Finite element approximations in 1D
corve study and solver

Generate finite element solver in 1D. It should be able to solve a given problem using piecewise linear and piecewise quadratic approximations. In 1D, using the proper FE-assembly routines is not a necessity, since inner products of the global basis functions can be computed by pen and paper (in particular with linear basis functions). However, it is very educational to build a FE-solver, which has the following structures:

Mesh
The mesh consists of nodes and elements. For example a mesh of linear elements, consisting of only two linear basis functions associated to nodes is

\[
\begin{align*}
\text{nodes} &= [ 0.0 \ 0.25 \ 0.5 \ 0.75 \ 1.0 ] \\
\text{elems} &= \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{bmatrix}
\end{align*}
\]

In other words, every element contains two consecutive nodes. Second order element contains three nodes and the mesh data would be

\[
\begin{align*}
\text{nodes} &= [ 0.0 \ 0.25 \ 0.5 \ 0.75 \ 1.0 ] \\
\text{elems} &= \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 3 & 5 \end{bmatrix}
\end{align*}
\]

Thus there are only two elements in the mesh albeit the amount of nodes is the same as in the linear case.

Reference element
A convenient reference element is \([-1,1]\), where nodes are at the endpoints. The basis functions on the reference element are to be solved (or found from the literature/google). Linear and quadratic local basis functions on the reference element are plotted in Figures 1 and 2, respectively. Integration quadratures for can be easily find by searching for Gaussian quadratures.

Affine mapping and assembly
The integration quadrature is to be computed only on the reference element and the inner products are computed using only the values of local basis functions on the reference element.
Figure 1: Local first order basis functions on a reference element

Figure 2: Local second order basis functions on a reference element
Exercise

There is some programming to do and it might be most convenient to do it in pieces.

1. Compute or find local basis functions on the reference element.

2. Compute the affine mapping $F_k : [-1, 1] \rightarrow [x_k, 1, x_k, \text{end}]$ from the reference element to a local element. Create some plots to make sure that you can indeed generate basis functions on any element using it.

3. Compute (pen and paper is useful) how the integration is transformed on a reference element.

4. Make the matrix assembly routine and a solver (again) for the problem

\[-u''(x) = f(x), \quad u(0) = u(1) = 0.\]

5. Use a known solution, e.g., $u(x) = \sin(k\pi x)$ and $f(x) = k^2 \pi^2 \sin(k\pi x)$, where $k \in \mathbb{N}$, to study the convergence rates for linear and quadratic elements.