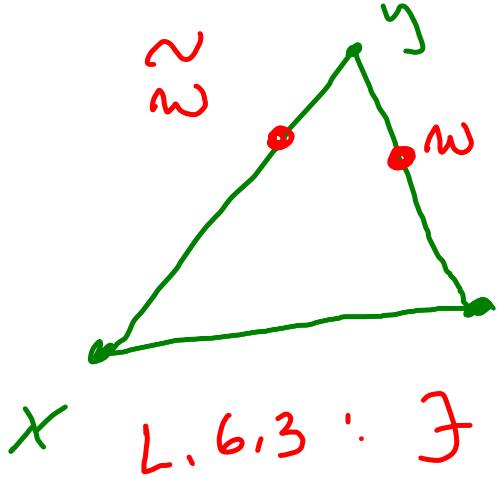
X metric space, $x, y, z \in X$. The Gromov product of x and y wrt. z is $(x|y)_z = \frac{1}{2}\big(d(z,x) + d(z,y) - d(x,y)\big) \geq 0$



$$(y|z)_x + (x|z)_y = \tfrac{1}{2}\Big(\underline{d(y,x)} + d(\cancel{z},\cancel{x}) - \cancel{d(y,z)}$$
$$+ \underline{d(x,y)} + \cancel{d(z,y)} - d(\cancel{x},z)\Big)$$
$$= d(x,y).$$

Lemma 6.4 Let X be a geod. space, Δ triangle w/ vertices x, y, z

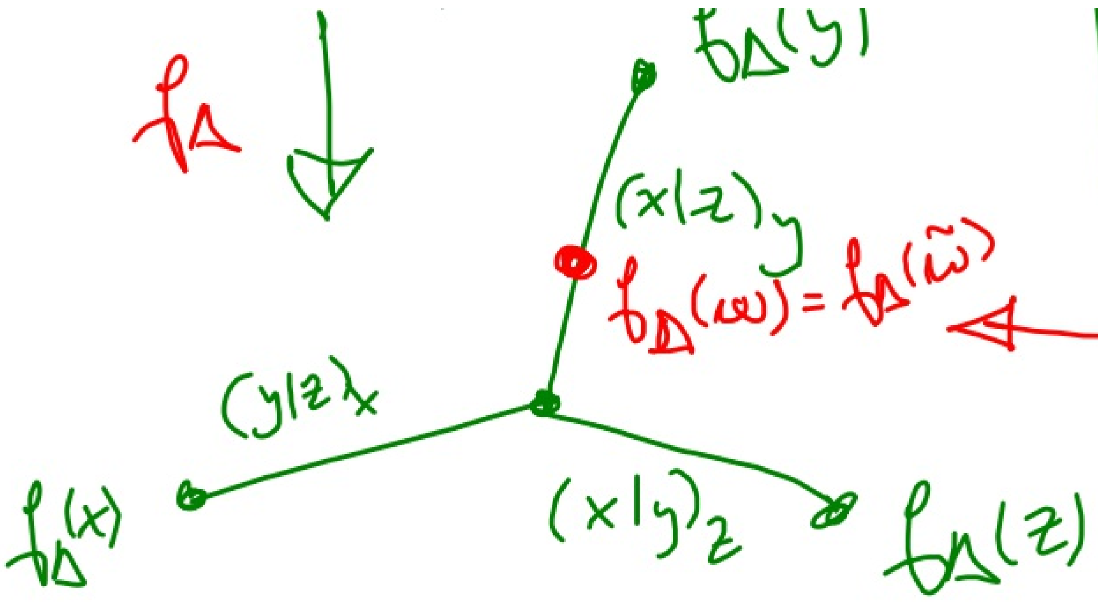
$$\boxed{(y|z)_x \leq d(x,[y,z])}$$

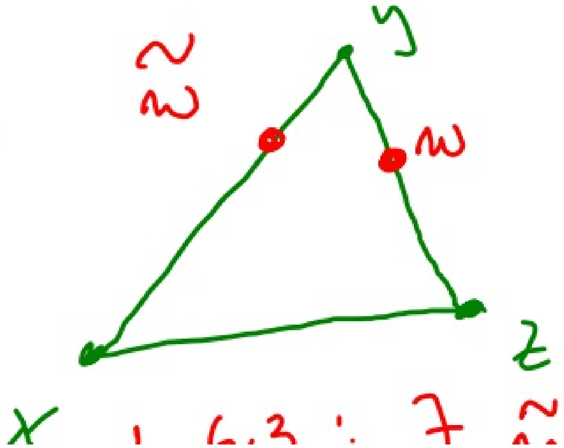


Proof. Let $w \in [y,z]$ s.t. $d(w,x) = d(x,[y,z])$.

