TIEA311 Tietokonegrafiikan perusteet

kevät 2019

("Principles of Computer Graphics" – Spring 2019)

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Adapted from: *Wojciech Matusik*, and *Frédo Durand*: 6.837 Computer Graphics. Fall 2012. Massachusetts Institute of Technology: MIT OpenCourseWare, https://ocw.mit.edu/.

Frontpage of the local course version, held during Spring 2019 at the Faculty of Information technology, University of Jyväskylä:
`http://users.jyu.fi/~nieminen/tgp19/`

TIEA311 - Today in Jyväskylä

Plan for today:

- Usual warm-up and group discussion
- Try to address the most urgent issues
- Break – reset the brain.
- Then continue with the theory.

TIEA311 - Today in Jyväskylä

We start by discussion, reflection and **questions**!

Work in groups of 3 students if possible:

- Fast warm-up: 90 seconds evenly split between group members (30s each in groups of 3), no interruptions from others: Foremost feelings right now?
- Reflection: Silent work, solo, 1 minute, **list words on paper**: What have you learned during the last week? Or since the course started?
- Interaction: 1.5 minutes group discussion: Compare if you learned the same or different things? Do those things feel useful? Why or why not?
  $\rightarrow$ Sum it up classwide.
- Interaction: Group work, 1.5 minutes or less if talk ends: At the moment, what would be the most helpful thing to help you (or others!)?
  $\rightarrow$ Sum it up classwide, and try to address the findings.

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What were the findings in group discussion?

What were found to be the most important issues to address right now?

$\rightarrow$ Classwide discussion is found on the lecture video.

NOTE: Even if you watch at home, please think about the same things and try to be in ”virtual dialogue” with those in classroom. Use pen and paper! I believe, more and more every day, that doing so will make your brain perform activities that help **your own learning**.

NOTE: Contemplate if you could watch the lecture videos with some friends who would also like to learn computer graphics? Get some pizza and coke if it helps you get to the mood(?).

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That’s All for Today, Folks

- Further reading
  - Buss, Chapters 7 and 8
  - Fun stuff to know about function/vector spaces
    - http://en.wikipedia.org/wiki/Vector_space
    - http://en.wikipedia.org/wiki/Functional_analysis
    - http://en.wikipedia.org/wiki/Function_space
- Inkscape is an open source vector drawing program for Mac/Windows. Try it out!

vectorportal.com

Polynomials as a Vector Space

- Polynomials $y(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n$
- Can be added: just add the coefficients

$$(y + z)(t) = (a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2 + \ldots + (a_n + b_n)t^n$$

- Can be multiplied by a scalar: multiply the coefficients

$$s \cdot y(t) = (s \cdot a_0) + (s \cdot a_1)t + (s \cdot a_2)t^2 + \ldots + (s \cdot a_n)t^n$$

Subset of Polynomials: Cubic

$$y(t) = a_0 + a_1\,t + a_2\,t^2 + a_3\,t^3$$

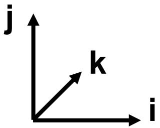
- Closed under addition & scalar multiplication
  - Means the result is still a cubic polynomial (verify!)
- Cubic polynomials also compose a vector space
  - A 4D **subspace** of the full space of polynomials
- The *x* and *y* coordinates of cubic Bézier curves belong to this subspace as functions of *t*.

Basis for Cubic Polynomials

More precisely:
What's a basis?

$$\vec{v} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$$

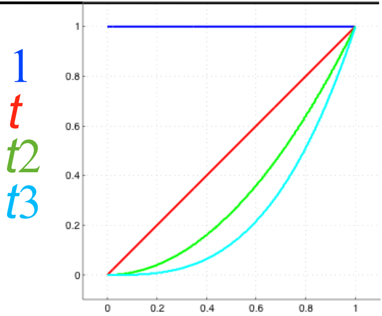
In 3D



- A set of "atomic" vectors
  - Called **basis vectors**
  - Linear combinations of basis vectors span the space
    - i.e. any cubic polynomial is a sum of those basis cubics
- Linearly independent
  - Means that no basis vector can be obtained from the others by linear combination
    - Example: **i**, **j**, **i**+**j** is not a basis (missing **k** direction!)

Canonical Basis for Cubics

$$\{1, t, t^2, t^3\}$$



- Any cubic polynomial is a linear combination of these:

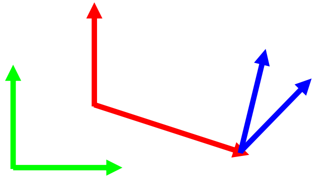
$a0+a1t+a2t2+a3t3 = a0*1+a1*t+a2*t2+a3*t3$

- They are linearly independent
  - Means you cannot write any of the four monomials as a linear combination of the others. (You can try.)

