Funktionaalianalyysi
Exercises 12, 16.4.2018

1. Let H be a Hilbert space. Prove that H' is a Hilbert space with the inner product
$$(x' \mid y')' = (T^{-1}y' \mid T^{-1}x')\,.$$

2. Let X be a Banach space and let $(x_k)_{k\in\mathbb{N}}$ be a weakly convergent sequence in X with weak limit $x \in X$. Prove that $x \in \overline{\langle x_k : k \in \mathbb{N}\rangle}$

3. Let X and Y be normed spaces. Let $T \in \mathrm{Lin}_b(X, Y)$ and let $x_k \in X$ for all $k \in \mathbb{N}$. Assume that $x_k \xrightharpoonup[k\to\infty]{} x$. Prove that $Tx_k \xrightharpoonup[k\to\infty]{} Tx$.[1]

4. Let X be a normed space. Prove that a compact set $K \subset X$ is weakly sequentially compact but that a weakly sequentially compact set $M \subset X$ may not be compact.

5. Let V be a normed space and let $v_k \in V$ for all $k \in \mathbb{N}$. Prove that a sequence $(v_k)_{k\in\mathbb{N}}$ converges weakly if and only if it converges in the weak topology.

6. A sequence $(x_k)_{k\in\mathbb{N}}$ in a normed space X is a *weak Cauchy sequence* if $(x'x_k)_{k\in\mathbb{N}}$ is a Cauchy sequence for all $x' \in X'$. Prove that a weak Cauchy sequence is bounded.

7. Let X be a normed space and let $x_k \in X$, $k \in \mathbb{N}$ such that $x_k \xrightharpoonup[k\to\infty]{} x \in X$. Prove that $\|x\| \le \liminf_{k\to\infty} \|x_k\|$.

8. The *left shift* $\sigma\colon \mathscr{F}(\mathbb{N}, \mathbb{C}) \to \mathscr{F}(\mathbb{N}, \mathbb{C})$ is defined by setting
$$\sigma(\omega)(n) = \omega(n+1)$$
for all $n \in \mathbb{N}$. Prove that σ is a linear mapping. Determine the eigenvalues of σ.

[1] If $g \in Y'$, then $g \circ T \in X'$.