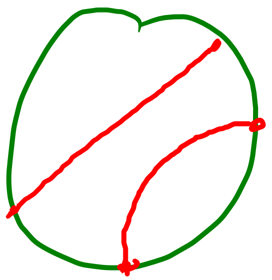
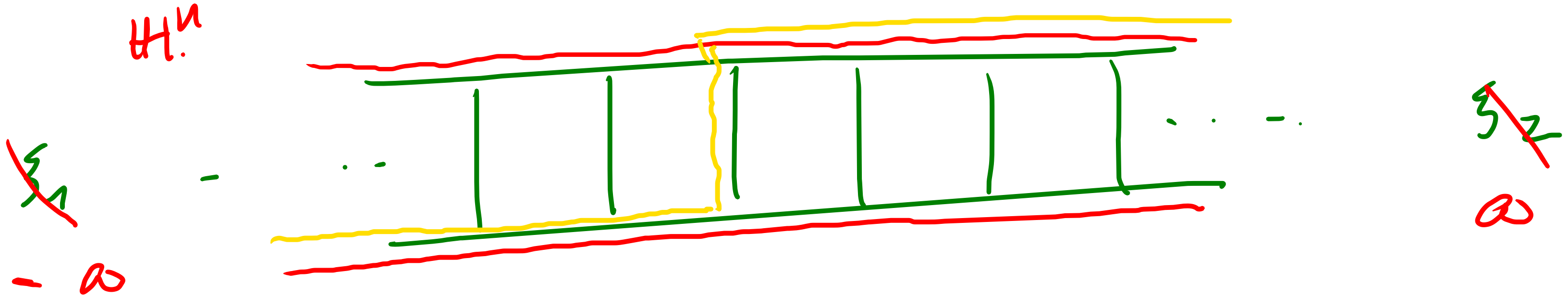
# Neg. curved spaces 12.11.2020

$X$ geod. metric space is a visibility space if $\forall\, \xi_-, \xi_+ \in \partial_\infty X$, $\xi_- \neq \xi_+$,
$\exists\, g \in \mathcal{G}(X)$ s.t. $g(-\infty) = \xi_-$, $g(+\infty) = \xi_+$.



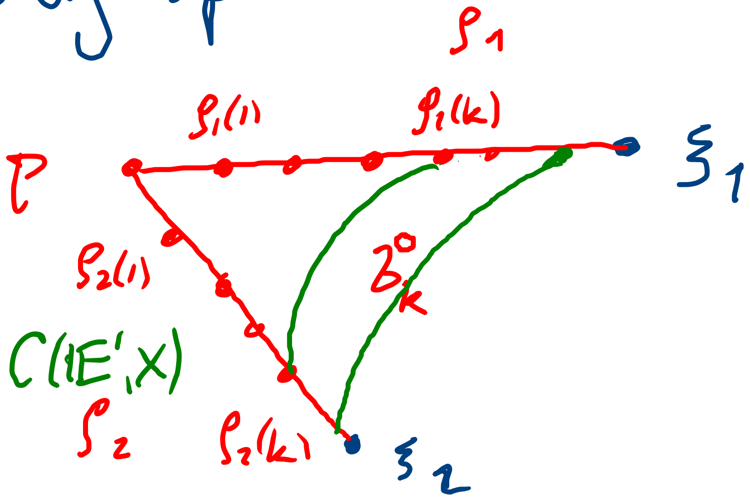

, trees are visibility spaces.




Ex. If $\gamma : \overset{[0,b]}{\overset{\shortparallel}{I}} \to X$ geod. segment

define $\check{\gamma}(t) = \begin{cases} \gamma(0) & \text{if } t \le 0 \\ \gamma(t) & \text{if } t \in I \\ \gamma(b) & \text{if } t \geq b. \end{cases}$

**Thm 9.8.** Proper Gromov-hyperbolic spaces are visibility spaces.




**Def$^{\text{n}}$** $\omega : \mathbb{R} \to X$ is a generalized geodesic line if $\exists$
closed interval $I \subset \mathbb{R}$ s.t. $\omega|_I$ is geodesic and $\omega$ is locally
① constant in $\mathbb{R} - I$. $\rightsquigarrow$ $\check{\mathcal{G}}(X) = \{\text{space of gen. geod lines}\} \subset C(\mathbb{R}, X)$