Different Basis

- For example:
  - {*1*, *1+t*, *1+t+t2*, *1+t-t2+t3*}
  - {*t3*, *t3+t2*, *t3+t*, *t3+1*}

2D examples



- These can all be obtained from $1, t, t^2, t^3$ by linear combination
- Infinite number of possibilities, just like you have an infinite number of bases to span R2

Matrix-Vector Notation

- For example:

$1,\ 1+t,\ 1+t+t^2,\ 1+t-t^2+t^3$

$t^3,\ t^3+t^2,\ t^3+t,\ t^3+1$

Change-of-basis matrix

"Canonical" monomial basis

These relationships hold for each value of *t*

$$\begin{pmatrix} 1 \\ 1+t \\ 1+t+t^2 \\ 1+t-t^2+t^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

$$\begin{pmatrix} t^3 \\ t^3+t^2 \\ t^3+t \\ t^3+1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

Matrix-Vector Notation

- For example:

1, 1+t, 1+t+t2, 1+t-t2+t3

t3, t3+t2, t3+t, t3+1

Change-of-basis matrix

"Canonical" monomial basis

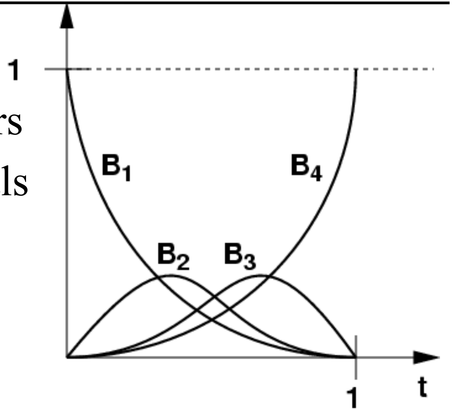
$$\begin{pmatrix} 1 \\ 1+t \\ 1+t+t^2 \\ 1+t-t^2+t^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

Not any matrix will do! If it's singular, the basis set will be linearly dependent, i.e., redundant and incomplete.

$$\begin{pmatrix} t^3 \\ t^3+t^2 \\ t^3+t \\ t^3+1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

Bernstein Polynomials

- For Bézier curves, the basis polynomials/vectors are Bernstein polynomials



- For cubic Bezier curve:

  $B1(t)=(1-t)^3$ $B2(t)=3t(1-t)^2$

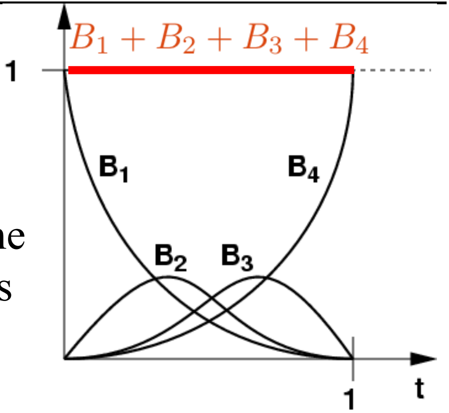
  $B3(t)=3t^2(1-t)$ $B4(t)=t^3$

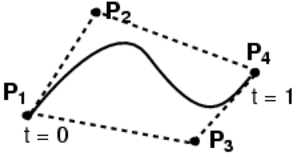
  (careful with indices, many authors start at 0)

- Defined for any degree

Properties of Bernstein Polynomials

- ≥ 0 for all $0 \leq t \leq 1$
- Sum to 1 for every *t*
  - called *partition of unity*
- These two together are the reason why Bézier curves lie within convex hull
- B1(0) =1
  - Bezier curve interpolates P1
- B4(1) =1
  - Bezier curve interpolates P4





Bézier Curves in Bernstein Basis

- $P(t) = P1B1(t) + P2B2(t) + P3B3(t) + P4B4(t)$
  - Pi are 2D points (xi, yi)
- P(t) is a linear combination of the control points with weights equal to Bernstein polynomials at t
- But at the same time, the control points (P1, P2, P3, P4) are the "coordinates" of the curve in the Bernstein basis
  - In this sense, specifying a Bézier curve with control points is exactly like specifying a 2D point with its x and y coordinates.

Two Different Vector Spaces!!!

