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Exercises 1, 15.1.2018

1. Let V be a $\mathbb{K}$-vector space. Let $I \neq \emptyset$ be an index set and let H_α be a vector subspace of V for every $\alpha \in I$. Prove that the intersection $\bigcap_{\alpha \in I} H_\alpha$ is a vector subspace.

2. Let V be a vector space and let $X \subset V$, $X \neq \emptyset$. Proe that

$$\langle X \rangle = \left\{ \sum_{i=1}^{k} \lambda_i x_i : \lambda_i \in \mathbb{K},\ x_i \in X,\ k \in \mathbb{N} - \{0\} \right\}.$$

3. Prove that $\ell^\infty(\mathbb{K})$ and $\ell^1(\mathbb{K})$ are vector subspaces of $\mathscr{F}(\mathbb{N}, \mathbb{K})$.

4. Prove that

$$c(\mathbb{K}) = \{\omega \in \mathscr{F}(\mathbb{N}, \mathbb{K}) : \exists \lim_{n\to\infty} \omega(n) \in \mathbb{R}\}$$

is a vector subspace of $\mathscr{F}(\mathbb{N}, \mathbb{K})$ and that $\lim \colon c(\mathbb{K}) \to \mathbb{K}$,

$$\lim \omega = \lim_{k\to\infty} \omega(k)\,,$$

is a linear mapping.

5. Prove that $\mathrm{C}^0\big([0,1], \mathbb{R}\big)$ is an infinite-dimensional real vector space.

6. Let U be a vector space and let $(W, \|\cdot\|_W)$ be a normed space. Let $L \colon U \to W$ be a linear bijection. Prove that

$$\|u\| = \|Lu\|_W$$

defines a norm in U.

7. Let $I \subset \mathbb{R}$ be a compact interval. Prove that

$$\|f\|_{\mathrm{C}^1,1} = \|f\|_\infty + \|f'\|_\infty$$

is a norm in $\mathrm{C}^1(I)$.

8. Let V be a normed space and let

$$S(0,1) = \{x \in V : \|x\| = 1\}.$$

Let $\mathrm{pr}_S \colon V - \{0\} \to S(0,1)$ be the mapping defined by setting $\mathrm{pr}_S(x) = \frac{x}{\|x\|}$ for all $x \in V - \{0\}$. Prove that pr_S is continuous.