**Lemma.** $\check{\mathcal{G}}(X)$ is closed.

**Thm 9.7** $X$ proper, $K \subset X$ compact. $\{ g \in \check{\mathcal{G}}(X) : g(0) \in K \}$ is compact.

**Proof.** Uses AA, similar to Thm. 9.2. $\Rightarrow$ Exercise.

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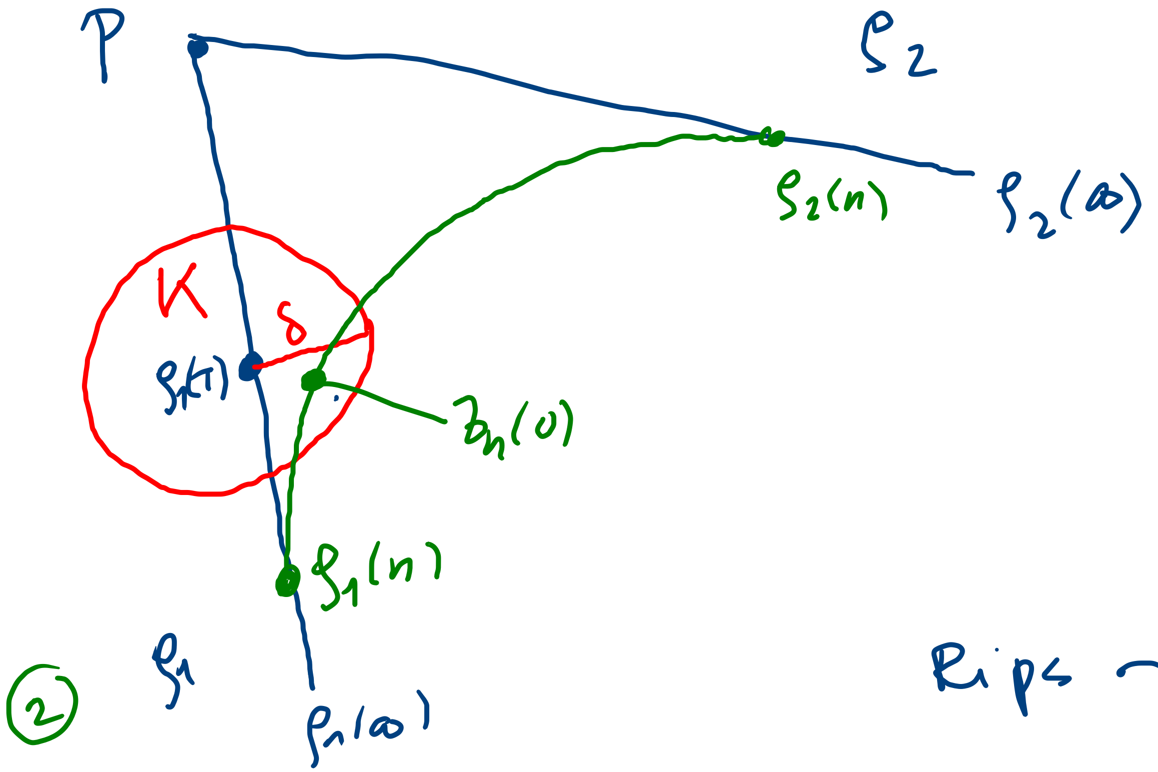
Let $X$ be $\delta$-hyperbolic.

**Proof of Thm 9.8.** Let $\rho_1, \rho_2 \in \mathcal{G}_+(X,p)$ for some $p \in X$.

Let $\gamma_n^0 : [0, b_n] \to X$ be a geod. segment s.t. $\gamma_n^0(0) = \rho_1(n)$, $\gamma_n^0(b_n) = \rho_2(n)$.

$\exists T \geq 0$ s.t. $d(\rho_1(T), \rho_2([0,\infty[)) > \delta$

$n > T$.




Change parametrization of $\gamma_n^0$ to get $\gamma_n$ with the same image and $\gamma_n(0) \in K = \overline{B}(\rho_1(T), \delta)$

Thm. 9.7: $(\gamma_n)_{n \in \mathbb{N}, n > T}$ has a convergent subsequence $\gamma_{n_k} \longrightarrow g \in \check{\mathcal{G}}(X)$

$\in \mathcal{G}(X)$

Rips $\Rightarrow$ $g(-\infty) = \rho_1(\infty)$
$g(+\infty) = \rho_2(\infty)$. $\square$

(2)

Recall Thm 7.8: If $X, Y$ are geod. metric spaces & quasi-isometric. Then $X$ Gromov-hyperbolic $\Leftrightarrow$ $Y$ is Gromov-hyperbolic.

What happens to the boundaries at $\infty$ under quasi-isometry?

$$QG_+(X) = \{ \text{quasi geod. rays } \rho : [0,\infty[ \to X \}$$

$$QG_+(X,p) = \{ \rho \in QG_+(X) : \rho(0) = p \},$$

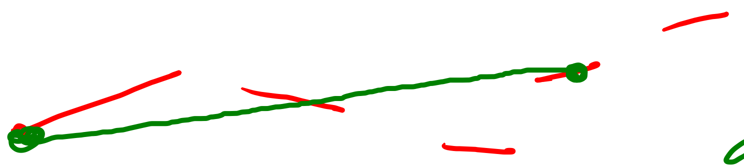
$\rho_1, \rho_2 \in QG_+(X)$ are <u>asymptotic</u> $\rho_1 \sim \rho_2$ if $d_{Haus}(\rho_1, \rho_2) < \infty$.

