Convection dominated problems

Next, we consider a special class of singularly perturbed problems. At first glance, they do not appear daunting, but the appearance of the “small parameter” complicates the numerical treatment of the problem.

Let us consider first the example

\[-\varepsilon u'' + bu' = f, \quad x \in (0, 1)\]
\[u(0) = u(1) = 0.\]

If \(b = 1\) and \(f = 1\), then it is easy to confirm that the solution is

\[u(x) = x - \frac{\exp(-\frac{1}{\varepsilon}) - \exp(-\frac{1}{\varepsilon})}{1 - \exp(-\frac{1}{\varepsilon})}.\]

The difficulties of “small \(\varepsilon\)” can be predicted from the following “inconsistency”. At points \(x_0 \in [0, 1)\) everything is fine, i.e.,

\[\lim_{x \to x_0} \lim_{\varepsilon \to 0^+} u(x) = x_0 \lim_{\varepsilon \to 0^+} \lim_{x \to x_0} u(x),\]

but at \(x = 1\) we have

\[\lim_{x \to 1} \lim_{\varepsilon \to 0^+} u(x) = 1, \quad \text{and} \quad \lim_{\varepsilon \to 0^+} \lim_{x \to x_0} u(x) = 0.\]

Presence of such problematic point means that the problem is singularly perturbed. Moreover, the abrupt change of the solution as \(x\) approaches 1 is called a boundary layer.

Central finite difference scheme

Consider a standard finite difference scheme, where the derivative is approximated by central differences

\[D_c u(x) := \frac{u(x + h) - u(x - h)}{2h},\]

and the second order derivative is approximated by

\[D^2 u(x) := \frac{u(x + h) - 2u(x) + u(x - h)}{h^2}.\]

If one recalls the Taylor polynomials used to derive the difference schemes, the accuracy of the central difference is \(O(h^2)\).
Upwind scheme

In the upwind scheme, depending on the sign of the convection term (here, we assume that \( b \neq 0 \), i.e., there are no “turning points”) one selects either a forward or backward difference approximation of the derivative depending on the “direction” of the convection term, i.e.,

\[
D_{uw} = \begin{cases} 
D_+ & \text{if } b < 0, \\
D_- & \text{if } b > 0,
\end{cases}
\]

where

\[
D_+ u(x) := \frac{u(x + h) - u(x)}{h} \quad \text{and} \quad D_- u(x) := \frac{u(x - h) - u(x)}{h}.
\]

These approximations of the derivatives are of order \( O(h) \).

Il’in-Allen-Southwell

The remedy is to introduce artificial diffusion to the discrete problem and using a fitting factor \( \sigma \).

\[-\varepsilon \sigma(q(x)) D^2 u + b D_0 u + cu = f,\]

where \( q(x) := \frac{b(x)h}{2\varepsilon} \). Note that if we set \( \sigma(q) = 1 + q \), then we end up with the standard upwind scheme. The literature concerning a good selection (rate of convergence, stability, uniform convergence) of \( \sigma \) is vast, but here we present only a selection

\[\sigma(q) = q \coth q,\]

which yields a uniformly stable and second order consistent Il’in-Allen-Southwell scheme.

What is the error doing?

If \( \varepsilon \gg 0 \), then both methods are stable and the central difference scheme (second order method) performs as well as expected. However, if \( \varepsilon \) is close to zero then (unless the mesh is extremely dense) the central difference method generates a poor approximate solution with spurious oscillations and the upwind scheme remains stable. However, the upwind scheme suffers from the property that the maximal nodal error

\[e_\infty = \max_i |u - u_h(x_i)|\]
decreases first, but when the mesh is so dense that node enters the boundary layer the error suddenly increases. Importantly, Il'in-Allen-Southwell converges uniformly and is stable.

Layer-adapted meshes

The idea of is to locate more nodes on the boundary layer, so that the abrupt change of the solution is captured, even with the central difference scheme. Here we consider meshes generated by a mesh generating function $\lambda : [0, 1] \rightarrow [0, 1]$, which satisfies $\lambda(0) = 0$ and $\lambda(1) = 1$. The grid points are defined by $x_i = \lambda(i/N)$, for $i = 0, 1, \ldots, N$.

Assume that like in our case we have an exponential boundary layer function, i.e., solution behaves like $y = \exp(-\gamma(1-x)/\varepsilon)$ for some fixed $\gamma$. The general idea of Bakhvalov 1969 was to use such mesh points $x_i$ that the corresponding values at $y_i$ are equidistantly distributed. This condition means that

$$\exp\left(-\frac{\gamma(1-x_i)}{\varepsilon}\right) = \frac{i}{N}$$

or

$$x_i = 1 + \varepsilon \frac{\gamma}{N} \ln\left(\frac{i}{N}\right).$$

Of course this behaviour occurs only on the boundary layer and the mesh generating function has two parts.

$$\lambda_{Ba}(t) = \begin{cases} 
\psi(t) = 1 + A \ln \left(1 - \frac{1-t}{q}\right), & t \in [1 - \tau, 1], \\
\pi(t) = \psi(1-t) + (t - 1 + \tau)\psi'(1-t), & t \in [0, 1 - \tau],
\end{cases}$$

where $\tau$ is a transition point defined only implicitly by

$$\psi(1 - \tau) + (\tau - 1)\psi'(1 - \tau) = 0.$$  

A more popular, but slightly less efficient, choice is a piecewise equidistant Shishkin mesh. It consists of two uniform meshes with, e.g., equal number of nodes. One is very dense near the boundary layer and the other one is very coarse.

Summary and Remarks

- Singularly perturbed problems have been rigorously analyzed by mathematicians using the perturbation theory and related expansions.
• If problem contains boundary layers, any classical methods relying on equidistant meshes require unacceptable amount of mesh nodes.

• Other methods to generate layer adaptive meshes rely on a monitor function or recursion.