Differential geometry 2023

Exercises 2

1. Let X be a topological space and let B be a base of the topology of X. Let $\sim$ be an equivalence relation in X such that the quotient map $\pi\colon X \to X/\sim$ is open. Prove that $\{\pi(E) : E \in B\}$ is a base of the topology of $X/\sim$.

Solution. Let $U \in X/\sim$ be an open set and let $p \in U$. Let $\widetilde{p} \in \pi^{-1}(U)$. By definition, $\pi^{-1}(U)$ is open, and there is some $E \in B$ such that $p \in E \subset U$. By assumption, the set $\pi(E) \subset \pi(\pi^{-1}(U)) = U$ is open, and it contains p. This implies the claim.

2. Prove that projective space is a second countable Hausdorff space.

Solution. Let us check that the quotient map $x \mapsto [x]$ is open: If $U \subset \mathbb{E}^n - \{0\}$ is open, then

$$\mathbb{R}^\times U = \{tx : t \in \mathbb{R}^\times,\ x \in U\}$$

is open. But note that $\pi^{-1}(\pi(U)) = \mathbb{R}^\times U$ implies that $\pi(U)$ is open by the definition of quotient topology. Corollary 1.19 implies that $\mathbb{P}^n$ is second countable.

Let $[x], [y] \in \mathbb{P}^n$, $[x] \neq [y]$. We may assume $x, y \in \mathbb{S}^n$, $x \neq \pm y$. Choose $r_x, r_y > 0$ such that

$$(B(x, r_x) \cup B(-x, r_x)) \cap (B(y, r_y) \cup B(-y, r_y)) = \emptyset\,.$$

The quotient map π is open, thus $\pi(B(x, r_x))$ and $\pi(B(y, r_y))$ are neighbourhoods of $[x]$ and $[y]$ respectively. Furthermore, these neighbourhoods are disjoint since π is surjective and their preimages are disjoint thanks to the following:

$$\pi^{-1}(\pi(B(x, r_x))) = B(x, r_x) \cup B(-x, r_x) \text{ and } \pi^{-1}(\pi(B(y, r_y))) = B(y, r_y) \cup B(-y, r_y).$$

3. Let M and N be smooth manifolds. Prove that a continuous mapping $F\colon M \to N$ is smooth at a point $p \in M$ if the mapping $\psi \circ F \circ (\phi|_{U \cap F^{-1}(V)})^{-1}$ is smooth at the point $\phi(p)$ for some smooth chart (U, ϕ) that contains p and for some smooth chart (V, ψ) that contains $F(p)$.

Solution. The smoothness of F at p is defined as the smoothness of the function $\widetilde{\psi} \circ F \circ (\widetilde{\phi}_{|\widetilde{U} \cap F^{-1}(\widetilde{V})})^{-1}$ at the point $\widetilde{\phi}(p)$ for all charts $(\widetilde{U}, \widetilde{\phi})$ of M and $(\widetilde{V}, \widetilde{\psi})$ of N such that $p \in \widetilde{U}$ and $F(p) \in \widetilde{V}$. Then, fact that it is sufficient to check the smoothness for only one pair of chart (here (U, ϕ) and (V, ψ)) comes from the definition of compatibility of charts since, for all such charts $(\widetilde{U}, \widetilde{\phi})$ of M and $(\widetilde{V}, \widetilde{\psi})$ of N, the set $U \cap \widetilde{U} \cap F^{-1}(V) \cap F^{-1}(\widetilde{V})$ is open, contains the point p and

$$\widetilde{\psi} \circ F \circ (\widetilde{\phi}_{|U \cap \widetilde{U} \cap F^{-1}(V) \cap F^{-1}(\widetilde{V})})^{-1} = \left(\widetilde{\psi} \circ \psi^{-1}\right) \circ \left(\psi \circ F \circ \phi^{-1}\right) \circ \left(\phi \circ (\widetilde{\phi}_{|U \cap \widetilde{U} \cap F^{-1}(V) \cap F^{-1}(\widetilde{V})})^{-1}\right).$$

4. Prove that the inclusion mapping $i\colon \mathbb{S}^1 \to \mathbb{E}^2$, $i(p) = p$, is smooth.

Solution. Let $(x_1, x_2) \in \mathbb{S}^1$. Thanks to exercise 3, we only ave to check the smoothness of $i \circ \phi^{-1}$ for one chart (U, ϕ) such that $(x_1, x_2) \in U$. If $x_1 > 0$, we can take the chart

$(U_1^+, \mathrm{pr}_{1\,|U_1^+})$ and compute

$$i \circ \mathrm{pr}_{1\,|U_1^+}^{-1} : x_2' \mapsto (\sqrt{1 - x_2'^2}, x_2')$$

which is indeed a smooth map at the point $y = \mathrm{pr}_1(x_1, x_2) < 1$. For other cases, the same argument works with the chart $(U_1^-, \mathrm{pr}_{1\,|U_1^+})$ if $x_1 < 0$, and the two charts $(U_2^\pm, \mathrm{pr}_{2\,|U_2^\pm})$ if $x_1 = 0$ depending on the sign of x_2.

5. Let $f\colon \mathbb{E}^2 \to \mathbb{E}^1$ be a smooth function. Prove that $f|_{\mathbb{S}^1}$ is smooth.

Solution. We have the formula $f_{|\mathbb{S}^1} = f \circ i$ with the notation i from Exercise 4. As the composition of two smooth mappings, it is also smooth.

6. Prove that the quotient map $\pi\colon \mathbb{E}^n \to \mathbb{T}^n$, $\pi(x) = [x] = x + \mathbb{Z}^n$, is smooth.

Solution. We use the canonical atlas in $\mathbb{E}^n$ and the atlas given by the local inverses of π on $\mathbb{T}^n$. Let $p \in \mathbb{E}^n$. The mapping π is injective in the open ball $B(p, \frac{1}{2})$ so we use the coordinate map $\phi = (\pi|_{B(p,\frac{1}{2})})^{-1}$ on $\pi(B(p, \frac{1}{2}))$. The composition $\phi \circ \pi_{|B(p,\frac{1}{2})} = \mathrm{id}\,|_{B(p,\frac{1}{2})}$ is smooth.