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Exercises 13, 23.4.2018

Let $\sigma, \rho \colon \ell^2(\mathbb{C}) \to \ell^2(\mathbb{C})$ be the left and right shifts defined by setting

$$\sigma\omega(k) = \omega(k+1)$$
$$\rho\omega(k) = \begin{cases} 0 & \text{, when } k = 0 \\ \omega(k-1) & \text{, when } k \geq 1 \end{cases}.$$

for all $\omega \in \ell^2$

1. Prove that $\|\sigma\| = \|\rho\| = 1$

2. Determine $\operatorname{spec}(\sigma)$.[1]

3. (a) Prove that $\rho - \lambda\,\mathrm{id}$ is injective for all $\lambda \in \mathbb{C}$.
(b) Prove that $\rho - \lambda\,\mathrm{id}$ is not surjective if $\lambda \in \mathbb{C}$ and $|\lambda| \leq 1$.[2]
(c) Determine $\operatorname{spec}(\rho)$.

4. Let X be a normed space and let $T \in \operatorname{Lin}(X, X)$. Prove that T is compact if and only if for any sequence $(x_k)_{k\in\mathbb{N}}$ in the closed unit ball of X the sequence $(Tx_k)_{k\in\mathbb{N}}$ has a convergent subsequence.

5. Prove that a compact operator is bounded.

6. Let X and Y be normed spaces. Prove that compact operators $T \colon X \to Y$ form a linear subspace of $\operatorname{Lin}_b(X, X)$. [3]

7. Let $S \colon X \to Y$ and $T \colon Y \to Z$ be bounded operators. Assume that S or T is compact. Prove that $T \circ S$ is compact.

8. Let $T \colon \ell^2(\mathbb{K}) \to \ell^2(\mathbb{K})$ be the linear mapping defined by

$$(T\omega)(k) = \frac{\omega(k)}{k+1}$$

for all $\omega \in \ell^2(\mathbb{K})$ all $k \in \mathbb{N}$. Prove that the operator T is compact[4]

[1] Problem 8 of Exercises 12 is useful.
[2] Prove that e_0 is not in the image.
[3] Lemma 12.8 may be useful.
[4] Corollary 12.8 may be useful.