Differential geometry 2023

Exercises 8

1. In this exercise, x and y are the canonical coordinates in Euclidean spaces.
(1) Let $\exp\colon \mathbb{E}^1 \to]0,\infty[\subset \mathbb{E}^1$, $\exp(x) = e^x$. Let

$$\omega_1 = \frac{dy}{y}\,.$$

Compute $\exp^* \omega$.
(2) Let $F\colon \{x \in \mathbb{E}^2 : \|x\| < 1\} \to \mathbb{E}^3 - \{0\}$,

$$F(x) = (x^1, x^2, \sqrt{1 - \|x\|^2})$$

and let $\omega_2 \in \mathfrak{X}^*(\mathbb{E}^3 - \{0\})$,

$$\omega_2 = (1 - (y^1)^2 - (y^2)^2)\, dy^3\,.$$

Compute $F^*\omega_2$.
(3) Compute the expression of $F^*\omega_2$ in polar coordinates of $B(0,1) \subset \mathbb{E}^2$.

Solution. (1) $\exp^* \frac{dy}{y} = \exp^*(\frac{1}{y}\,dy) = \frac{1}{e^x}\frac{d\,e^x}{dx}dx = dx$.
(2)

$$\begin{aligned} F^*(1 - (y^1)^2 - (y^2)^2)\, dy^3 &= (1 - \|x\|^2)\Big(\frac{\partial\sqrt{1 - \|x\|^2}}{\partial x^1}dx^1 + \frac{\partial\sqrt{1 - \|x\|^2}}{\partial x^2}dx^2\Big) \\ &= -\sqrt{1 - \|x\|^2}(x^1\, dx^1 + x^2\, dx^2)\,. \end{aligned}$$

(3) Thanks to the result of Exercise 2, the polar change of variable $\phi : (r,\theta) \to (r\cos(\theta), r\sin(\theta))$ gives

$$\phi^*(F_2^*\omega_2) = (F_2 \circ \phi)^*\omega_2 = (1 - r^2)\frac{\partial\sqrt{1 - r^2}}{\partial r}\,dr + 0\,d\theta = -r\sqrt{1 - r^2}\,dr$$

2. Let $F_1\colon M_1 \to M_2$ and $F_2\colon M_2 \to M_3$ be smooth mappings and let $\omega \in \mathfrak{X}^*(M_3)$. Prove that

$$(F_2 \circ F_1)^*\omega = F_1^*(F_2^*\omega)\,.$$

Solution. By definition of the pullback of a covector field

$$(F_2 \circ F_1)^*\omega = \omega \circ (F_2 \circ F_1) = (\omega \circ F_2) \circ F_1 = F_1^*(\omega \circ F_2) = F_1^*(F_2^*\omega).$$

3. Let M be a smooth manifold and let S be a regular level set of a smooth function $f \in \mathfrak{F}(M)$. Prove that df restricts to the 1-form $0 \in \mathfrak{X}^*(S)$ on the submanifold S.

Solution. Let $S = f^{-1}(c)$ for some $c \in \mathbb{E}^1$. If $i\colon S \to M$ is the inclusion map, then $f \circ i = c$, and we have $i^*df = d(f \circ i) = dc = 0$ because the differential of a constant function is zero: $dc(v) = vc = 0$ for all $v \in T_pS$ for all $p \in S$.

4. Prove that the tensor product of real-valued linear mappings is a multilinear mapping.

Solution. Let $V_1, \ldots, V_n$ be real vector spaces and $\omega_1 \in V_1^*, \ldots, \omega_n \in V_n^*$. Their tensor product is defined from $V_1 \times \ldots V_n$ to $\mathbb{R}$ by the formula

$$\omega_1 \otimes \omega_2 \ldots \otimes \omega_n(v_1, \ldots, v_n) = \omega_1(v_1)\omega_2(v_2)\ldots\omega_n(v_n),$$

which is the formula of a multilinear map since all the maps ω_i are linear.

5. Express the evaluation tensor $E \in T^{(1,1)}(\mathbb{R}^n)$,

$$E(\omega, v) = \omega v \ ,$$

using tensor products of the standard basis of $\mathbb{R}^n$ and its dual basis.

Solution. Recall that $T^{(1,1)}(\mathbb{R}^n)$ is the space of bilinear forms on $(\mathbb{R}^n)^* \times \mathbb{R}^n$. Denote the canonical basis of $\mathbb{R}^n$ by $(e_1, \ldots, e_n)$. Its dual base will be denoted by $(e_1^*, \ldots, e_n^*)$. We recall that $\mathbb{R}^n$ is identified with its bidual space, in particular every e_i is identified with the evaluation linear map $w \mapsto w(e_i)$ defined on $(\mathbb{R}^n)^*$. We compute, for all $\omega \in (\mathbb{R}^n)^*$ and $v \in \mathbb{R}^n$,

$$E(\omega, v) = \omega v = \sum_{i=1}^{n} e_i^*(v)\omega(e_i) = \Big(\sum_{i=1}^{n} e_i \otimes e_i^*\Big)(\omega, v).$$

Thus $E = \sum_{i=1}^{n} e_i \otimes e_i^*$.

6. Let V be a real vector space. Let $A_1, A_2 \in T^{(r_1,s_1)}(V)$, $B_1, B_2 \in T^{(r_2,s_2)}(V)$ and let $a_1, a_2, b_1, b_2 \in \mathbb{R}$. Prove that

$$(a_1A_1 + a_2A_2) \otimes (b_1B_1 + b_2B_2) = a_1b_1A_1 \otimes B_1 + a_1b_2A_1 \otimes B_2 + a_2b_1A_2 \otimes B_1 + a_2b_2A_2 \otimes B_2 \, .$$