Various solvers

Next we consider various solution methods, which generate approximations lying outside of the proper solution space. For example, they may not be differentiable enough, may not satisfy boundary conditions etc.

Nonconforming FEM

Consider first a second order PDE. After shifting derivatives by the integration by parts formula, elementwise differentiability and continuity over the element edges is required.

- A classical example of a nonconforming element is a triangle, where nodes are not located at the corners but to the midpoints on edges (Ciarlet 1978).

- Rotated Q1-element (rectangular) has also degrees of freedom at nodal values (or mean values of edge integrals) and the basis functions belong to the space span\{1, x, y, x^2 - y^2\}.

In the case of fourth order PDEs, motivated by the plate bending problem, the differentiability requirements are severe. Global functions should have continuous first order derivatives. This requires (at least 18 DOF) on each element, which is computationally costly and unpleasant to implement. The oldest nonconforming plate element is the Morley element. It is a triangle with following local DOFs: nodal values at corners and normal derivatives at the midpoints of edges. There are plenty of different plate elements.

Fictitious domain method

Consider the Poisson problem

$$-\text{div}(A \nabla u) = f, \quad \text{in } \Omega,$$
$$u = 0, \quad \text{on } \partial\Omega \cup \partial\omega,$$

where the domain $\Omega$ contains hole(s) $\omega \subset \Omega$. The idea is to solve the problem in the entire domain instead, but to modify it in such a way that the
original boundary conditions (and the rest of the problem) are approximatively satisfied. Let \( \widehat{\Omega} := \Omega \cup \omega \) and define the following variational problem: Find \( \widehat{u} \in H^1_0(\widehat{\Omega}) \) s.t.,
\[
\int_{\widehat{\Omega}} \left( \widehat{A}_\varepsilon \nabla \widehat{u} \cdot \nabla \widehat{w} + \widehat{b}_\varepsilon \widehat{u} \widehat{w} \right) \, dx = \int_{\widehat{\Omega}} \widehat{f} \widehat{w} \, dx, \quad \forall \widehat{w} \in H^1_0(\widehat{\Omega}),
\]
where, e.g.,
\[
\widehat{A}_\varepsilon = \begin{cases} A & \text{in } \Omega \\ \frac{1}{\varepsilon} I & \text{in } \omega \end{cases}, \quad \widehat{b}_\varepsilon = \begin{cases} 0 & \text{in } \Omega \\ \frac{1}{\varepsilon} & \text{in } \omega \end{cases}, \quad \widehat{f} = \begin{cases} f & \text{in } \Omega \\ 0 & \text{in } \omega \end{cases}.
\]
Note that \( g \) is some continuation of \( f \) on \( \omega \). If the problem is written in the minimization form
\[
\min_{\widehat{u} \in H^1_0(\Omega)} \left\{ \int_{\Omega} \left( A \nabla \widehat{u} \cdot \nabla \widehat{u} - 2f \widehat{u} \right) \, dx + \int_{\omega} \left( \frac{1}{\varepsilon} \left| \nabla \widehat{u} \right|^2 + \frac{1}{\varepsilon} \widehat{u}^2 \right) \, dx \right\}
\]
one can see that the last term is related to the penalization of the function being other than zero in \( \omega \). This does not force the boundary condition at \( \partial \omega \) to be correct, but approximates it somehow.

**Treffts method**

This method is applicable, if one can come up with a basis that satisfies the differential equation exactly. Typically constant coefficients are required. Consider the problem,
\[
-\Delta u = 0, \quad \text{in } \Omega \\
u = g, \quad \text{in } \partial \Omega.
\]
Now we can use harmonic basis functions \( \Delta \psi_i = 0 \) to generate a harmonic approximation
\[
\tilde{u} = \sum_{i=1}^N \alpha_i \psi_i.
\]
However, \( \tilde{u} \) does not satisfy boundary conditions. The coefficients \( \alpha_i \) are selected in such a way that the quantity \( \| \nabla (u - \tilde{u}) \|^2 \) is minimized. Expanding the squared norm and omitting \( \| \nabla u \|^2 \) (since it does not depend on \( \alpha_i \)'s) leads to a minimization problem
\[
\min_{\tilde{u} \in H^1_0(\Omega)} J(\tilde{u}) := \int_{\Omega} \left( |\nabla \tilde{u}|^2 - 2 \nabla \tilde{u} \cdot \nabla u \right) \, dx.
\]
The last term depending on unknown exact solution \( u \) can be rewritten using the properties of the basis functions and the boundary condition. By the Gauss-Ostrogradskii Theorem

\[
\int_{\Omega} \nabla \tilde{u} \cdot \nabla u \, dx = -\int_{\Omega} (\Delta \tilde{u}) u \, dx + \int_{\partial \Omega} \tilde{u} \nabla \cdot n \, ds.
\]

Substituting this to the minimization problem yields

\[
\min_{\tilde{u} \in H^1_0(\Omega)} J(\tilde{u}) := \int_{\Omega} |\nabla \tilde{u}|^2 - 2 \int_{\partial \Omega} g \nabla \tilde{u} \cdot n \, ds,
\]

where everything is known. Minimization leads to a linear problem

\[
S \alpha = b,
\]

where

\[
S_{ij} = \int_{\Omega} \nabla \psi_j \cdot \nabla \psi_j \, dx, \quad \text{and} \quad b_i = \int_{\partial \Omega} g \nabla \psi_i \cdot n \, ds.
\]

Note that if \( g \) can be extended to the whole \( \Omega \) and is differentiable, then (again, by the Gauss-Ostrogradskii theorem) \( b \) can be computed also as follows:

\[
b_i = \int_{\partial \Omega} g \nabla \psi_i \cdot n \, ds = \int_{\Omega} (\Delta \psi_i g + \nabla \psi_i \cdot \nabla g) \, dx = \int_{\Omega} \nabla \psi_i \cdot \nabla g \, dx.
\]

**Finite volume method (FVM)**

The finite volume method is focused on the conservation law. In our model problem

\[
\begin{align*}
- \text{div} p &= f, \quad \text{in } \Omega \\
p &= \kappa \nabla u, \quad \text{in } \Omega
\end{align*}
\]

it means the equilibrium condition (1). The idea is to discretize the conservation law expressed via control volumes, i.e., for any \( T \subset \Omega \) holds

\[
- \int_{T} p \cdot n \, ds = \int_{T} f \, dx.
\]

The approximation is based on the following ingredients, which can be selected in various ways.
• Dividing $\Omega$ into a finite set of control volumes $T_j$.

• Approximating $p$ by it’s values on control volume edges requiring that edge values satisfy (3).

• Approximating condition (2) by connecting the edge values of the flux and the value of approximated $u$ inside the control domains.

The main advantage of FVM is that the conservation law is satisfied discretely. Moreover, it can be applied to very unstructured meshes. There are two main ways how to select the control volumes: Consider a scheme, where $u$ is approximated by a single nodal value inside every control volume

• **cell-centered**: First define control volumes and set nodes at the center. This way the nodal value represents the mean over the control volume well.

• **vertex-centered**: First define nodes and set volume boundaries to the middle between nodes. This way the difference of nodal values represents the value of the gradient at the edge well.

This method is widely used in the engineering community, in particular in fluid flow simulations. This is due to the emphasis on the conservation law.