L. 6.3: $\exists\ \tilde{w} \in [x,y] \cup [x,z]$: $f_\Delta(\tilde{w}) = f_\Delta(w)$

Δ triangle w/



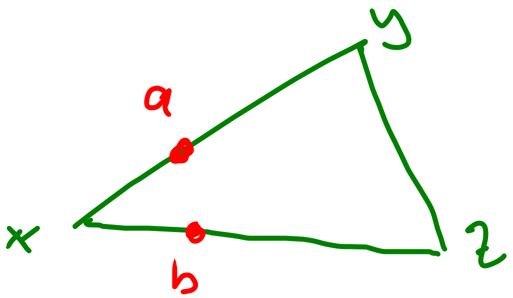


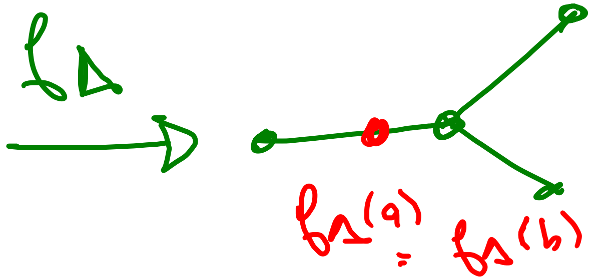
triangle ineq

$$\underline{(y|z)_x} \le d(x,\tilde w) = d(x,y) - d(y,\tilde w) = d(x,y) - d(y,w) \overset{\text{triangle ineq}}{\le} d(x,w) = \underline{d(x,[y,z])}$$

Def^n X geod space, $\delta > 0$. A triangle Δ is $\underline{\delta\text{-thin}}$ if

(3)





Ex

$d(a,b) \le \delta \quad \forall\, a,b \in \Delta$ s.t. $f_\Delta(a) = f_\Delta(b)$

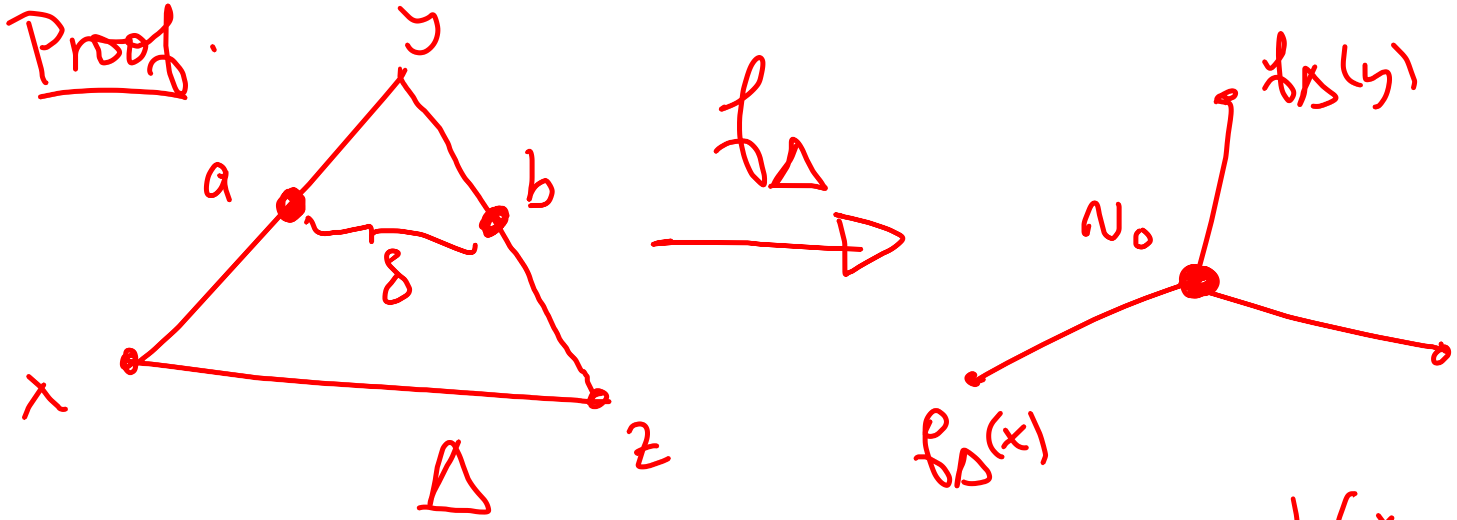
L. 6.6. δ thin $\Rightarrow$ Rips with δ

P. 6.7 all triangles in δ-hyp. space are 4δ-thin.

Lemma 6.5 X geod. space Δ δ-thin triangle, vertices x, y, z.

$$(y|z)_x \le d(x,[y,z]) \le (y|z)_x + \delta.$$

Proof.