<u>Prop. 9.9.</u> $X$ proper Gromov-hyp. $q \in X$. $\forall \rho \in QG_+(X,q)$ $\exists \bar{\rho} \in G_+(X,q)$ s.t. $\rho(\infty) = \bar{\rho}(\infty)$.

<u>Proof.</u> Follows proof of Prop. 9.3. Note $d(\rho(0), \rho(t)) \xrightarrow{t \to \infty} \infty$.

$\Rightarrow \exists \bar{\rho}$. <u>Asymptoticity</u>:

Thm 7.10



$d_{Haus}(\rho|_{[0,b]}, [\rho(0), \rho(b)]) < M(\lambda, c, \delta)$

$\square$

**Corollary** If $X$ proper Gromov-hyp. space $\partial_\infty X$ is "naturally" identified with $QG_+(X)/\sim$ and $QG_+(X,p)/\sim$. $\boxtimes$

**Prop. 9.11.** $X$ metric space, $Y$ proper Gromov-hyp. space. $g_1, g_2 \in G_+(X)$, $F: X \to Y$ quasi-isom. embedding. Then
$g_1(\infty) = g_2(\infty) \iff F\circ g_1(\infty) = F\circ g_2(\infty)$.

$$F_\infty(\underbrace{g(\infty)}_{\in \partial_\infty X}) = (F\circ g)(\infty).$$

**Proof.** Exercise.

**Corollary** $F$ induces an inj mapping $F_\infty: \partial_\infty X \longrightarrow \partial_\infty Y$. $\square$

**Prop. 9.13** Let $X, Y$ be proper Gromov-hyp. spaces, $F: X \to Y$ quasi-isometry. Then $F_\infty$ is a bijection.

④

**Proof.** Let $\bar{F} : Y \to X$ be a quasi-inverse of $F$, $K \geq 0$

$$\boxed{d(\bar{F}\circ F(x), x) \leq K}, \quad d(F\circ\bar{F}(y), y) \leq K. \quad \forall\, x\in X,\ y \in Y.$$

$\rho \in G_+(X)$ $\quad \underbrace{\bar{F}\circ F}_{q.i}\circ\rho \in QG_+(X) \Rightarrow d(\bar{F}\circ F\circ\rho(t), \rho(t)) \leq K \ \forall\, t \geq 0$

$$\Rightarrow \boxed{\begin{array}{c}(\bar{F}\circ F\circ\rho)(\infty) = \rho(\infty) \\ \| \\ \bar{F}_\infty(F_\infty(\rho(\infty)))\end{array}} \Rightarrow \bar{F}_\infty \circ F_\infty = id$$

similarly get $F_\infty \circ \bar{F}_\infty = id$ $\quad\square$

**Ex.** $\mathbb{E}^1$ $\qquad$ bi-infinite ladder. $B$


$j$ isom. embedding

quasi-isometry.

$j(\mathbb{E}^1)$

$d(b, j(\mathbb{E}^1)) \leq 3/2 \ \forall\, b \in B$

## A topology on $\boxed{X \cup \partial_\infty X}$

$X$ is geodesic so if we fix $p \in X$ $\forall\, x \in X$ $\exists$ geod. segment $\gamma_x : [0, b_x] \to X$, $\gamma_x(0) = p$, $\gamma_x(b_x) = x$. But this geod. segment is not nec. unique.

Also $\forall\, \xi \in \partial_\infty X$ $\exists\, \rho \in \mathcal{G}_+(X,p)$ s.t. $\xi = \rho(\infty)$ if $X$ is proper

Define $E : \check{\mathcal{G}}_+(X,p) \longrightarrow X \cup \partial_\infty X$

$$E(\rho) = \begin{cases} \rho(\infty) & \forall\, \rho \in \mathcal{G}_+(X,p) \\ \underbrace{\lim_{t\to\infty} \rho(t)}_{= \rho(\infty)} & \forall\, \rho \in \check{\mathcal{G}}_+(X,p) - \mathcal{G}(X,p) \end{cases}$$

$\check{\mathcal{G}}_+(X,p) / \underset{E}{\sim}$

identified with $X \cup \partial_\infty X$

$E(\rho(\infty)) = \rho(\infty)$.

equiv. class

$E \rightsquigarrow$ equivalence relation $\underset{E}{\sim}$ : $\rho_1 \underset{E}{\sim} \rho_2 \iff \underline{\underline{E(\rho_1) = E(\rho_2)}}$.