Funktionaalianalyysi
Exercises 14, 30.4.2018

1. Let H be a Hilbert space. Prove that the mapping $^\dagger\colon \mathrm{Lin}_b(H,H) \to \mathrm{Lin}_b(H,H)$ is conjugate linear, and that $(T^\dagger)^\dagger = T$ for all $T \in \mathrm{Lin}_b(H,H)$.

2. Let $V\colon L^2([0,1]) \to L^2([0,1])$,

$$Vf(x) = \int_0^x f(t)dt\,.$$

The operator V is a Hilbert-Schmidt integral operator. Determine the kernel $k \in L^2([0,1]\times[0,1])$for which $V = F_k$. Determine $V^\dagger$.

3. Let H be a Hilbert space and let $T \in \mathrm{Lin}_b(H,H)$. Prove that

$$\ker T = T^\dagger(H)^\perp\,.$$

4. Let H be a Hilbert space. Prove that Hermitian operators form a closed real subspace of the normed space $\mathrm{Lin}_b(H,H)$.

5. Let H be a complex Hilbert space. Let $Q\colon H \to H$ be an operator for which $(Qz \mid z) = 0$ for all $z \in H$. Prove that $Q = 0$.[1] Show that the corresponding statement does not hold in real Hilbert spaces.

6. Let H be a Hilbert space and let $T\colon H \to H$ be a Hermitian operator. Prove that 0 is not in the residual spectrum of T.

Let $\sigma, \rho\colon \ell^2(\mathbb{C}) \to \ell^2(\mathbb{C})$ be the left and right shifts defined by setting

$$\sigma\omega(k) = \omega(k+1)$$
$$\rho\omega(k) = \begin{cases} 0 & \text{, when } k = 0 \\ \omega(k-1) & \text{, when } k \geq 1 \end{cases}.$$

for all $\omega \in \ell^2$

7. Compute the adjoint operators of the left and right shifts.

8. Determine the point spectrum, continuous spectrum and residual spectrum of the left and right shifts.[2]

[1] Let $v, w \in H$. Consider the cases $z = v + w$ and $z = v + iw$.
[2] Use exercise 3 and the properties of the orthogonal complement. What is $(\sigma - \lambda)^\dagger$?