Let $a \in [x,y]$, $b \in [y,z]$ s.t. $f_\Delta(a) = f_\Delta(b) = v_0$

Δ is δ-thin $\Rightarrow d(a,b) \le \delta$.

$$d(x,[y,z]) \le \underbrace{d(x,a)}_{(y|z)_x} + \underbrace{d(a,b)}_{\le \delta} \le (y|z)_x + \delta$$

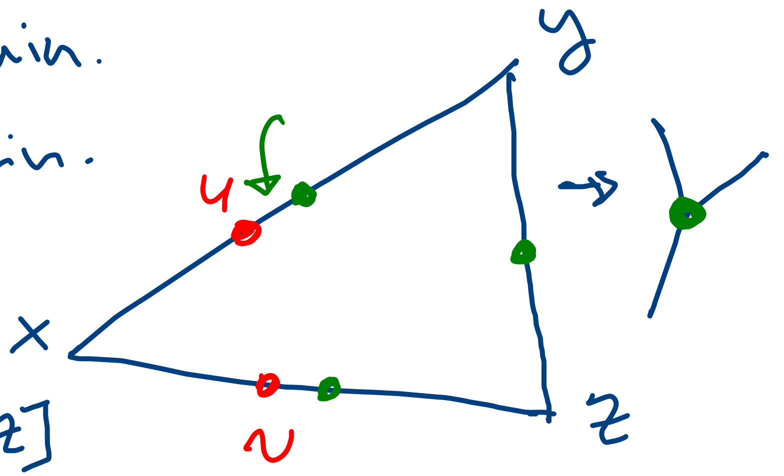
$\square$

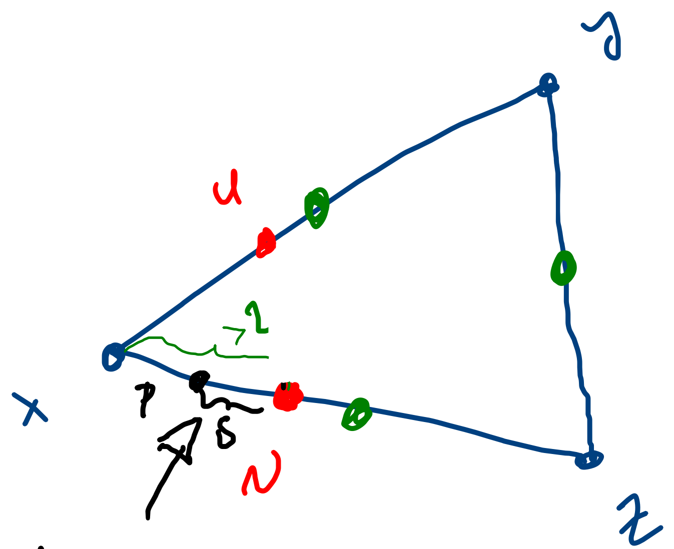
Prop. 6.7. X δ-hyp $\Rightarrow$ All triangles in X are 4δ-thin.

Proof. Let $\Delta \subset X$ be a triangle that is not 4δ-thin.

$\Rightarrow \exists$ 2 pts u, v : $f_\Delta(u) = f_\Delta(v)$ and $d(u,v) > 4\delta$

Let x, y, z be the vertices of Δ and $u \in [x,y]$, $v \in [x,z]$





$d(x,u) = d(x,v) < (y|z)_x$.

can assume

find a point here that is not in $\bar{N}_\delta([x,y]) \cup \bar{N}_\delta([y,z])$

$$d(v,[x,y]) = \min\Big(\underbrace{d(v,[x,u])}_{\geq (x|u)_v},\ \underbrace{d(v,[u,y])}_{\geq (u|y)_v}\Big)$$

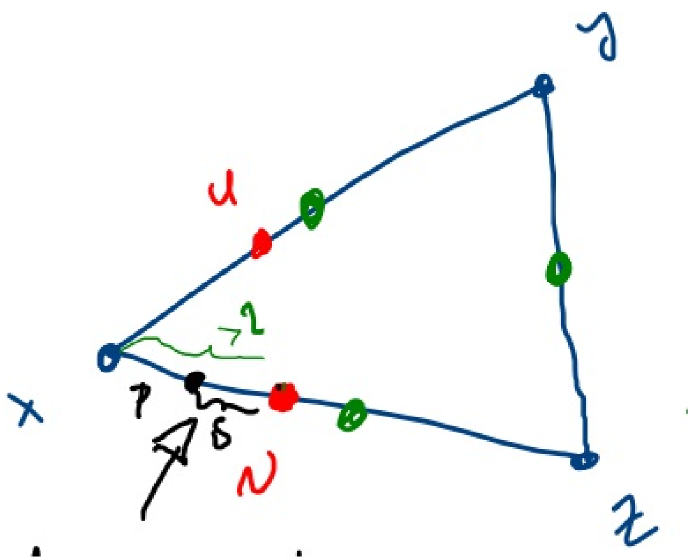
$$\boxed{2(x|u)_v} = d(v,x) + d(v,u) - d(x,u) = d(v,u)$$

