Differential geometry 2023

Exercises 1

1. Let
$$X = (\mathbb{R} \times \{0\}) \cup (\{0\} \times \mathbb{R}) \subset \mathbb{E}^2$$
with the relative topology. Prove that X is a second countable Hausdorff space but not a manifold.[1]

Solution. The Hausdorff property and second countability are inherited from $\mathbb{E}^2$.

For a proof by contradiction that X is not a manifold, assume it is. Then, it has to be 1-dimensional because, for example, the subset $]0,1[\, \times \{0\}$ is open (since it is equal to $]0,1[^2 \cap X$ and using the definition of relative topology) and is homeomorphic to $]0,1[$. Let (U,ϕ) be a chart of X around 0. Up to taking the intersection with a small ball in $\mathbb{E}^2$ centered at 0, we can assume that U is connected. Since $\phi\colon U \to \phi(U) \subset \mathbb{E}^1$ is a homeomorphism, its image $\phi(U)$ is connected as well. This implies that $\phi(U)$ is an interval. Then, the set $U - \{0\}$ has 4 components, while its image $\phi(U) - \{\phi(0)\}$ only has 2 components, a contradiction.

2. Let $T > 0$ and let
$$S_T = \Big\{(\sin(t), \sin(2t)) : 0 < t < T\Big\} \subset \mathbb{E}^2 .$$
Prove that S_T is a manifold if $0 < T \leq \pi$, and that S_T is not a manifold if $T > \pi$.

Solution. Begin by a drawing to help the intuition (with the points $(\sin(t), \sin(2t))$ for usual trigonometric values of t between 0 and π, and a bit above π to see the problem when $T > \pi$).

First, assume that $0 < T \leq \pi$. We will show that S_T is a (smooth) submanifold of $\mathbb{E}^2$. Define $f : t \mapsto (\sin(t), \sin(2t))$, then observe that $S_T = f(]0,T[)$. It is then sufficient to show that $f|]0,T[$ is an embedding, that is to say an immersion at each point t of the interval $]0,T[$ (in fact at each $t \in \mathbb{R}$), and a homeomorphism onto its image (i.e. onto S_T).

The function f is smooth and its first derivative is given by $f' : t \mapsto (\cos(t), 2\cos(2t))$. For each $t \in \mathbb{R}$, since the differential $df(t) : h \mapsto hf'(t)$ is defined on $\mathbb{R}$, for f to be a immersion at t is equivalent to having $f'(t) = 0$ (for other dimension, e.g. on $\mathbb{R}^2$ instead of $\mathbb{R}$, this condition is not sufficient to get an immersion though). It is clearly the case ($\cos(t) = 0 \iff t \in \frac{\pi}{2} + \pi\mathbb{Z}$ and for such t's, we have $\cos(2t) = -1 \neq 0$), thus f is an immersion.

A trigonometric direct study shows that f is injective on $[0,T]$ (in fact, on the whole interval $]0, 2\pi[$). To prove that S_T is a manifold, it remains to show that it is an open map (for the relative topology of S_T), which we don't do in this sheet. Hint for this: show that $f(]a,b[)$ is open, beginning with the case $0 < a < b < \frac{\pi}{2}$, then $\frac{\pi}{2} < a < b < T$ (you will have to use the hypothesis $T \leq \pi$ here), then in the last case $0 < a < \frac{\pi}{2} < b < T$.

Now assume that $T > \pi$. For a proof by contradiction, let (U,ϕ) be a chart of S_T around $(\sin(\pi), \sin(2\pi)) = (0,0)$. Thus, $\phi(U)$ is an open neighbourhood of $\phi(0,0)$ in $\mathbb{E}^1$. Up to taking the intersection with a small ball of $\mathbb{E}^1$ centered at $\phi(0,0)$, we can assume that $\phi(U)$ is connected, hence it is an open interval. Since ϕ is a homeomorphism, the

[1] Give a careful argument to show that 0 does not have a neighborhood in X which is homeomorphic to an open subset of $\mathbb{E}^1$.

same goes for $\phi(U)$. But any small enough neighborhood of $(0,0)$ in S_T has (at least) 3 connected components after removing $(0,0)$: the one of a point $(\sin(t), \sin(2t))$ with $0 < t \ll 1$, the one for $t \lesssim \pi$ and the one for $t \gtrsim \pi$, giving a contradiction because $\phi(U) - \{\phi(0,0)\}$ only has 2 connected components.

3. Let (U_1, ϕ_1), (U_2, ϕ_2), (U_3, ϕ_3) be charts such that (U_1, ϕ_1) and (U_2, ϕ_2) are compatible and (U_2, ϕ_2) and (U_3, ϕ_3) are compatible. Prove that $(U_1 \cap U_2, \phi_1)$ and $(U_3 \cap U_2, \phi_3)$ are compatible.

Solution. $\phi_1 \circ (\psi_3|_{U_1 \cap U_2 \cap U_3})^{-1} = (\phi_1 \circ \phi_2^{-1}) \circ (\phi_2 \circ (\phi_3|_{U_1 \cap U_2 \cap U_3})^{-1})$ is smooth as the composition of two smooth functions.

4. Let $\mathscr{U}$ be a smooth atlas on a topological manifold M, and let (V, ψ) be compatible with $\mathscr{U}$. Prove that ψ is smooth in the smooth structure that $\mathscr{U}$ determines.

Solution. The assumptions imply that $\mathscr{U} \cup \{(V, \psi)\}$ is a smooth atlas that contains $\mathscr{U}$. It determines a unique maximal atlas that has to be the unique maximal atlas that contains $\mathscr{U}$.

5. Prove that the stereographic projection $\mathscr{S}\colon \mathbb{S}^2 - \{\mathbf{e}_3\} \to \mathbb{E}^2$ is compatible with the standard smooth structure of $\mathbb{S}^2$.

Solution. We recall the formulae (a drawing is helpful here)

$$S : (x, y, z) \mapsto \frac{1}{1-z}(x, y) \quad \text{and} \quad S^{-1} : u \mapsto \frac{1}{1 + \|u\|^2}(2u, \|u\|^2 - 1).$$

We have to show that for every chart (U, ϕ) in the standard structure of $\mathbb{S}^2$, the maps $S \circ \phi^{-1}$ and $\phi \circ S^{-1}$ are smooth. Thanks to Exercise 4, we only have to prove this for the charts of the form $(U_k^{\pm}, \mathrm{pr}_k)$ (notations from the lecture notes) for all $k \in \{1, 2, 3\}$. Let us check the compatibility of S only with (U_1^+, pr_1). We have $U_1^+ \subset \mathbb{S}^2 - \{e_3\}$, and for all (y, z) in the disk $\mathrm{pr}_1(U_1^+) = \{(y, z) \in \mathbb{E}^2 \ : \ y^2 + z^2 < 1\}$, we compute

$$S \circ \mathrm{pr}_1^{-1}(y, z) = S\Big(\sqrt{1 - y^2 - z^2}, y, z\Big) = \frac{1}{1-z}\Big(\sqrt{1 - y^2 - z^2}, y\Big)$$

and for all $(x, y) \in S(U_1^+)$,

$$\mathrm{pr}_1 \circ S^{-1}(x, y) = \frac{1}{1 + x^2 + y^2}(2y, x^2 + y^2 - 1)$$

which are indeed the formulae of smooth maps.