- The plane where the curve lies, a 2D vector space
- The space of cubic polynomials, a 4D space
- Don't be confused!
- The 2D control points can be replaced by 3D points – this yields space curves.
  - The math stays the same, just add $z(t)$.
- The cubic basis can be extended to higher-order polynomials
  - Higher-dimensional vector space
  - More control points

Questions?

Change of Basis

- How do we go from Bernstein basis to the canonical monomial basis $1, t, t^2, t^3$ and back?
  - With a matrix!

- $B1(t)=(1-t)^3$
- $B2(t)=3t(1-t)^2$
- $B3(t)=3t^2(1-t)$
- $B4(t)=t^3$

$$\begin{pmatrix} B_1(t) \\ B_2(t) \\ B_3(t) \\ B_4(t) \end{pmatrix} = \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

New basis vectors

How You Get the Matrix

Cubic Bernstein:

- $B1(t)=(1-t)^3$
- $B2(t)=3t(1-t)^2$
- $B3(t)=3t^2(1-t)$
- $B4(t)=t^3$

Expand these out and collect powers of *t*. The coefficients are the entries in the matrix B!

$$\begin{pmatrix} B_1(t) \\ B_2(t) \\ B_3(t) \\ B_4(t) \end{pmatrix} = \overbrace{\begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}}^{\boldsymbol{B}} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

Change of Basis, Other Direction

- Given B1...B4, how to get back to canonical 1, t, t^2, t^3 ?

$$\begin{pmatrix} B_1(t) \\ B_2(t) \\ B_3(t) \\ B_4(t) \end{pmatrix} = \overbrace{\begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}}^{\boldsymbol{B}} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

Change of Basis, Other Direction

That's right, with the inverse matrix!

- Given B1...B4, how to get back to canonical 1, t, t^2, t^3 ?

$$\begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix} = \overbrace{\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1/3 & 2/3 & 1 \\ 0 & 0 & 1/3 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}}^{\boldsymbol{B}^{-1}} \begin{pmatrix} B_1(t) \\ B_2(t) \\ B_3(t) \\ B_4(t) \end{pmatrix}$$

Recap

- Cubic polynomials form a 4D vector space.
- Bernstein basis is canonical for Bézier.
  - Can be seen as influence function of data points
  - Or data points are coordinates of the curve in the Bernstein basis
- We can change between basis with matrices.

Questions?

More Matrix-Vector Notation

$$P(t) = \sum_{i=1}^{4} P_i \, B_i(t) = \sum_{i=1}^{4} \left[\begin{pmatrix} x_i \\ y_i \end{pmatrix} B_i(t) \right]$$

Bernstein polynomials
(4x1 vector)

$$P(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix} \begin{pmatrix} B_1(t) \\ B_2(t) \\ B_3(t) \\ B_4(t) \end{pmatrix}$$

point on curve
(2x1 vector)

matrix of
control points (2 x 4)

Flashback

$$\begin{pmatrix} B_1(t) \\ B_2(t) \\ B_3(t) \\ B_4(t) \end{pmatrix} = \overbrace{\begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}}^{\boldsymbol{B}} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

Cubic Bézier in Matrix Notation

point on curve
(2x1 vector)

$$P(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} =$$

Canonical
monomial basis

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix} \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

"Geometry matrix"
of control points P1..P4
(2 x 4)

"Spline matrix"
(Bernstein)

General Spline Formulation

$$Q(t) = \mathbf{GBT(t)} = \text{Geometry } \mathbf{G} \cdot \text{Spline Basis } \mathbf{B} \cdot \text{Power Basis } \mathbf{T(t)}$$

- Geometry: control points coordinates assembled into a matrix (P1, P2, …, Pn+1)
- Spline matrix: defines the type of spline
  - Bernstein for Bézier
- Power basis: the monomials (1, t, ..., *t*n)
- Advantage of general formulation
  - Compact expression
  - Easy to convert between types of splines
  - Dimensionality (plane or space) does not really matter

A Cubic Only Gets You So Far

- What if you want more control?