$$\boxed{2(u|y)_v} = d(v,u) + d(v,y) - \underbrace{d(u,y)}_{d(x,y)-d(x,u)} = d(v,u) + \Big(\underbrace{d(v,y) + \overbrace{d(x,u)}^{d(x,v)} - d(x,y)}_{2(x|y)_v \geq 0}\Big) \geq d(u,v)$$

$$\Rightarrow d(v,[x,y]) \geq \tfrac{1}{2} d(u,v) > 2\delta .$$

$\exists\, p \in [x,v]$: $d(p,v) = \delta$, $d(p,x) > \delta$.

$\boxed{d(p,[x,y]) > \delta}$.



Estimate $d(p,[y,z])$:

$$\underline{\underline{d(p,[y,z])}} \overset{\text{triangle ineq.}}{\geq} d(x,[y,z]) - d(p,x)$$

$$\geq \underline{\underline{(y|z)_x}} > d(x,v) \ \cancel{= d(x,u)}$$

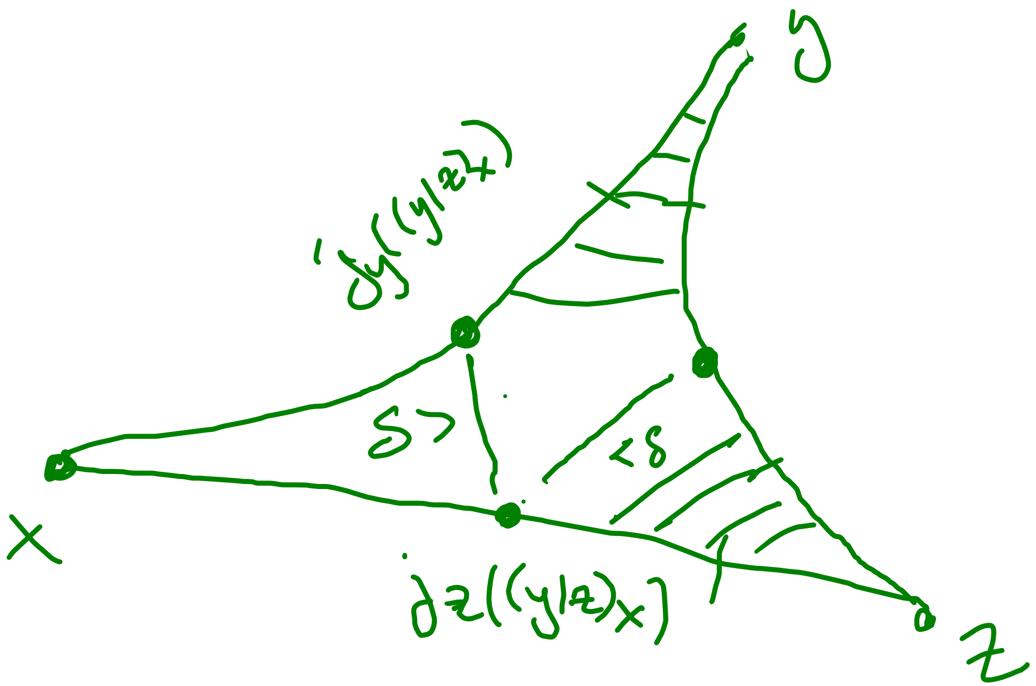
$$> d(x,v) - d(p,x) = d(p,v) = \underline{\underline{\delta}}$$

$\square$

$\rightsquigarrow$ $\underline{\underline{d(p,[x,y]),\ d(p,[y,z]) > \delta}}$

$\rightsquigarrow$ Could use δ-thinness to define Gromov-hyperbolicity.

Let Δ be a triangle with long sides, vertices x, y, z.



Assume $d(x,y) = d(x,z)$

Δ δ-thin $\Rightarrow$ geodesics from x to y and z stay close until time $(y|z)_x$ because if

$j_y: [0, d(x,y)] \to X$

$j_z: [0, d(x,z)] \to X$

are the geod. segments connecting x to y and z and

$t \in [0, (y|z)_x]$, $f_\Delta(j_z(t)) = f_\Delta(j_y(t))$

$\Rightarrow d(j_y(t), j_z(t)) \le \delta$

If $t > (y|z)_x$, then the distance $d(j_y(t), j_z(t))$ grows "very quickly" because $j_y(t), j_z(t)$ are now δ-close to $[y,z]$.