Consider the Lotka-Volterra predator prey model

\[
\begin{bmatrix}
x_1'(t) \\
x_2'(t)
\end{bmatrix}
= f(t) :=
\begin{bmatrix}
ax_1(t) - bx_1(t)x_2(t) \\
cx_1(t)x_2(t) - dx_2(t)
\end{bmatrix},
\]

where \(x_1(t)\) and \(x_2(t)\) are the prey and predator populations, respectively. Parameters have the following interpretation:

- \(a\) is the exponential growth rate of the prey population.
- \(x_1x_2\) is related to the encounter probability of species, this encounter decreases the prey population by rate \(b\) and increases the predator population by rate \(c\).
- \(d\) is the death rate of the predator population.

Combined with some initial condition

\[
x(0) = x_0
\]

this system models the interplay of the population in time.

1. Implement the explicit Euler method to generate approximations of the problem (1)-(2). Try different step lengths and parameters (good start is \(a = b = c = d = 1\)). Plot results as the evolution of the time components and in the phase plane, i.e., \(x_1x_2\)-coordinates. In the phase plane, plot also the vector field generated by \(f(x_1, x_2)\) (hint: \texttt{help quiver}) and the initial point \(x_0\). The resulting figure should resemble Figure 1.

2. How does the solution look like? What happens if the step length is not small enough?

3. Implement also Runge-Kutta 4, the implicit Euler method, and the implicit midpoint rule, use, e.g., Newton method to solve the nonlinear system arising in implicit method. Compare the behavior of different time integrators changing the length of the time step.
Explicit Euler method: $a=1$ $b=1$ $c=1$ $d=1$ $T=50$ $h=0.010002$

Figure 1: Approximation in the phase plane.