Funktionaalianalyysi
Exercises 8, 5.3.2018

1. Prove that the norm of the Banach space $C^0([0,1])$ is not defined by an inner product.

---

2. Let $(x_k)_{k=1}^\infty$ be a sequence in a Hilbert space H and let $x \in H$ such that

$$\lim_{k\to\infty} (x_k \mid x) = (x \mid x)$$

and

$$\lim_{k\to\infty} \|x_k\| = \|x\|.$$

Prove that the sequence $(x_k)_{k=1}^\infty$ converges to x.

3. Give an example of a Hilbert space H and a sequence $(h_k)_{k\in\mathbb{N}}$, $h_k \in H$ for all $k \in \mathbb{N}$, such that $\lim_{k\to\infty} (h_k \mid x) = (0 \mid x)$ and $(h_k)_{k\in\mathbb{N}}$ does not converge to 0.

---

4. Let $(V, (\cdot \mid \cdot)$ be an inner product space, let W be a vector space and let $\Phi\colon W \to V$ be a linear bijection. Prove that $\langle w_1 \mid w_2 \rangle = (\Phi(w_1) \mid \Phi(w_2))$ defines an inner product in W.

5. Let

$$A = \{f \in C^0([0,1]) : f(1) = 1\}.$$

is a closed convex subset of the normed space $C^0([0,1])$ that has infinitely many elements of minimal norm.

6. Let

$$B = \Big\{f \in C^0([0,1]) : f(0) = 0, \int_0^1 f(t)\,dt = 1\Big\}.$$

Prove that B is a closed convex subset of the normed space $C^0([0,1])$ that does not have any element of minimal norm.

7. Let H be a Hilbert space. Prove that $A \subset (A^\perp)^\perp$ for all $A \subset H$. Give an example of a subspace A of some inner product space, such that $(A^\perp)^\perp \neq A$.

8. Let H be a real inner product space and let $T\colon H \to H'$ be the mapping

$$(Tx)(y) = (y \mid x)$$

Prove that the mapping T is a linear isometric embedding.