Differential geometry 2023

Exercises 11

1. Compute the expression of the form $dx^1 \wedge dx^2 \wedge dx^3$ in spherical coordinates.

Solution. The spherical coordinates of a point $x \in \mathbb{E}^3 \setminus \{0\}$ are given by

$$x = (r\cos\theta_1 \sin\theta_2, r\sin\theta_1\sin\theta_2, r\cos\theta_2)\,.$$

Thus,

$$\begin{aligned} dx^1 &= \cos\theta_1\sin\theta_2 dr - r\sin\theta_1\sin\theta_2 d\theta_1 + r\cos\theta_1\cos\theta_2 d\theta_2\ , \\ dx^2 &= \sin\theta_1\sin\theta_2 dr + r\cos\theta_1\sin\theta_2 d\theta_1 + r\sin\theta_1\cos\theta_2 d\theta_2\ , \\ dx^3 &= \cos\theta_2 dr - r\sin\theta_2 d\theta_2 \end{aligned}$$

Thus,

$$\begin{aligned} &dx^1 \wedge dx^2 \wedge dx^3 = \cos\theta_1\sin\theta_2 dr \wedge r\cos\theta_1\sin\theta_2 d\theta_1 \wedge (-r\sin\theta_2 d\theta_2) \\ &- r\sin\theta_1\sin\theta_2 d\theta_1 \wedge (\sin\theta_1\sin\theta_2 dr \wedge (-r\sin\theta_2 d\theta_2) + r\sin\theta_1\cos\theta_2 d\theta_2 \wedge \cos\theta_2 dr) \\ &+ r\cos\theta_1\cos\theta_2 d\theta_2 \wedge r\cos\theta_1\sin\theta_2 d\theta_1 \wedge \cos\theta_2 dr \\ &= r^2\cos^2\theta_1\sin^3\theta_2 dr \wedge d\theta_1 \wedge d\theta_2 + r^2\sin^2\theta_1\sin^3\theta_2 d\theta_1 \wedge dr \wedge d\theta_2 \\ &- r^2\sin^2\theta_1\sin^2\theta_2\cos\theta_2 d\theta_1 \wedge d\theta_2 \wedge dr + r^2\cos^2\theta_1\cos^2\theta_2\sin\theta_2 d\theta_2 \wedge d\theta_1 \wedge dr \\ &= -r^2(\cos^2\theta_1\sin^3\theta_2 + \sin^2\theta_1\sin^3\theta_2 + \sin^2\theta_1\sin^2\theta_2\cos\theta_2 + r^2\cos^2\theta_1\cos^2\theta_2\sin\theta_2) dr d\theta_1 d\theta_2 \\ &= -r^2\sin\theta_2 dr d\theta_1 d\theta_2 \end{aligned}$$

2. Let $\mathbb{E}^3 \times \mathbb{E}^1$ be the 4-dimensional *spacetime*, where the first 3-dimensional component x of (x,t) corresponds to *space* and the fourth component t is *time*. Let $\mathbf{E}\colon \mathbb{E}^3 \times \mathbb{E}^1 \to \mathbb{E}^3$ be the *electric field* and let $\mathbf{B}\colon \mathbb{E}^3 \times \mathbb{E}^1 \to \mathbb{E}^3$ be the *magnetic field*. *Maxwell's equations* in the vacuum without charge or current are

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}\,, \qquad \nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t}\,, \qquad \nabla \cdot \mathbf{E} = 0\,, \qquad \nabla \cdot \mathbf{B} = 0\,. \tag{1}$$

In these equations, the curl and the divergence are taken with respect to the space coordinates. Maxwell's equations can be formulated using differential forms and the exterior derivative if we define a 1-form E using the components of the electric field

$$E = E_1 dx^1 + E_2 dx^2 + E_3 dx^3\,,$$

a 2-form B using the components of the magnetic field

$$B = B_1 dx^2 \wedge dx^3 + B_2 dx^3 \wedge dx^1 + B_3 dx^1 \wedge dx^2\,,$$

and in the 4-dimensional spacetime the 2-form

$$F = E \wedge dt + B\,.$$

The equation $dF = 0$ corresponds to two of Maxwell's equations (1). Which equations are these two?

Solution.

