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Exercises 7, 26.2.2018

1. Let $\mathscr{A}$ be the collection of linearly independent subsets of a vector space V that contain a linearly independent set E as a subset. Let C be a chain in the partially ordered set $(\mathscr{A}, \subset)$. Prove that

$$M = \bigcup_{B \in C} B \subset V$$

is linearly independent.

2. Let $(V, (\cdot|\cdot))$ be an inner product space. Assume that for soem $u, v \in V$ the equation

$$(x \mid u) = (x \mid v)$$

holds for all $x \in V$. Prove that $u = v$.

3. Prove that an inner product defines a norm by setting

$$\|x\| = \sqrt{(x \mid x)}\,.$$

4. Let $(V, (\cdot \mid \cdot))$ be a complex inner product space. Prove that

$$\operatorname{Im}(x \mid y) = \frac{1}{4}\big(\|x + iy\|^2 - \|x - iy\|^2\big)$$

for all $x, y \in V$.

5. Let H_1 and H_2 be inner product spaces. Let $\phi \colon H_1 \to H_2$ be an isometric linear bijection. Prove that

$$(\phi(x) \mid \phi(y)) = (x \mid y)$$

for all $x, y \in H_1$.

6. Prove that the parallelogram rule

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

holds for the norm of an inner product space.

7. Let $(V, (\cdot \mid \cdot))$be an inner product space. Prove that the norm given by the inner product is strictly subadditive:

$$\|x + y\| < \|x\| + \|y\|\,,$$

for all $x, y \in V$ unless $x = \lambda y$ or $y = \lambda x$ for some $\lambda \geq 0$.

8. Prove that $\| \cdot \|_1$ is not a strictly subadditive norm in $\mathbb{R}^2$.