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Exercises 6, 19.2.2018

1. Let $T_k \colon d^\infty(\mathbb{K}) \to \mathbb{K}$, $T_k\omega = k\,\omega(k)$. Use the mappings T_k to show that the theorem of Banach and Steinhaus does not hold without the assumption that the domain of definition of the mappings is a Banach space.

2. Prove that a uniformly convergent sequence of operators converges strongly.

3. Let X be a Banach space and let Y be a normed space. Let $T_k \in \mathrm{Lin}_b(X,Y)$ such that $\sup_{k\in\mathbb{N}} \|T_k\| = \infty$. Prove that there is a point $x_0 \in X$, for which $\sup_{k\in\mathbb{N}} \|T_k x_0\|_Y = \infty$.

4. Let X and Ybe normed spaces. Prove that a linear mapping $T \colon X \to Y$ is open if 0 is an interior point of $T(B(0,1))$.

5. Let $i_{12} \colon \ell^1(\mathbb{R}) \to \ell^2(\mathbb{R})$ be the mapping $\iota(\omega) = \omega$. By testricting the target space we get a mapping $i_{12} \colon \ell^1(\mathbb{R}) \to i_{12}(\ell^1(\mathbb{R}))$ which is a bounded linear bijection. Prove that this mapping is not open.[1]

6. Let X be a Banach space and let $T \in \mathrm{Lin}_b(X,X)$ such that $\|T\| < 1$. Prove that the series $\sum_{k=0}^\infty T^k$ converges and determines the inverse mapping of $\mathrm{id}_X - T$. Prove that

$$\|(\mathrm{id}_X - T)^{-1}\| \le \frac{1}{1 - \|T\|}\,.$$

Let X and Ybe Banach spaces and let $T \in \mathrm{Lin}_b(X,Y)$ be injective.

7. Prove that if $T(X)$ is closed, then $\|Tx\| \ge \beta\|x\|$ holds for some $\beta > 0$ and all $x \in X$.[2]

8. Prove that $T(X)$ is closed if $\|Tx\| \ge \beta\|x\|$ holds for some $\beta > 0$ and all $x \in X$. [3]

[1] It may be useful to consider the subspace $d^2(\mathbb{R}) \subset \ell^2(\mathbb{R})$.
[2] Use the open mapping theorem.
[3] What does the assumption tell about $T^{-1} \colon T(X) \to X$?