$$\begin{aligned}
dF &= dE_1 \wedge dx^1 \wedge dt + dE_2 \wedge dx^2 \wedge dt + dE_3 \wedge dx^3 \wedge dt + \\
&\qquad dB_1 \wedge dx^2 \wedge dx^3 + dB_2 \wedge dx^3 \wedge dx^1 + dB_3 \wedge dx^1 \wedge dx^2 \\
&= \frac{\partial E_1}{\partial x^2} dx^2 \wedge dx^1 \wedge dt + \frac{\partial E_1}{\partial x^3} dx^3 \wedge dx^1 \wedge dt + \frac{\partial E_2}{\partial x^1} dx^1 \wedge dx^2 \wedge dt + \frac{\partial E_2}{\partial x^3} dx^3 \wedge dx^2 \wedge dt \\
&+ \frac{\partial E_3}{\partial x^1} dx^1 \wedge dx^3 \wedge dt + \frac{\partial E_3}{\partial x^2} dx^2 \wedge dx^3 \wedge dt + \frac{\partial B_1}{\partial x^1} dx^1 \wedge dx^2 \wedge dx^3 + \frac{\partial B_1}{\partial t} dt \wedge dx^2 \wedge dx^3 \\
&+ \frac{\partial B_2}{\partial x^2} dx^2 \wedge dx^3 \wedge dx^1 + \frac{\partial B_2}{\partial t} dt \wedge dx^3 \wedge dx^1 + \frac{\partial B_3}{\partial x^3} dx^3 \wedge dx^1 \wedge dx^2 + \frac{\partial B_3}{\partial t} dt \wedge dx^1 \wedge dx^2 \\
&= \left(\frac{\partial E_3}{\partial x^2} - \frac{\partial E_2}{\partial x^3} + \frac{\partial B_1}{\partial t} \right) dx^2 \wedge dx^3 \wedge dt + \left(\frac{\partial E_1}{\partial x^3} - \frac{\partial E_3}{\partial x^1} + \frac{\partial B_2}{\partial t} \right) dx^3 \wedge dx^1 \wedge dt \\
&+ \left(\frac{\partial E_2}{\partial x^1} - \frac{\partial E_1}{\partial x^2} + \frac{\partial B_3}{\partial t} \right) dx^1 \wedge dx^2 \wedge dt + \left(\frac{\partial B_1}{\partial x^1} + \frac{\partial B_2}{\partial x^2} + \frac{\partial B_3}{\partial x^3} \right) dx^1 \wedge dx^2 \wedge dx^3
\end{aligned}$$

Thus, $dF = 0$ is equivalent with the equations $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ and $\nabla \cdot \mathbf{B} = 0$.

3. Let M be a smooth manifold. Let $\omega \in \Omega^k(M)$ be an exact form and let $\tau \in \Omega^\ell(M)$ be a closed form. Prove that $\omega \wedge \tau$ is an exact form.

Solution. Let $\widehat{\omega} \in \Omega^{k-1}(M)$ be a form such that $d\widehat{\omega} = \omega$. The form $\omega \wedge \tau$ is exact because

$$d(\widehat{\omega} \wedge \tau) = d\widehat{\omega} \wedge \tau + (-1)^k \widehat{\omega} \wedge d\tau = \omega \wedge \tau + (-1)^k \widehat{\omega} \wedge 0 = \omega \wedge \tau \,.$$

4. Let M be a compact manifold. Let $\omega \in \Omega^1(M)$ be a 1-form that has no zeros.[1] Prove that ω is not an exact form.

Solution. By contradiction, let us assume that ω is an exact form, hence we can choose a real valued smooth function $f \in \mathscr{F}(M)$ such that $\omega = df$. Since M is compact, the function f has a maximum at some point $p \in M$. Since the exterior derivative on $\mathscr{F}(M)$ is the differential, we have $df_p = 0$ i.e. p is a zero of $\omega = df$.

5. (1) Prove that the restrictions of the coordinate forms of $\mathbb{E}^3$ to the submanifold $\mathbb{S}^2 \subset \mathbb{E}^3$ satisfy[2] $x^1 dx^1 + x^2 dx^2 + x^3 dx^3 = 0$.
(2) Let $\omega = x^1 dx^2 \wedge dx^3 + x^2 dx^3 \wedge dx^1 + x^3 dx^1 \wedge dx^2 \in \Omega^2(\mathbb{S}^2)$. Prove that

$$\omega = \begin{cases} \dfrac{dx^2 \wedge dx^3}{x^1} \,, & \text{when } x^1 \neq 0 \\ \dfrac{dx^3 \wedge dx^1}{x^2} \,, & \text{when } x^2 \neq 0 \\ \dfrac{dx^1 \wedge dx^2}{x^3} \,, & \text{when } x^3 \neq 0 \end{cases}$$

Solution. (1) By Exercise 3 of week 8,[3], since $\mathbb{S}^2$ is the regular level set $f^{-1}(1)$ of the smooth function $f = (x^1)^1 + (x^2)^2 + (x^3)^2$, then on $\mathbb{S}^2$ we have

$$0 = d((x^1)^2 + (x^2)^2 + (x^3)^2) = 2(x^1 dx^1 + x^2 dx^2 + x^3 dx^3) \,.$$

[1] $p \in M$ is a *zero* of ω if $\omega_p = 0$.
[2] Exercise 3 of week 8 may be useful here.
[3] Exercise 6.12 in the Finnish text.

(2) Assume that $x^1 \neq 0$. Part (1) implies that $x^1 dx^1 = -(x^2 dx^2 + x^3 dx^3)$. Therefore,

$$x^2 dx^3 \wedge dx^1 = -\frac{(x^2)^2 dx^3 \wedge dx^2}{x^1} = \frac{(x^2)^2 dx^2 \wedge dx^3}{x^1}$$

and

$$x^3 dx^1 \wedge dx^2 = -\frac{(x^3)^2 dx^3 \wedge dx^2}{x^1} = \frac{(x^3)^2 dx^2 \wedge dx^3}{x^1}\,.$$

Therefore,

$$\begin{aligned}\omega &= x^1 dx^2 \wedge dx^3 + \frac{(x^2)^2 dx^2 \wedge dx^3}{x^1} + \frac{(x^3)^2 dx^2 \wedge dx^3}{x^1} \\ &= \frac{(x^1)^2 + (x^2)^2 + (x^3)^2)\, dx^2 \wedge dx^3}{x^1} = \frac{dx^2 \wedge dx^3}{x^1}\,.\end{aligned}$$

The other cases are treated in the same way.