Higher-Order Bézier Curves

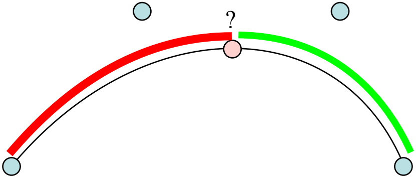
- $>$ 4 control points
- Bernstein Polynomials as the basis functions
  - For polynomial of order *n*, the *ith* basis function is

$$B_i^n(t) = \frac{n!}{i!(n-i)!} t^i (1-t)^{n-i}$$

- Every control point affects the entire curve
  - Not simply a local effect
  - More difficult to control for modeling
- You will not need this in this class

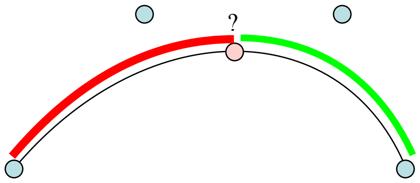
Subdivision of a Bezier Curve

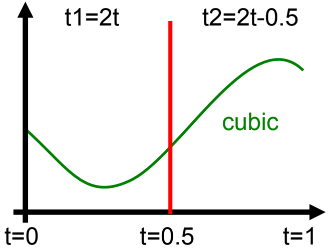
- Can we split a Bezier curve in the middle into two Bézier curves?
  - This is useful for adding detail
  - It avoids using nasty higher-order curves



Subdivision of a Bezier Curve

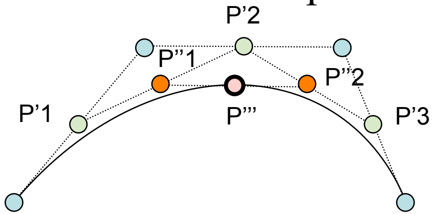
- Can we split a Bezier curve in the middle into two Bézier curves?
  - The resulting curves are again a cubic (Why? A cubic in t is also a cubic in $2t$)
  - Hence it must be representable using the Bernstein basis. So yes, we can!





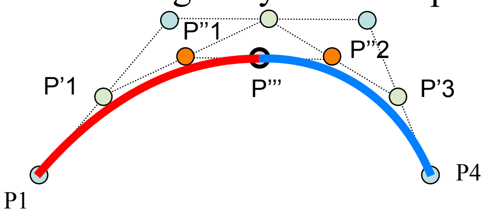
De Casteljau Construction

- Take the middle point of each of the 3 segments
- Construct the two segments joining them
- Take the middle of those two new segments
- Join them
- Take the middle point P'''



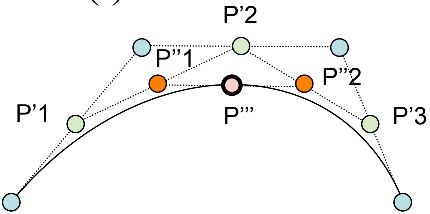
Result of Split in Middle

- The two new curves are defined by
  - P1, P'1, P''1, and P'''
  - P''', P''2, P'3, and P4
- Together they exactly replicate the original curve!
  - Originally 4 control points, now 7 (more control)



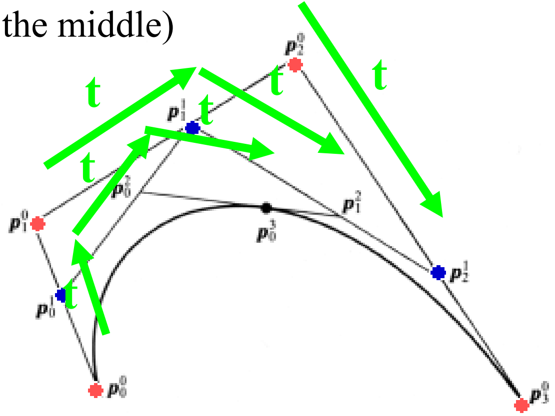
Sanity Check

- Do we actually get the middle point?
- $B1(t)=(1-t)^3$
- $B2(t)=3t(1-t)^2$
- $B3(t)=3t^2(1-t)$
- $B4(t)=t^3$



De Casteljau Construction

- Actually works to construct a point at any *t,* not just 0.5
- Just subdivide the segments with ratio (1-*t*), *t* (not in the middle)



Recap

- Bezier curves: piecewise polynomials
- Bernstein polynomials
- Linear combination of basis functions
  - Basis: control points weights: polynomials
  - Basis: polynomials weights: control points
- Subdivision by de Casteljau algorithm
- All linear, matrix algebra

That's All for Today, Folks

- Further reading
  - Buss, Chapters 7 and 8

  - Fun stuff to know about function/vector spaces
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    - http://en.wikipedia.org/wiki/Functional_analysis
    - http://en.wikipedia.org/wiki/Function_space

- Inkscape is an open source vector drawing program for Mac/Windows. Try it out!

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The time allotted for this lecture is now over.

Now: Break until tomorrow morning. Sleep if you have time.

But also **try to wake up and come to the lecture**!

We will pick up our thoughts soon enough!