Semidiscrete Galerkin method

In time dependent problems, the spatial domain can be approximated using the Galerkin (Bubnov/Petrov) method, while the temporal (time related) derivatives are approximated by differences. This approach is called semidiscrete Galerkin, we present here the classical model examples and apply FEM to generate the Galerkin basis.

Heat equation

Consider a heat equation (parabolic model problem)

\[
\frac{\partial u}{\partial t} - \Delta u = f, \quad \text{in } \Omega \times [0, T]
\]

\[
u = 0, \quad \text{on } \partial \Omega \times [0, T]
\]

\[
u(x, 0) = u_0, \quad \text{in } \bar{\Omega}.
\]

As before we use the integration (over \(\Omega\)) by parts formula to obtain a weak formulation (for fixed \(t \in [0, T]\)) suitable for Galerkin approximation. Find \(u\) s.t.

\[
\int_{\Omega} \frac{\partial u}{\partial t} w \, dx + \int_{\Omega} \nabla u \cdot \nabla w \, dx = \int_{\Omega} fw \, dx, \quad \forall w
\]

\[
u(x, 0) = u_0, \quad \text{in } \bar{\Omega}.
\]

The semidiscrete Galerkin approximation is obtained, by selecting some finite dimension subspace spanned by functions \(\{\psi_1, \ldots, \psi_N\}\). Then

\[
u_h(x, t) := \sum_{k=1}^{N} q_k(t) \psi_k(x)
\]

and the weak formulation (initial condition is discussed later) becomes

\[
\sum_{k=1}^{N} \frac{\partial q_k}{\partial t} \int_{\Omega} \psi_k \psi_j \, dx + q_k \int_{\Omega} \nabla \psi_k \cdot \nabla \psi_j \, dx = \int_{\Omega} f \psi_j \, dx, \quad \forall j \in \{1, \ldots, N\}.
\]
This can be written in a matrix form (recalling the definitions of global FE-matrices) as
\[ \mathbf{M} \mathbf{q}'(t) + \mathbf{S} \mathbf{q}(t) = \mathbf{f}(t). \]

Observe that we started with a time-dependent PDE, but ended up with an ODE, which is a considerable simplification. Next, we construct the time grid \( \{t_0, t_1, \ldots, t_M\} \) and are interested only on (“time-nodal”) values \( q_j^k = q_k(t_j), \ j \in \{1, \ldots, M\} \). Then, we can apply any earlier discussed time stepping method to approximate \( q_k \), for example
\[ \frac{\mathbf{M} \mathbf{q}_j^{i+1} - \mathbf{q}_j^i}{\Delta t} + \mathbf{S} \mathbf{q}_j^i = \mathbf{f}_j^i \quad \text{Explicit Euler method} \]
\[ \frac{\mathbf{M} \mathbf{q}_j^{i+1} - \mathbf{q}_j^i}{\Delta t} + \mathbf{S} \mathbf{q}_j^{i+1} = \mathbf{f}_j^i \quad \text{Implicit Euler method}. \]

The remaining task is to compute \( \mathbf{q}^0 \), i.e., \( q(0) \). It is defined by the initial condition \( u(x, 0) = u_0(x) \). Thus,
\[ u_h(x, 0) = \sum_{k=1}^{N} q_k(0) \psi_k(x) = u_0(x). \tag{1} \]

In general, we could again multiply by \( \psi_j \) and integrate, which would result in a system
\[ \mathbf{M} \mathbf{q}(0) = \mathbf{v}, \]
where
\[ v_j = \int_{\Omega} u_0(x) \psi_j(x) \, dx. \]

However, if basis functions \( \psi_k \) have nodal degrees of freedom, meaning that each of them is one at a particular node and zero in others, then we simply require (1) at each node \( x_k \) and obtain
\[ q_k(0) = u_0(x_k), \quad \forall k \in \{1, \ldots, \text{NOFnodes}\}. \]
Wave equation

Consider the wave equation (hyperbolic model problem)

\[
\frac{\partial^2 u}{\partial t^2} - \Delta u = f, \quad \text{in } \Omega \times [0, T],
\]

\[
u = 0, \quad \text{on } \partial \Omega_D \times [0, T],
\]

\[
n \cdot \nabla u = 0, \quad \text{on } \partial \Omega_N \times [0, T],
\]

\[
u(x, 0) = u_0, \quad \text{in } \bar{\Omega},
\]

\[
\frac{\partial u}{\partial t}|_{t=0} = u'_0, \quad \text{in } \bar{\Omega}.
\]

Repeating the procedure leads to a system

\[
Mq''(t) + Sq(t) = f(t),
\]

\[
q'(0) = q'_0,
\]

\[
q(0) = q_0.
\]

Again, there are multiple ways to numerically approximate “the time step”. Consider a following family of methods,

\[
M \frac{q^{j+1} - 2q^j + q^{j-1}}{\Delta t} + Sq^{j, \theta} = f^{j, \theta},
\]

where

\[
q^{j, \theta} := \theta q^{j+1} + (1 - 2\theta)q^j + \theta q^{j-1}.
\]

Selecting \( \theta = \frac{1}{4} \) yields a second order unconditionally stable implicit method and selecting \( \theta = 0 \) leads to a conditionally stable “leap-frog” method. Note that in order to start the method, one needs the solution at \( t^{-1} = -\Delta t \). It can be approximated (assuming that the wave equation holds at \( t = 0 \)) from the Taylor expansion as follows:

\[
q(-\Delta t) = q(0) - q'(0)\Delta t + \frac{1}{2}q''(0)(\Delta t)^2 + \mathcal{O}((\Delta t)^3)
\]

\[
= q_0 - q'_0\Delta t + \frac{1}{2}(\Delta t)^2M^{-1}(f^0 - Sq^0) + \mathcal{O}((\Delta t)^3).
\]